The Secret Topological Life of Shared Information

Tom Mainiero

Rutgers University

String Math
July 29, 2020

What's the Big Idea?

Multipartite
State

- $\psi \in \bigotimes_{s \in P} \mathcal{H}_s$
- $\hat{\rho} \in \mathrm{Dens}\left(\bigotimes_{s \in P} \mathcal{H}_s\right)$
- $\mu : \prod_{s \in P} \Omega_s \longrightarrow \mathbb{R}_{\geq 0}$

What's the Big Idea?

$$\text{Multipartite State} \xrightarrow{\text{Mutual Info.}} \mathbb{R}$$

- $\psi \in \bigotimes_{s \in P} \mathcal{H}_s$
- $\hat{\rho} \in \mathrm{Dens}\left(\bigotimes_{s \in P} \mathcal{H}_s\right)$
- $\mu : \prod_{s \in P} \Omega_s \longrightarrow \mathbb{R}_{\geq 0}$

What's the Big Idea?

$$\begin{tikzcd}
\text{Multipartite State} \arrow[r, squiggly] \arrow[dr, "\text{Mutual Info.}"'] \arrow[r, phantom, shift right=4ex, "\circlearrowright"] & \text{Graded Vector Space} \arrow[d, "\text{Euler Char.}"] \\
& \mathbb{R}
\end{tikzcd}$$

- $\psi \in \bigotimes_{s \in P} \mathcal{H}_s$
- $\hat{\rho} \in \mathrm{Dens}\left(\bigotimes_{s \in P} \mathcal{H}_s\right)$
- $\mu : \prod_{s \in P} \Omega_s \longrightarrow \mathbb{R}_{\geq 0}$

What's the Big Idea?

$$\begin{tikzcd}
\text{Multipartite State} \arrow[r, squiggly] \arrow[dr, "\text{Mutual Info.}"'] & \text{Graded Vector Space} \arrow[d, "\text{Euler Char.}"] \arrow[dl, phantom, "\circlearrowright"] \\
& \mathbb{R}
\end{tikzcd}$$

How much Info. is shared

- $\psi \in \bigotimes_{s \in P} \mathcal{H}_s$
- $\hat{\rho} \in \mathrm{Dens}\left(\bigotimes_{s \in P} \mathcal{H}_s\right)$
- $\mu : \prod_{s \in P} \Omega_s \to \mathbb{R}_{\geq 0}$

What's the Big Idea?

- What Info. is Shared
- May be non-trivial even when mutual info. vanishes

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\text{Multipartite State} \arrow[r, squiggly] \arrow[dr, "\text{Mutual Info.}"'] & \text{Graded Vector Space} \arrow[d, "\text{Euler Char.}"] \\
& \mathbb{R}
\end{tikzcd}$$

How much Info. is Shared

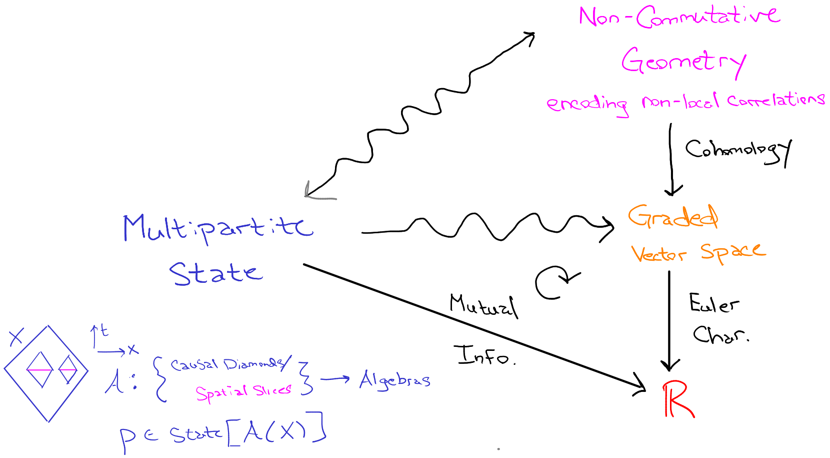
- $\psi \in \bigotimes_{s \in P} \mathcal{H}_s$
- $\hat{\rho} \in \mathrm{Dens}\left(\bigotimes_{s \in P} \mathcal{H}_s\right)$
- $\mu : \prod_{s \in P} \Omega_s \to \mathbb{R}_{\geq 0}$

What's the Big Idea?

$$\begin{tikzcd}
 & \text{Non-Commutative Geometry encoding non-local correlations} \arrow[d, "\text{Cohomology}"] \arrow[dl, leftrightsquigarrow] \\
\text{Multipartite State} \arrow[r, rightsquigarrow] \arrow[dr, "\text{Mutual Info.}"'] \arrow[r, phantom, shift right=6ex, "\circlearrowright"] & \text{Graded Vector Space} \arrow[d, "\text{Euler Char.}"] \\
 & \mathbb{R}
\end{tikzcd}$$

- $\psi \in \bigotimes_{s \in P} \mathcal{H}_s$
- $\hat{\rho} \in \mathrm{Dens}\left(\bigotimes_{s \in P} \mathcal{H}_s\right)$
- $\mu : \prod_{s \in P} \Omega_s \to \mathbb{R}_{\geq 0}$

What's the Big Idea?



Why is this cool?

$$N\text{-partite state} \rightsquigarrow \bigoplus_{k=0}^{N-1} H^k\left[N\text{-partite state}\right]$$

$$H^k\left[N\text{-partite state}\right] = \left\{\begin{matrix}\text{tuples of } (k+1)\text{-body operators} \\ \text{exhibiting correlations}\end{matrix}\right\} \Big/ \left\{\begin{matrix}\text{trivial} \\ \text{correlations}\end{matrix}\right\}, \; k < N-1$$

Why is this cool?

$$N\text{-partite state} \rightsquigarrow \bigoplus_{k=0}^{N-1} H^k\,[N\text{-partite state}]$$

$$H^k\,[N\text{-partite state}] = \left\{\begin{matrix}\text{tuples of } (k+1)\text{-body operators} \\ \text{exhibiting correlations}\end{matrix}\right\} \Big/ \left\{\begin{matrix}\text{trivial} \\ \text{correlations}\end{matrix}\right\}, \; k < N-1$$

$$H^k(\psi_1 \otimes \cdots \otimes \psi_N) = 0, \; k < N-1$$

Why is this cool?

