

GLSMs, localisation, and the stringy Kähler moduli space

[with M. Romo, E. Scheidegger: arXiv:2003.00182[hep-th]] [with D. Erkinger: in progress]

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Outline

Overview

GLSMs and localisation

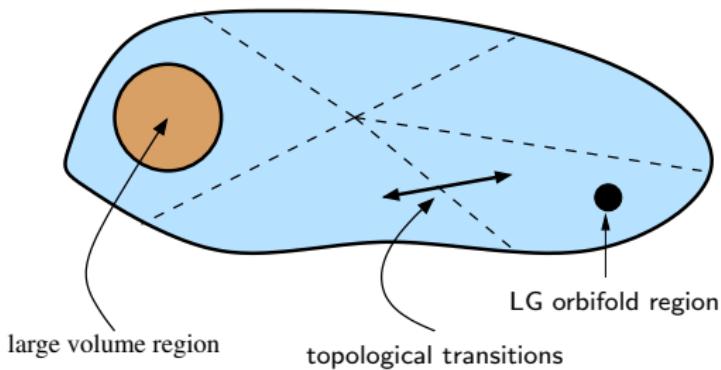
Hemisphere

Sphere

Conclusions

Stringy Kähler moduli space \mathcal{M}_K

- This talk: Calabi-Yau threefolds in a type II setting (+ branes)
- \mathcal{M}_K is divided into chambers



- Quantum corrections due to worldsheet instantons.
- One can cross the chamber boundaries.
- Categories associated to chambers (conjecturally) equivalent.
- Study \mathcal{M}_K in limiting regions that are not large volume.

\mathcal{M}_K via GLSMs

- The $2D (2,2)$ **gauged linear sigma model (GLSM)** provides a common UV description of the CFTs in \mathcal{M}_K . [Witten 93]
- **Phases:** different low-energy (IR) configurations depending on the values of the FI-theta parameters:

$$t = \zeta - i\theta \quad \leftrightarrow \quad t \in \mathcal{M}_K \quad \leftrightarrow \quad \text{chambers}$$

- Use the GLSM to map out \mathcal{M}_K .
- **SUSY localisation** computes (quantum) exact expressions.
 - Can be evaluated in **any phase**.
 - t -dependence \leftrightarrow worldsheet instanton corrections.
 - Compute quantum corrections directly in the GLSM.

Beyond geometry

- Geometric phases are well-understood and we know how to do computations.
 - Toric geometry, topological strings, mirror symmetry, enumerative invariants, etc.
- The structures we know in geometry should also be visible in non-geometric settings. Why?
 - The worldsheet CFT of the phases does not care whether there is a geometry.
 - Whatever is computed in non-geometric phases using localisation has the same UV origin as the geometric results.
- If we evaluate localisation results in phases of the GLSM, how do we interpret the result in the IR theory?

Universal structures in phases of GLSMs

- **Claim:** When evaluating the GLSM partition functions in different types of phases the (exact) result has the same structure in every phase.
- To see this, we need **structures that are available beyond geometric settings**, e.g.
 - Worldsheet: CFT, topological gravity, tt^* -geometry
 - FJRW theory
 - Givental's mirror construction
 - Categories
- **Ingredients** that appear in every phase:
 - State space and pairing
 - I/J-function
 - Gamma class
 - D-brane data

[Witten 92][Fan-Jarvis-Ruan 07]

[Givental 96-03]

[Hosono 00][Iritani 07][Katzarkov-Kontsevich-Pantev 08]

Results

- Hemisphere partition function:

[Romo-Scheidegger-JK 20]

$$Z_{D^2}^{phases}(\mathcal{B}) = \langle \text{ch}(\mathcal{B}), \Gamma \cdot I \rangle$$

- Works in geometry and Landau-Ginzburg orbifold phases.
- Sphere partition function:

[Erkinger-JK in progress]

$$Z_{S^2}^{phases} = \langle \bar{I}, I \rangle, \quad \langle \bar{I} | = (-1)^{\text{Gr}} \frac{\Gamma}{\Gamma^*} \bar{I}$$

- Works in hybrid phases that are LG orbifolds fibered over some base manifold.

GLSM data

- G ... a compact Lie group (gauge group)
- V ... space of chiral fields $\phi_i \in V$
- $\rho_V : G \rightarrow GL(V)$... faithful complex representation
 - CY condition: $G \rightarrow SL(V)$
- $R : U(1)_V \rightarrow GL(V)$... vector R-symmetry
 - R_i ... R-charges
- $T \subset G$... maximal torus
 - Lie algebras: $\mathfrak{g} = Lie(G)$, $\mathfrak{t} = Lie(T)$
 - $Q_i^a \in \mathfrak{t}_{\mathbb{C}}^*$... gauge charges of chiral fields

GLSM Data (ctd.)