$$N\text{-partite state} \rightsquigarrow \bigoplus_{k=0}^{N-1} H^k\,[N\text{-partite state}]$$

$$H^k\,[N\text{-partite state}] = \left\{\begin{matrix}\text{tuples of } (k+1)\text{-body operators}\\ \text{exhibiting correlations}\end{matrix}\right\} \Big/ \left\{\begin{matrix}\text{trivial}\\ \text{correlations}\end{matrix}\right\},\ k < N-1$$

$$H^k(\psi_1 \otimes \cdots \otimes \psi_N) = 0,\ k < N-1$$

$$[(|0_\mathrm{A}\rangle\langle 0_\mathrm{A}|\,, |0_\mathrm{B}\rangle\langle 0_\mathrm{B}|)] \in H^0(|0_\mathrm{A}0_\mathrm{B}\rangle + |1_\mathrm{A}1_\mathrm{B}\rangle)$$

Why is this cool?

$$N\text{-partite state} \rightsquigarrow \bigoplus_{k=0}^{N-1} H^k\,[N\text{-partite state}]$$

$$H^k\,[N\text{-partite state}] = \left\{ \begin{matrix} \text{tuples of } (k+1)\text{-body operators} \\ \text{exhibiting correlations} \end{matrix} \right\} \Big/ \left\{ \begin{matrix} \text{trivial} \\ \text{correlations} \end{matrix} \right\},\ k < N-1$$

$$H^k(\psi_1 \otimes \cdots \otimes \psi_N) = 0,\ k < N-1$$

$$[(|0_A\rangle\langle 0_A|, |0_B\rangle\langle 0_B|)] \in H^0(|0_A 0_B\rangle + |1_A 1_B\rangle)$$

$$[(r_A, r_B)] \in H^0(\widehat{\rho}_{AB}) \iff \operatorname{Tr}[\widehat{\rho}_{AB}\, x (r_A \otimes 1_B - 1_A \otimes r_B)] = 0,\ \forall x \in B(\mathcal{H}_{AB})$$

Why is this cool?

$$\text{N-partite state} \rightsquigarrow \bigoplus_{k=0}^{N-1} H^k\,[\text{N-partite state}]$$

$$H^k\,[N\text{-partite state}] = \left\{ \substack{\text{tuples of } (k+1)\text{-body operators} \\ \text{exhibiting correlations}} \right\} \Big/ \left\{ \substack{\text{trivial} \\ \text{correlations}} \right\},\ k < N-1$$

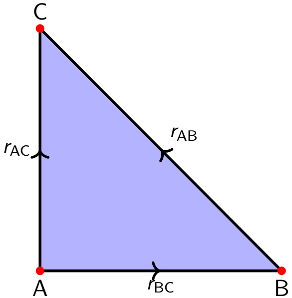
$$H^k(\psi_1 \otimes \cdots \otimes \psi_N) = 0,\ k < N-1$$

$$[(|0_A\rangle\langle 0_A|\,,|0_B\rangle\langle 0_B|)] \in H^0(|0_A 0_B\rangle + |1_A 1_B\rangle)$$

$$[(r_A, r_B)] \in H^0(\widehat{\rho}_{AB}) \iff \underbrace{\mathrm{Tr}[\widehat{\rho}_{AB}\, x(r_A \otimes 1_B - 1_A \otimes r_B)] = 0,\ \forall x \in B(\mathcal{H}_{AB})}_{\substack{r_A \underset{AB}{\sim} r_B \\ \text{“}r_A \text{ and } r_B \text{ are maxmly. correlated”}}}$$

1-cochains for a tripartite state

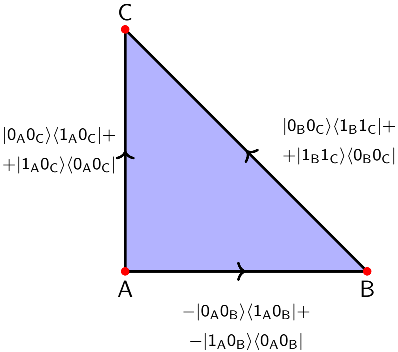
$$\psi \in \mathcal{H}_{\mathsf{A}} \otimes \mathcal{H}_{\mathsf{B}} \otimes \mathcal{H}_{\mathsf{C}}$$



$$[(r_{\mathsf{AB}}, r_{\mathsf{AC}}, r_{\mathsf{BC}})] \in H^1(\psi) \iff \widetilde{r}_{\mathsf{BC}} + \widetilde{r}_{\mathsf{AB}} \underset{\mathsf{ABC}}{\sim} \widetilde{r}_{\mathsf{AC}}$$

1-cochains for the GHZ state

$$\psi = |\mathsf{GHZ}_3\rangle = |0_A 0_B 0_C\rangle + |1_A 1_B 1_C\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$$



Why is this cool?

Cohomology has advantages over mutual information

Why is this cool?

Cohomology has advantages over mutual information

$$I_2[\substack{\text{bipartite state}\\ \text{on } \{\text{A, B}\}}] = S_{\text{A}} + S_{\text{B}} - S_{\text{AB}} \in \mathbb{R}_{\geq 0}$$

Measures how much information is shared by A and B.

Why is this cool?

Cohomology has advantages over mutual information

$$I_2\left[\substack{\text{bipartite state}\\ \text{on }\{\text{A, B}\}}\right] = S_{\mathrm{A}} + S_{\mathrm{B}} - S_{\mathrm{AB}} \in \mathbb{R}_{\geq 0}$$

Measures how much information is shared by A and B.

Multipartite mutual information:

$$I_n = \sum_{T \subseteq P} (-1)^{|T|} S_T \in \mathbb{R}$$

Is a (sometimes unreliable) measure of information shared by every $T \subseteq P$.

Why is this cool?

Cohomology has advantages over mutual information

$$I_2\left[\substack{\text{bipartite state}\\ \text{on } \{\mathsf{A},\ \mathsf{B}\}}\right] = S_{\mathsf{A}} + S_{\mathsf{B}} - S_{\mathsf{AB}} \in \mathbb{R}_{\geq 0}$$

Measures how much information is shared by A and B.

Multipartite mutual information:

$$I_n = \sum_{T \subseteq P} (-1)^{|T|} S_T \in \mathbb{R}$$

Is a (sometimes unreliable) measure of information shared by every $T \subseteq P$.

$$I_3[|000\rangle + |111\rangle] = I_3[|001\rangle + |010\rangle + |100\rangle] = 0$$

Why is this cool?

Cohomology has advantages over mutual information

$$I_2[{}^{\text{bipartite state}}_{\text{on }\{\text{A, B}\}}] = S_{\text{A}} + S_{\text{B}} - S_{\text{AB}} \in \mathbb{R}_{\geq 0}$$

Measures how much information is shared by A and B.

Multipartite mutual information:

$$I_n = \sum_{T \subseteq P} (-1)^{|T|} S_T \in \mathbb{R}$$

Is a (sometimes unreliable) measure of information shared by every $T \subseteq P$.

$$I_3[|000\rangle + |111\rangle] = I_3[|001\rangle + |010\rangle + |100\rangle] = 0$$

while for $\bullet = 0, 1$

$$H^\bullet[|000\rangle + |111\rangle] \neq 0,$$
$$H^\bullet[|001\rangle + |010\rangle + |100\rangle] \neq 0$$

Mutual Info. as an Euler Char. (kinda...it's better)

$$N\text{-partite state} \rightsquigarrow \mathrm{Geom}\left[\begin{smallmatrix} N\text{-partite} \\ \text{state} \end{smallmatrix}\right]$$

$$\mathrm{Geom}\left[\begin{smallmatrix} N\text{-partite} \\ \text{state} \end{smallmatrix}\right] \xrightarrow{\text{Euler Char.}} \text{State Index}\left[\begin{smallmatrix} N\text{-partite} \\ \text{state} \end{smallmatrix}\right] \in \underbrace{\mathcal{O}(\mathbb{C}_\alpha \times \mathbb{C}_q \times \mathbb{C}_r)}_{\substack{\text{Holomorphic functions} \\ \text{in 3-parameters}}};$$

Mutual Info. as an Euler Characteristic (kinda...it's better)

$$\begin{tikzcd}
 & \text{State Index} \in \mathcal{O}(\mathbb{C}_\alpha \times \mathbb{C}_q \times \mathbb{C}_r) \arrow[dl, "\alpha \to 0", "\times \frac{1}{r(q-1)}"'] \arrow[dr, "q \to 0", "{\alpha,\, r \in \mathbb{Z}}"'] & \\
\text{Tsallis/Rényi Deformed Mutual Information} \in \mathcal{O}(\mathbb{C}_q \times \mathbb{C}_r) \arrow[d, "q \to 1"'] & & \text{Euler Characteristics of Complexes of Vector Spaces} \in \mathbb{Z} \\
\text{Mutual Information} \in \mathbb{R} & &
\end{tikzcd}$$

Mutual Info. as an Euler Characteristic (kinda...it's better)

We should think of

$$I_3[|000\rangle + |111\rangle] = I_3[|001\rangle + |010\rangle + |100\rangle] = 0$$

while for $\bullet = 0, 1$

$$H^\bullet[|000\rangle + |111\rangle] \neq 0,$$
$$H^\bullet[|001\rangle + |010\rangle + |100\rangle] \neq 0$$

as analogous to how Euler(Compact 3-Manifold) $= 0$ while cohomology can be non-vanishing.

Mutual Info. as an Euler Characteristic (kinda...it's better)

We should think of

$$I_3[|000\rangle + |111\rangle] = I_3[|001\rangle + |010\rangle + |100\rangle] = 0$$

while for $\bullet = 0,\, 1$

$$H^\bullet[|000\rangle + |111\rangle] \neq 0,$$
$$H^\bullet[|001\rangle + |010\rangle + |100\rangle] \neq 0$$

as analogous to how Euler(Compact 3-Manifold) $= 0$ while cohomology can be non-vanishing.

Cohomology can detect how things are glued together, Euler characteristics only count how many things are glued together.

Before We Define Things

This is a talk about structures in *basic* Quantum Mechanics (or probability theory).

Before We Define Things

This is a talk about structures in *basic* Quantum Mechanics (or probability theory).

No *a priori* geometry on the set of subsystems/tensor factors. Geometry is emergent.

Before We Define Things

This is a talk about structures in *basic* Quantum Mechanics (or probability theory).

No *a priori* geometry on the set of subsystems/tensor factors. Geometry is emergent.

The generality suggests something deep is to be learned.

Before We Define Things

This is a talk about structures in *basic* Quantum Mechanics (or probability theory).

No *a priori* geometry on the set of subsystems/tensor factors. Geometry is emergent.

The generality suggests something deep is to be learned.

Possibly new link invariants: $L \subset S^3$ a link with N-components;[1]

$$\psi_L := \mathcal{Z}_{\mathrm{CS}}[S^3 - L] \in \mathcal{Z}_{\mathrm{CS}}[\mathbb{T}]^{\otimes N}$$

Corresponding cohomology, Poincaré polynomials, and state indices are frame-equivariant/independent *link* invariants.

[1] Based on conversations with Greg Moore. See work of Salton-Swingle-Walter 1611.01516 and Balasubramanian, et. al.: 1801.01131.

Closely Related Work

- Baez-Fritz-Leinster: *Entropy as a Functor*.
- P. Baudot and D. Bennequin: *The Homological Nature of Entropy*. Mutual information (and their Tsallis q-deformations) arise as non-trivial cochains of some complex of functions on spaces of probability measures. J.P. Vigneaux provides an excellent exposition in 1709.07807.
- Drummond-Cole, Park, and Terilla: *Homotopy probability theory*. A_∞/L_∞-techniques applied to probability theory.

What's a state?

$$\text{“state”} = (\text{normal})\ \text{positive linear functional on a}\ \overbrace{W^*\text{-algebra}}^{\text{“von Neumann algebra”}}\ R.$$

$$\rho : R \longrightarrow \mathbb{C}$$

What's a state?

$$\text{“state”} = (\text{normal}) \text{ positive linear functional on a } \overbrace{W^*\text{-algebra}}^{\text{“von Neumann algebra”}} R.$$

$$\rho : R \longrightarrow \mathbb{C}$$

Algebra R of Random Variables	State ρ
$B\mathcal{H}$	$\rho(r) = \mathrm{Tr}_{\mathcal{H}}[\widehat{\rho} r]$

What's a state?

$$\text{“state”} = \text{(normal) positive linear functional on a } \overbrace{W^*\text{-algebra}}^{\text{“von Neumann algebra”}} R.$$

$$\rho : R \longrightarrow \mathbb{C}$$

Algebra R of Random Variables	State ρ
$B\mathcal{H}$	$\rho(r) = \mathrm{Tr}_{\mathcal{H}}[\widehat{\rho} r]$
$\mathrm{Fun}_{\mathbb{C}}(\Omega) \cong \mathbb{C}^{\lvert\Omega\rvert}$	$\rho(f) = \sum_{\omega\in\Omega} \underbrace{\mu_\omega}_{\mu:\Omega\longrightarrow\mathbb{R}_{\geq 0}} f(\omega)$