- $t^a \in \mathfrak{g}_{\mathbb{C}}^*$... FI-theta parameters
 - $t^a = \zeta^a - i\theta^a$ ζ^a : real, θ^a : 2π -periodic
 - $t^a \leftrightarrow$ Kähler moduli of the CY
- $\sigma_a \in \mathfrak{g}_{\mathbb{C}}$... scalar components of the vector multiplet
- $W \in \text{Sym}V^*$... superpotential
 - G -invariant
 - R -charge 2
 - non-zero for **compact** CYs

D-branes in GLSMs

- D-branes (B-type) in the GLSM are G -invariant **matrix factorisations** of the GLSM potential with R -charge 1

[Herbst-Hori-Page 08][Honda-Okuda,Hori-Romo 13]

- Data:

- **Sym V^* -module** (Chan-Paton space): $M = M^0 \oplus M^1$
- **Matrix Factorisation**: $Q \in End^1(M)$ with

$$Q^2 = W \cdot \text{id}_M$$

- **G-action**: $\rho : G \rightarrow GL(M)$ with

$$\rho(g)^{-1} Q(g\phi) \rho(g) = Q(\phi) \quad g \in G$$

- **R-action**: $r_* : u(1)_V \rightarrow gl(M)$ with

$$\lambda^{r_*} Q(\lambda^R \phi) \lambda^{-r_*} = \lambda Q(\phi) \quad \lambda \in U(1)_V$$

Sphere partition function Z_{S^2}

- Sphere partition function

[Benini-Cremonesi 12][Doroud-Gomis-LeFloch-Lee 12]

$$\begin{aligned} Z_{S^2} = & C \sum_{m \in \mathbb{Z}^{\text{rk } G}} \int_{\gamma} d^{\text{rk } G} \sigma \prod_{\alpha > 0} \left(\frac{\alpha(m)^2}{4} + \alpha(\sigma)^2 \right) \\ & \cdot \prod_{j=1}^{\dim V} \frac{\Gamma \left(iQ_j(\sigma) - \frac{Q_j(m)}{2} + \frac{R_j}{2} \right)}{\Gamma \left(1 - iQ_j(\sigma) + \frac{Q_j(m)}{2} - \frac{R_j}{2} \right)} e^{-4\pi i \zeta(\sigma) - i\theta(m)} \end{aligned}$$

- $\alpha > 0$ positive roots
- γ ... integration contour (s.t. integral is convergent)
- Z_{S^2} computes the exact Kähler potential on \mathcal{M}_K .

[Jockers-Kumar-Lapan-Morrison-Romo 12][Gomis-Lee 12]

[Gerchkovitz-Gomis-Komargodski 14][Gomis-Hsin-Komargodski-Schwimmer-Seiberg-Theisen 15]

Hemisphere partition function

- Hemisphere partition function: [Sugishita-Terashima][Honda-Okuda][Hori-Romo 13]

$$\begin{aligned} Z_{D^2}(\mathcal{B}) &= C \int_{\gamma} d^{\text{rk}_G} \sigma \prod_{\alpha > 0} \alpha(\sigma) \sinh(\pi \alpha(\sigma)) \\ &\quad \cdot \prod_{j=1}^{\dim V} \Gamma \left(iQ_j(\sigma) + \frac{R_j}{2} \right) e^{it(\sigma)} f_{\mathcal{B}}(\sigma) \end{aligned}$$

- Brane factor

$$f_{\mathcal{B}}(\sigma) = \text{tr}_M \left(e^{i\pi \mathbf{r}_*} e^{2\pi \rho(\sigma)} \right)$$

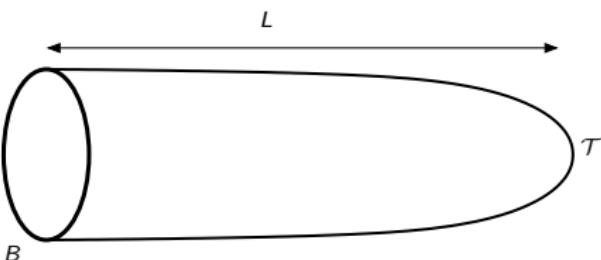
- Z_{D^2} computes the exact D-brane central charge.

D-brane central charge – worldsheet perspective

- The central charge connects A/B-branes with $(c, c)/(a, c)$ -operators:

[Ooguri-Oz-Yin 96][Hori-Iqbal-Vafa 00]

$$Z(B) = \langle B | \mathbf{0} \rangle$$



- Here we consider B-branes \mathcal{B} and (a, c) -operators.
 - Note:** “B-branes in the A-model”

Z_{D^2} in LG orbifold phases

- We want to collect evidence that

[Romo-Scheidegger-JK 20]

$$Z_{D^2}^{phases}(\mathcal{B}) = \langle \text{ch}(\mathcal{B}), \Gamma \cdot I \rangle.$$

- $\langle \cdot, \cdot \rangle$... pairing (state space)
- Γ ... Gamma class
- I ... I -function
- $\text{ch}(\mathcal{B})$... Chern character
- Focus on Landau-Ginzburg orbifold phases.
- Plan:
 - Define the objects on the right-hand side for LG orbifolds.
 - Show that the results match with Z_{D^2} and FJRW theory.

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GLSMs and localisation
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Conclusions
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LG data

- Landau-Ginzburg orbifold: $(W, G, \bar{\rho}_m, U(1)_{L/R})$

[Vafa 89][Intriligator-Vafa 90]

- $W(x_i)$... superpotential
- G ... orbifold group (assume $\mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \times \dots$)
- $\bar{\rho}_m$... matter representation
- $U(1)_{L/R}$... left/right R-symmetry
- State space
 - $(c, c), (a, c), \dots$ - chiral rings $\leftrightarrow \gamma$ -twisted sectors ($\gamma \in G$) with basis elements ϕ_γ
 - Restriction to one-dimensional (“narrow”) sectors ($\delta \in G$)
 - (Topological) pairing on the (a, c) -ring:

$$\langle \phi_\delta, \phi_{\delta'} \rangle = \frac{1}{|G|} \delta_{\delta, \delta'^{-1}}$$

- Note: Spectral flow required to map (c, c) to (a, c) .

LG branes

- **LG orbifold B-brane:** $\overline{\mathcal{B}} = (\overline{M}, \overline{Q}, \bar{\rho}_\gamma, \bar{r}_*)$

[Kapustin-Li 02][Brunner-Herbst-Lerche-Scheuner 03][Walcher 04]

- \overline{M} ... Chan-Paton space
- \overline{Q} ... matrix factorisation of W
- $\bar{\rho}_\gamma, \bar{r}_*$... representation of G and $u(1)_V$ on the boundary

- **Chern character:**

$$\text{ch}(\overline{\mathcal{B}})_\gamma = \frac{1}{n_\gamma!} \text{Res}_{W_\gamma} (\Phi_\gamma \cdot \text{Tr}_{\overline{M}} [\bar{\rho}_\gamma (\partial \overline{Q}_\gamma)^{\wedge n_\gamma}])$$

- Φ_γ ... state(s) in the γ -twisted sector \mathcal{H}_γ (to be precise, this is the (c, c) -ring)
- n_γ ... $\dim \text{Fix}_\gamma$
- $W_\gamma = W|_{\text{Fix}_\gamma}$

Gamma class

- Taking into account **deformations** away from the LG point, one can define a $h \times (N + h)$ matrix q .
 - N ... number of chirals x_i .
 - h ... number of marginal deformations (narrow sectors).
- **Gamma class:**

$$\Gamma\phi_\delta = \Gamma_\delta\phi_\delta \quad \Gamma_\delta = \prod_{j=1}^N \Gamma \left(1 - \left\langle \sum_{a=1}^h k_a q_{a,h+j} \right\rangle \right).$$

- $k_a \in \mathbb{Z}_{\geq 0}^h$: each gets associated to a sector δ .
- $\langle x \rangle = x - [x]$

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I-function

- *I*-function

$$I_{LG}(u) = \sum_{\delta \in G} I_\delta(u) \phi_\delta$$

$$\begin{aligned} I_\delta(u) &= - \sum_{\substack{k_1, \dots, k_h \geq 0 \\ k'_i = \delta_i \mod d_i}} \frac{u^k}{\prod_{a=1}^h \Gamma(k_a + 1)} \\ &\times \prod_{j=1}^N \frac{(-1)^{\langle -\sum_{a=1}^h k_a q_{a,h+j} + q_j \rangle} \Gamma(\langle \sum_{a=1}^h k_a q_{a,h+j} - q_j \rangle)}{\Gamma(1 + \sum_{a=1}^h k_a q_{a,h+j} - q_j)} \end{aligned}$$

- u ... local coordinates/deformation parameters
- q_j ... left R-charges of the x_j

J-function

- One obtains the *J*-function from the *I*-function through a transformation to **flat coordinates**.
 - These correspond to the deformation parameters of the marginal deformations in the worldsheet CFT.
- Select components of the I_{δ_a} ($a = 1, \dots, h$) of the *I*-function associated to the subspace of (narrow) marginal deformations $\leftrightarrow (a, c)$ states with $(q, \bar{q}) = (-1, 1)$ and the unique component I_0 with $(q, \bar{q}) = (0, 0)$.
- The flat coordinates and the *J*-function are

$$t_a(u) = \frac{I_{\delta_a}(u)}{I_0(u)} \quad J(t) = \frac{I_{LG}(u(t))}{I_0(u(t))}$$

- This coincides with the **mirror map**.