What's a state?

$$\text{“state”} = \text{(normal) positive linear functional on a } \overbrace{W^*\text{-algebra}}^{\text{“von Neumann algebra”}}\ R.$$

$$\rho : R \longrightarrow \mathbb{C}$$

Algebra R of Random Variables	State ρ
$B\mathcal{H}$	$\rho(r) = \mathrm{Tr}_{\mathcal{H}}[\widehat{\rho} r]$
$\mathrm{Fun}_{\mathbb{C}}(\Omega) \cong \mathbb{C}^{\lvert\Omega\rvert}$	$\rho(f) = \sum_{\omega \in \Omega} \underbrace{\mu_\omega}_{\mu:\Omega \longrightarrow \mathbb{R}_{\geq 0}} f(\omega)$
$L^\infty(\mathbb{X})$	$\rho(f) = \int_{\mathbb{X}} f d\mu$

What's a state?

$$\text{“state”} = \text{(normal) positive linear functional on a } \overbrace{W^*\text{-algebra}}^{\text{“von Neumann algebra”}} R.$$

$$\rho : R \longrightarrow \mathbb{C}$$

Algebra R of Random Variables	State ρ
$B\mathcal{H}$	$\rho(r) = \mathrm{Tr}_{\mathcal{H}}[\widehat{\rho} r]$
$\mathrm{Fun}_{\mathbb{C}}(\Omega) \cong \mathbb{C}^{\lvert\Omega\rvert}$	$\rho(f) = \sum_{\omega\in\Omega} \underbrace{\mu_\omega}_{\mu:\Omega\longrightarrow\mathbb{R}_{\geq 0}} f(\omega)$
$L^\infty(\mathbb{X})$	$\rho(f) = \int_{\mathbb{X}} f d\mu$
$\prod_{i=1}^{n} \mathrm{End}(\mathcal{H}_i)$	$\rho(r_1, \cdots, r_n) = \sum_i \mathrm{Tr}_{\mathcal{H}_i}[\widehat{\rho}^{(i)} r_i]$

$$\underset{\prod_i \mathrm{End}(\mathcal{H}_i)}{\text{State on}} \leftrightarrow \underset{(\widehat{\rho}^{(1)}, \cdots, \widehat{\rho}^{(n)})}{\text{Tuple of density states}}$$

What's a state?

$$\text{“state”} = (\text{normal}) \text{ positive linear functional on a } \overbrace{W^*\text{-algebra}}^{\text{“von Neumann algebra”}} R.$$

$$\rho : R \longrightarrow \mathbb{C}$$

Algebra R of Random Variables	State ρ
$B\mathcal{H}$	$\rho(r) = \mathrm{Tr}_{\mathcal{H}}[\widehat{\rho} r]$
$\mathrm{Fun}_{\mathbb{C}}(\Omega) \cong \mathbb{C}^{\lvert\Omega\rvert}$	$\rho(f) = \sum_{\omega \in \Omega} \underbrace{\mu_\omega}_{\mu : \Omega \longrightarrow \mathbb{R}_{\geq 0}} f(\omega)$
$L^\infty(\mathbb{X})$	$\rho(f) = \int_{\mathbb{X}} f d\mu$
$\prod_{i=1}^n \mathrm{End}(\mathcal{H}_i)$	$\rho(r_1, \cdots, r_n) = \sum_i \mathrm{Tr}_{\mathcal{H}_i}[\widehat{\rho}^{(i)} r_i]$

$$\begin{array}{c}\text{State on}\\ \prod_i \mathrm{End}(\mathcal{H}_i)\end{array} \leftrightarrow \begin{array}{c}\text{Tuple of density states}\\ (\widehat{\rho}^{(1)}, \cdots, \widehat{\rho}^{(n)})\end{array}$$

$$\begin{array}{c}\text{State on}\\ \prod_{i=1}^n \mathbb{C}\end{array} \leftrightarrow \begin{array}{c}\text{Tuple of non-negative reals}\\ (\mu^{(1)}, \cdots, \mu^{(n)})\end{array}$$

What's a Bipartite State? (roughly)

$$\text{“bipartite state”} = \begin{array}{c} R_\mathsf{A},\ R_\mathsf{B} \text{ a pair of algebras} \\ + \\ \rho : R_\mathsf{A} \otimes R_\mathsf{B} \longrightarrow \mathbb{C} \text{ a state} \end{array}$$

What's a Bipartite State? (roughly)

$$\text{“bipartite state”} = \underbrace{\begin{matrix} R_A,\ R_B \text{ a pair of algebras} \\ + \\ \rho : R_A \otimes R_B \longrightarrow \mathbb{C} \text{ a state} \end{matrix}}_{\underline{\rho_{AB}}}$$

We have homomorphisms

$$\begin{aligned} \epsilon_A : R_A &\longrightarrow R_A \otimes R_B \\ a &\longmapsto a \otimes 1 \end{aligned} \qquad \begin{aligned} \epsilon_B : R_B &\longrightarrow R_A \otimes R_B \\ b &\longmapsto 1 \otimes b \end{aligned}$$

What's a Bipartite State? (roughly)

$$\text{“bipartite state”} = \underbrace{\begin{array}{c} R_A,\ R_B \text{ a pair of algebras} \\ + \\ \rho : R_A \otimes R_B \longrightarrow \mathbb{C} \text{ a state} \end{array}}_{\underline{\rho_{AB}}}$$

We have homomorphisms

$$\begin{aligned} \epsilon_A : R_A &\longrightarrow R_A \otimes R_B \\ a &\longmapsto a \otimes 1 \end{aligned} \qquad \begin{aligned} \epsilon_B : R_B &\longrightarrow R_A \otimes R_B \\ b &\longmapsto 1 \otimes b \end{aligned}$$

Giving us the reduced states (“partial traces” / “partial measures”)

$$\begin{aligned} \rho_A := \rho \circ \epsilon_A : R_A &\longrightarrow \mathbb{C} \\ a &\longmapsto \rho(a \otimes 1) \end{aligned} \qquad \begin{aligned} \rho_B := \rho \circ \epsilon_B : R_B &\longrightarrow \mathbb{C} \\ b &\longmapsto \rho(1 \otimes b) \end{aligned}$$

What is Factorizability?

Bipartite ρ is *factorizable* if $\rho(1)\rho = \rho_{\mathsf{A}} \otimes \rho_{\mathsf{B}}$.

What is Factorizability?

Bipartite ρ is *factorizable* if $\rho(1)\rho = \rho_{\mathsf{A}} \otimes \rho_{\mathsf{B}}$.

$\psi \in \mathcal{H}_{\mathsf{A}} \otimes \mathcal{H}_{\mathsf{B}}$ is factorizable $\quad\Longleftrightarrow\quad$ $\mathrm{Tr}_{\mathcal{H}_{\mathsf{A}} \otimes \mathcal{H}_{\mathsf{B}}}[\psi \otimes \psi^{\vee}(-)]$ is factorizable

What is Factorizability?

Bipartite ρ is *factorizable* if $\rho(1)\rho = \rho_{\mathsf{A}} \otimes \rho_{\mathsf{B}}$.

$\psi \in \mathcal{H}_{\mathsf{A}} \otimes \mathcal{H}_{\mathsf{B}}$ is factorizable	$\iff$	$\mathrm{Tr}_{\mathcal{H}_{\mathsf{A}} \otimes \mathcal{H}_{\mathsf{B}}}[\psi \otimes \psi^{\vee}(-)]$ is factorizable
$\mu : X \times Y \longrightarrow [0,1]$ a probability measure describes independent random variables.	$\iff$	its expectation value is factorizable

What is Factorizability?

Bipartite ρ is *factorizable* if $\rho(1)\rho = \rho_{\mathsf{A}} \otimes \rho_{\mathsf{B}}$.

$\psi \in \mathcal{H}_{\mathsf{A}} \otimes \mathcal{H}_{\mathsf{B}}$ is factorizable	$\Longleftrightarrow$	$\mathrm{Tr}_{\mathcal{H}_{\mathsf{A}} \otimes \mathcal{H}_{\mathsf{B}}}[\psi \otimes \psi^{\vee}(-)]$ is factorizable
$\mu : X \times Y \longrightarrow [0,1]$ a probability measure describes independent random variables.	$\Longleftrightarrow$	its expectation value is factorizable

What's a Multipartite State?

$$\text{“multipartite state”} \quad \text{“}{=}\text{”} \quad \begin{array}{c} (R_p)_{p \in P} \text{ tuple of algebras} \\ + \\ \rho : \bigotimes_{p \in P} R_p \longrightarrow \mathbb{C} \text{ a state} \end{array}$$

What's a Multipartite State?

$$\text{“multipartite state”} \quad \text{“}=\text{”} \quad \underbrace{\begin{array}{c} (R_p)_{p\in P} \text{ tuple of algebras} \\ + \\ \rho : \bigotimes_{p\in P} R_p \longrightarrow \mathbb{C} \text{ a state} \end{array}}_{\underline{\rho_P}}$$

For any subset $T \subseteq P$ we have algebras $R_T := \bigotimes_{t\in T} R_t$ ($R_\emptyset = \mathbb{C}$), and maps

$$\epsilon_T : R_T \longrightarrow R_P$$

Define the reduced states

$$\rho_T := \rho \circ \epsilon_T : R_T \to \mathbb{C}$$

Everything is a local automorphism invariant

Because everything in this talk is functorial, all interesting quantities associated to a multipartite state $\rho : \bigotimes_{p \in P} R_p \longrightarrow \mathbb{C}$ are invariant (or equivariant) under "local automorphisms":

$$\rho \longmapsto \rho \circ \bigotimes_{p \in P} A_p,$$

where $(A_p : R_p \longrightarrow R_p)_p$ is a collection of algebra automorphisms.

Everything is a local automorphism invariant

Because everything in this talk is functorial, all interesting quantities associated to a multipartite state $\rho : \bigotimes_{p\in P} R_p \longrightarrow \mathbb{C}$ are invariant (or equivariant) under "local automorphisms":

$$\rho \longmapsto \rho \circ \bigotimes_{p\in P} A_p,$$

where $(A_p : R_p \longrightarrow R_p)_p$ is a collection of algebra automorphisms. E.g. for pure states this includes local unitary transformations.

$$\psi \longmapsto U_1 \otimes \cdots \otimes U_n \psi.$$

Why Geometry? Homological Obstructions, that's why

$$\rho : R_{\mathsf{A}} \otimes R_{\mathsf{B}} \longrightarrow \mathbb{C}$$

Factorizability $\leftrightsquigarrow$ Descent of data to subsystems: all global data comes from gluing local data: $\rho(\sum_{ij} r^i_{\mathsf{A}} \otimes r^j_{\mathsf{B}}) = \frac{1}{\rho(1)} \sum_{ij} \rho_{\mathsf{A}}(r^i_{\mathsf{A}})\rho_{\mathsf{B}}(r^j_{\mathsf{B}})$.

$$\Updownarrow$$

Failure to Factorize $\leftrightsquigarrow$ Obstruction to descent: $\rho(r_{\mathsf{A}} \otimes r_{\mathsf{B}}) \neq \frac{1}{\rho(1)} \rho_{\mathsf{A}}(r_{\mathsf{A}})\rho_{\mathsf{B}}(r_{\mathsf{B}})$ for some $(r_{\mathsf{A}}, r_{\mathsf{B}})$

Why Geometry? Homological Obstructions, that's why

$$\rho : R_{\mathsf{A}} \otimes R_{\mathsf{B}} \longrightarrow \mathbb{C}$$

Factorizability $\longleftrightarrow$ Descent of data to subsystems: all global data comes from gluing local data: $\rho(\sum_{ij} r^i_{\mathsf{A}} \otimes r^j_{\mathsf{B}}) = \frac{1}{\rho(1)} \sum_{ij} \rho_{\mathsf{A}}(r^i_{\mathsf{A}})\rho_{\mathsf{B}}(r^j_{\mathsf{B}})$.

$\Updownarrow$

Homological Alarm Bells!