Consistency checks

- The LG quantities **consistent with FJRW theory**.

[Chiodo-Ruan 08][Chiodo-Iritani-Ruan 12]

- This gives an independent check for our results.
- Our results **generalise results from FJRW theory**: more moduli, more general G .
- The **hemisphere partition function** of the GLSM reproduces the results from the central charge formula when evaluated at LG points.
 - GLSM gauge group broken to orbifold group G
 - Matrix q is related to the gauge charges of the GLSM fields
 - **GLSM branes**: matrix factorisations of the GLSM superpotential reduce to LG matrix factorisations

[Herbst-Hori-Page 08][Clarke-Guffin 10]

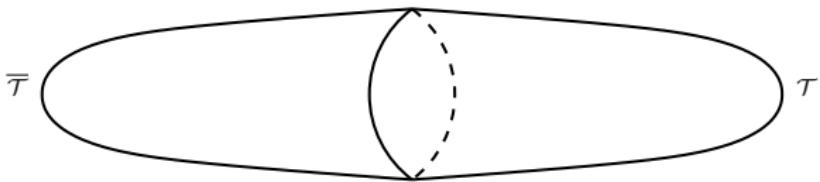
- Tested in **examples** with up to four Kähler moduli, including computation of **FJRW invariants**.

Kähler potential – worldsheet perspective

- The Kähler potential of \mathcal{M}_K is

[Cecotti-Vafa '91]

$$e^{-K(t,\bar{t})} = \langle \bar{0}|0\rangle$$



- $\langle \bar{0}|$ and $|0\rangle$ are related by CPT conjugation.

Hybrids

- Consider a GLSM phase that is a Landau-Ginzburg orbifold with orbifold group G fibered over some base manifold B .
- This includes:
 - Calabi-Yau complete intersections in toric ambient spaces
 - Landau-Ginzburg orbifolds
- The **sphere partition function** evaluates to [Erkinger-JK in progress]

$$Z_{S^2}^{phases}(t) = \sum_{\delta \in G} \int_B (-1)^{\text{Gr}} \frac{\Gamma_\delta(H)}{\Gamma_\delta^*(H)} |I_\delta(t, H)|^2 = \langle \bar{I}, I \rangle$$

- $\delta \in G \dots$ narrow sectors
- Basis $H \in H^2(B)$
- Gr. . . grading operator on the state space

Z_{S^2} in phases

- Landau-Ginzburg phases:
 - B is a point (no H)
 - Definitions of I/J , Gr, pairing, and the Gamma class coincide with definitions above
- Geometric phases:
 - B is a complete intersection CY X , G is trivial
 - The pairing is $\langle \alpha, \beta \rangle = \int_X \alpha \wedge \beta$, $\alpha, \beta \in H^{\text{even}}(X)$.
 - I/J , Gr, and Gamma class coincide with results in the literature
- Results suggest a definition [Iritani 07][Halverson-Jockers-Lapan-Morrison 13]

$$\langle \bar{I} | = (-1)^{\text{Gr}} \frac{\Gamma}{\Gamma^*} I(\bar{t}).$$

Consistency checks

- Tested for GLSMs associated to **14 one-parameter complete intersections** in toric ambient phases.
 - The small radius phases are Landau-Ginzburg, hybrids, and pseudo-hybrids. [Aspinwall-Plesser 09]
 - The structure is observed in all phases, even pseudo-hybrids, where we have a sum of terms.
 - I -functions and Gamma class in hybrid phases **match with FJRW theory**. [Clader 13]
- **Two-parameter** CY hypersurface with geometric, LG, and hybrid phase.
 - New conjectural results for the I -function and the Gamma class in multi-parameter hybrid models.

Summary

- We conjectured universal expressions for the hemisphere and sphere partition functions for phases of (abelian) GLSMs.
- Evidence that this works for geometric, Landau-Ginzburg and hybrid phases.
- Results match with mathematical results from FJRW theory and mirror symmetry, where available.
- Tested for lots of examples.

Open Questions

- Gamma class from the worldsheet perspective?
- Non-abelian GLSMs.
- Broad sectors.
- Hybrids, in particular with branes and enumerative invariants.
- More localisation results.
- Mathematical proofs.