Failure to Factorize $\longleftrightarrow$ Obstruction to descent: $\rho(r_{\mathsf{A}} \otimes r_{\mathsf{B}}) \neq \frac{1}{\rho(1)} \rho_{\mathsf{A}}(r_{\mathsf{A}})\rho_{\mathsf{B}}(r_{\mathsf{B}})$ for some $(r_{\mathsf{A}}, r_{\mathsf{B}})$

Why Geometry? Homological Obstructions, that's why

$$\rho : R_{\mathsf{A}} \otimes R_{\mathsf{B}} \longrightarrow \mathbb{C}$$

Factorizability $\Longleftrightarrow$ Descent of data to subsystems: all global data comes from gluing local data: $\rho(\sum_{ij} r^i_{\mathsf{A}} \otimes r^j_{\mathsf{B}}) = \frac{1}{\rho(1)} \sum_{ij} \rho_{\mathsf{A}}(r^i_{\mathsf{A}}) \rho_{\mathsf{B}}(r^j_{\mathsf{B}})$.

$$\Updownarrow$$

Failure to Factorize $\Longleftrightarrow$ Obstruction to descent: $\rho(r_{\mathsf{A}} \otimes r_{\mathsf{B}}) \neq \frac{1}{\rho(1)} \rho_{\mathsf{A}}(r_{\mathsf{A}}) \rho_{\mathsf{B}}(r_{\mathsf{B}})$ for some $(r_{\mathsf{A}}, r_{\mathsf{B}})$

$$\text{“}\; H^0(\rho) = \{(r_{\mathsf{A}}, r_{\mathsf{B}}) \in R_{\mathsf{A}} \times R_{\mathsf{B}} : \rho(1)\rho(r_{\mathsf{A}} \otimes r_{\mathsf{B}}) \neq \rho_{\mathsf{A}}(r_{\mathsf{A}})\rho_{\mathsf{B}}(r_{\mathsf{B}})\} \;\text{”}$$

Why Geometry? Homological Obstructions, that's why

$$\rho : R_{\mathsf{A}} \otimes R_{\mathsf{B}} \longrightarrow \mathbb{C}$$

Factorizability $\leftrightsquigarrow$ Descent of data to subsystems: all global data comes from gluing local data: $\rho(\sum_{ij} r^i_{\mathsf{A}} \otimes r^j_{\mathsf{B}}) = \frac{1}{\rho(1)} \sum_{ij} \rho_{\mathsf{A}}(r^i_{\mathsf{A}})\rho_{\mathsf{B}}(r^j_{\mathsf{B}})$.

$$\Updownarrow$$

Failure to Factorize $\leftrightsquigarrow$ Obstruction to descent: $\rho(r_{\mathsf{A}} \otimes r_{\mathsf{B}}) \neq \frac{1}{\rho(1)}\rho_{\mathsf{A}}(r_{\mathsf{A}})\rho_{\mathsf{B}}(r_{\mathsf{B}})$ for some $(r_{\mathsf{A}}, r_{\mathsf{B}})$

$$H^0[\rho] = \{(r_{\mathsf{A}}, r_{\mathsf{B}}) \in R_{\mathsf{A}} \times R_{\mathsf{B}} : r_{\mathsf{A}} \text{ and } r_{\mathsf{B}} \text{ are mxmly. correlated}\}/\mathbb{C}\langle(1,1)\rangle$$

Why? Because Mutual Info. Looks Like an Euler Char.

Mutual Information:

$$I(\underline{\rho_{\mathsf{AB}}}) = S(\rho_{\mathsf{A}}) + S(\rho_{\mathsf{B}}) - S(\rho_{\mathsf{AB}}) \in \mathbb{R}_{\geq 0}$$

Why? Because Mutual Info. Looks Like an Euler Char.

Mutual Information:

$$I(\underline{\rho_{\mathsf{AB}}}) = S(\rho_{\mathsf{A}}) + S(\rho_{\mathsf{B}}) - S(\rho_{\mathsf{AB}}) \in \mathbb{R}_{\geq 0}$$

$$\text{States on } \prod_{i=1}^{n} \mathrm{End}(\mathcal{H}_i) \longleftrightarrow \left\{ \underbrace{(\widehat{\rho}^{(1)}, \cdots, \widehat{\rho}^{(n)})}_{\text{tuple of density states}} \right\}$$

$$S[(\widehat{\rho}^{(1)}, \cdots, \widehat{\rho}^{(n)})] := -\sum_{i=1}^{n} \mathrm{Tr}[\widehat{\rho}^{(i)} \log \widehat{\rho}^{(i)}],$$

When $\sum_i \mathrm{Tr}[\widehat{\rho}_i] = 1$.

Why? Because Mutual Info. Looks Like an Euler Char.

Mutual Information:

$$I(\underline{\rho_{\mathrm{AB}}}) = S(\rho_{\mathrm{A}}) + S(\rho_{\mathrm{B}}) - S(\rho_{\mathrm{AB}}) \in \mathbb{R}_{\geq 0}$$

Multipartite Mutual information:

$$I(\underline{\rho_P}) = \sum_{T \subseteq P} (-1)^{|T|-1} S(\rho_T) \in \mathbb{R}$$

Why? Because Mutual Info. Looks Like an Euler Char.

Mutual Information:

$$I(\underline{\rho_{\mathrm{AB}}}) = S(\rho_{\mathrm{A}}) + S(\rho_{\mathrm{B}}) - S(\rho_{\mathrm{AB}}) \in \mathbb{R}_{\geq 0}$$

Multipartite Mutual information:

$$I(\underline{\rho_P}) = \sum_{T \subseteq P} (-1)^{|T|-1} S(\rho_T) \in \mathbb{R}$$

$$I(\underline{\rho_Q} \otimes \underline{\rho_S}) = 0$$

Non-vanishing N-partite mutual information $\Rightarrow$ no system can "decouple"

Why? Because Mutual Info. Looks Like an Euler Char.

Mutual Information:

$$I(\underline{\rho_{\mathrm{AB}}}) = S(\rho_{\mathrm{A}}) + S(\rho_{\mathrm{B}}) - S(\rho_{\mathrm{AB}}) \in \mathbb{R}_{\geq 0}$$

Multipartite Mutual information:

$$I(\underline{\rho_P}) = \sum_{T \subseteq P} (-1)^{|T|-1} S(\rho_T) \in \mathbb{R}$$

$$I(\underline{\rho_Q} \otimes \underline{\rho_S}) = 0$$

Non-vanishing N-partite mutual information $\Rightarrow$ no system can "decouple"

$\not\Leftarrow_{N \geq 3}$

Why? Because Mutual Info. Looks Like an Euler Char.

Mutual Information:

$$I(\underline{\rho_{\mathrm{AB}}}) = S(\rho_{\mathrm{A}}) + S(\rho_{\mathrm{B}}) - S(\rho_{\mathrm{AB}}) \in \mathbb{R}_{\geq 0}$$

Multipartite Mutual information:

$$I(\underline{\rho_P}) = \sum_{T \subseteq P} (-1)^{|T|-1} S(\rho_T) \in \mathbb{R}$$

$$I(\underline{\rho_Q} \otimes \underline{\rho_S}) = 0$$

Non-vanishing N-partite mutual information $\Rightarrow$ no system can "decouple"

$\not\Leftarrow_{N \geq 3}$

(Try $\alpha \left|0_{\mathrm{A}} 0_{\mathrm{B}} 0_{\mathrm{C}}\right\rangle + \sqrt{1-\alpha^2} \left|1_{\mathrm{A}} 1_{\mathrm{B}} 1_{\mathrm{C}}\right\rangle$ for any $\alpha \in \mathbb{C}$).

Why? Because Mutual Info. Looks Like an Euler Char.

$$
\begin{aligned}
I(\underline{\rho}_P) &= \sum_{T \subseteq P} (-1)^{|T|-1} S(\rho_T) \\
&= \sum_{k=0} (-1)^{k-1} \left[\sum_{|T|=k+1} S(\rho_T) \right]
\end{aligned}
$$

Why? Because Mutual Info. Looks Like an Euler Char.

$$I(\underline{\rho}_P) = \sum_{T \subseteq P} (-1)^{|T|-1} S(\rho_T)$$

$$= \sum_{k=0} (-1)^{k-1} \underbrace{\left[\sum_{|T|=k+1} S(\rho_T) \right]}_{\text{``dim[}\mathrm{Geom}^k(\underline{\rho_P})\text{]''}}$$

$$\mathrm{Geom}(\underline{\rho_P}) = \bigcup_{k=0}^{N-1} \mathrm{Geom}^k(\underline{\rho_P})$$

Why? Because Mutual Info. Looks Like an Euler Char.

$$I(\underline{\rho}_P) = \sum_{T \subseteq P} (-1)^{|T|-1} S(\rho_T)$$

$$= \sum_{k=0} (-1)^{k-1} \underbrace{\left[\sum_{|T|=k+1} S(\rho_T) \right]}_{\text{“dim[}\mathrm{Geom}^k(\underline{\rho_P})\text{]”}}$$

$$\mathrm{Geom}(\underline{\rho_P}) = \bigcup_{k=0}^{N-1} \underbrace{\mathrm{Geom}^k(\underline{\rho_P})}_{\substack{\text{Encodes all data about} \\ \{\rho_T\}_{|T|=k+1}}}$$

Think “simplicial complex, CW complex, (co)chain complex.”

What's an Euler Characteristic?

C a sufficiently nice category of geometric objects: a $\otimes$-category with an ability to glue objects (all pushouts)

What's an Euler Characteristic?

C a sufficiently nice category of geometric objects: a $\otimes$-category with an ability to glue objects (all pushouts)

An Euler characteristic (valued in a ring D) is an assignment that takes in any object X of **C** and outputs $\chi(X) \in D$ such that:

- $\chi(X)$ only depends on X up to iso.
- $\chi(X \otimes Y) = \chi(X)\chi(Y)$
- $\chi(X \coprod_Z Y) = \chi(X) + \chi(Y) - \chi(Z)$

What's an Euler Characteristic?

C a sufficiently nice category of geometric objects: a $\otimes$-category with an ability to glue objects (all pushouts)

An Euler characteristic (valued in a ring D) is an assignment that takes in any object X of **C** and outputs $\chi(X) \in D$ such that:

- $\chi(X)$ only depends on X up to iso.
- $\chi(X \otimes Y) = \chi(X)\chi(Y)$
- $\chi(X \coprod_Z Y) = \underbrace{\chi(X) + \chi(Y) - \chi(Z)}_{\text{Doesn't care about gluing details along } Z}$

What's an Euler Characteristic?

C a sufficiently nice category of geometric objects: a $\otimes$-category with an ability to glue objects (all pushouts)

An Euler characteristic (valued in a ring D) is an assignment that takes in any object X of **C** and outputs $\chi(X) \in D$ such that:

- $\chi(X)$ only depends on X up to iso.
- $\chi(X \otimes Y) = \chi(X)\chi(Y)$
- $\chi(X \coprod_Z Y) = \underbrace{\chi(X) + \chi(Y) - \chi(Z)}_{\substack{\text{Doesn't care about} \\ \text{gluing details along } Z}}$

An Euler characteristic valued in D is a homomorphism

$$\chi : K_0(\mathbf{C}) \to D$$

What is an Euler Characteristic

Category	Euler Characteristic
Finite Sets	Cardinality
Finite Vector Spaces	Dimension

What is an Euler Characteristic

Category	Euler Characteristic
Finite Sets	Cardinality
Finite Vector Spaces	Dimension
Bounded Graded Vector Spaces	$\chi(V^\bullet) = \sum_I (-1)^I \dim V^I$
Bounded cochain complexes	$\chi(C^\bullet) = \sum_I (-1)^I \dim C^I$

What is an Euler Characteristic

Category	Euler Characteristic
Finite Sets	Cardinality
Finite Vector Spaces	Dimension
Bounded Graded Vector Spaces	$\chi(V^\bullet) = \sum_l (-1)^l \dim V^l$
Bounded cochain complexes	$\chi(C^\bullet) = \sum_l (-1)^l \dim C^l$
Pairs (V, f) of a vector space and an endomorphism $f : V \longrightarrow V$	$\dim_n(V, f) = \mathrm{Tr}(f^n),\ n \in \mathbb{Z}_{\geq 1}$
Pairs $(V^\bullet, f^\bullet)$ of a (bdd.) graded vector space and a degree 0 endomorphism $f : V^\bullet \longrightarrow V^\bullet$	$\dim_n(V^\bullet, f^\bullet) = \sum_k (-1)^k \mathrm{Tr}_{V^k}[(f^{(k)})^n]$

The Euler Characteristic of a Multipartite State

Suppose there is a category of multipartite states with isomorphisms given by local automorphisms. Assume $\underline{\rho_P} \longmapsto \mathsf{Geom}(\underline{\rho_P})$ is a tautological equivalence (or duality) of categories.

The Euler Characteristic of a Multipartite State

Suppose there is a category of multipartite states with isomorphisms given by local automorphisms. Assume $\underline{\rho_P} \longmapsto \mathsf{Geom}(\underline{\rho_P})$ is a tautological equivalence (or duality) of categories.

$$\chi(\underline{\rho_P}) = \chi\left[\mathsf{Geom}(\underline{\rho_P})\right]$$

The Euler Characteristic of a Multipartite State

Suppose there is a category of multipartite states with isomorphisms given by local automorphisms. Assume $\underline{\rho_P} \longmapsto \mathsf{Geom}(\underline{\rho_P})$ is a tautological equivalence (or duality) of categories.

$$\chi(\underline{\rho_P}) = \chi\left[\mathsf{Geom}(\underline{\rho_P})\right]$$

$$\underline{\rho_{\mathsf{AB}}} = \underbrace{\rho_{\mathsf{A}} \coprod_{\rho_{\mathsf{AB}}} \rho_{\mathsf{B}}}_{\substack{\text{Glue together unipartite states} \\ \text{to make a bipartite state}}}$$

The Euler Characteristic of a Multipartite State

Suppose there is a category of multipartite states with isomorphisms given by local automorphisms. Assume $\underline{\rho_P} \longmapsto \mathsf{Geom}(\underline{\rho_P})$ is a tautological equivalence (or duality) of categories.

$$\chi(\underline{\rho_P}) = \chi\left[\mathsf{Geom}(\underline{\rho_P})\right]$$

$$\underline{\rho_{\mathsf{AB}}} = \underbrace{\rho_{\mathsf{A}} \coprod_{\rho_{\mathsf{AB}}} \rho_{\mathsf{B}}}_{\substack{\text{Glue together unipartite states} \\ \text{to make a bipartite state}}}$$

$$\chi(\underline{\rho_{\mathsf{AB}}}) = \chi(\rho_{\mathsf{A}}) + \chi(\rho_{\mathsf{B}}) - \chi(\rho_{\mathsf{AB}})$$

Can repeat recursively for N-partite states

$$\chi(\underline{\rho_P}) = \sum_{\emptyset \neq T \subseteq P} (-1)^{|T|-1} \underbrace{\chi(\rho_T)}_{\substack{\text{Euler characteristic} \\ \text{of unipartite state}}}$$

The Euler characteristic of a unipartite state

$$\chi(\underline{\rho_P}) = \sum_{\emptyset \neq T \subseteq P} (-1)^{|T|-1} \underbrace{\chi(\rho_T)}_{\substack{\text{Euler characteristic} \\ \text{of} \\ \text{unipartite state}}}$$

Define

$$\mathrm{dim}(\rho) = \chi_{\mathrm{unipartite}}(\rho).$$

The Euler characteristic of a unipartite state

$$\chi(\underline{\rho_P}) = \sum_{\emptyset \neq T \subseteq P} (-1)^{|T|-1} \underbrace{\chi(\rho_T)}_{\substack{\text{Euler characteristic} \\ \text{of} \\ \text{unipartite state}}}$$

Define

$$\dim(\rho) = \chi_{\text{unipartite}}(\rho).$$

$\chi(\underline{\rho_P} \otimes \underline{\rho_Q}) = \chi(\underline{\rho_P})\chi(\underline{\rho_Q}) \Rightarrow$

$$\dim(\rho \otimes \varphi) = \dim(\rho)\dim(\varphi).$$

$\dim(\rho) = S(\rho)$ does not satisfy this! $\dim(\rho) = e^{S(\rho)}$ has its owns set of subtle issues as well!

The Euler characteristic of a unipartite state

For density states on finite dimensional Hilbert spaces we can take dim to be valued in $\mathcal{O}(\mathbb{C}^3)$ (everywhere holomorphic functions in three parameters) with:

$$\dim_{\alpha,q,r}[(\mathcal{H},\widehat{\rho})] = \{\dim(\mathcal{H})^{\alpha}\,\mathrm{Tr}[(\widehat{\rho})^{q}]\}^{r}$$

Extend to any state on a finite dimensional algebra $\prod_{i=1}^{n}\mathrm{End}(\mathcal{H}_i)$ via

$$\dim[\underbrace{((\mathcal{H}_1,\widehat{\rho}^{(1)}),\cdots,(\mathcal{H}_n,\widehat{\rho}^{(n)}))}_{\text{``}\boxplus_{i=1}^{n}\widehat{\rho}^{(i)}\text{''}}] = \sum_{i}\dim[(\mathcal{H}_i,\widehat{\rho}^{(i)})].$$

Multipartite Information From the State Index

We define the State Index $\mathfrak{X}$:

$$\mathfrak{X}_{\alpha,q,r}(\rho_P) = -[\underbrace{\dim(\rho_\emptyset)}_{\rho(1)^{qr}1} + \chi(\rho_P)]$$

For a density state:

$$\mathfrak{X}_{\alpha,q,r}(\rho_P) = \sum_{\emptyset \subseteq T \subseteq P} (-1)^{|T|} \dim(\mathcal{H}_T)^\alpha \left[\mathrm{Tr}(\widehat{\rho}_T)^q\right]^r$$

It obeys the nice relation

$$\mathfrak{X}(\underline{\widehat{\rho}_P} \otimes \underline{\widehat{\rho}_Q}) = \mathfrak{X}(\underline{\widehat{\rho}_P})\mathfrak{X}(\underline{\widehat{\rho}_Q})$$

Multipartite Information From the State Index

We define the State Index $\mathfrak{X}$:

$$\mathfrak{X}_{\alpha,q,r}(\rho_P) = -[\underbrace{\dim(\rho_\emptyset)}_{\rho(1)^{qr}1} + \chi(\rho_P)]$$

For a density state:

$$\mathfrak{X}_{\alpha,q,r}(\rho_P) = \sum_{\emptyset \subseteq T \subseteq P} (-1)^{|T|} \dim(\mathcal{H}_T)^\alpha \left[\mathrm{Tr}(\widehat{\rho}_T)^q\right]^r$$

It obeys the nice relation

$$\mathfrak{X}(\underline{\widehat{\rho}_P} \otimes \underline{\widehat{\rho}_Q}) = \mathfrak{X}(\underline{\widehat{\rho}_P})\mathfrak{X}(\underline{\widehat{\rho}_Q})$$

And rescalings capture deformed mutual information:

$$\frac{\mathfrak{X}_{0,q,r}(\underline{\widehat{\rho}_P})}{r(1-q)} = \sum_{\emptyset \neq T \subseteq P} (-1)^{|T|-1} \underbrace{S^{\mathrm{TR}}_{q,r}(\widehat{\rho}_T)}_{(1-\mathrm{Tr}[\rho_T^q])^r}$$

with $q \longrightarrow 1$ recovering mutual information.

Euler characteristics of complexes of vector spaces?

State Index:
$$\mathfrak{X}_{\alpha,q,r}(\underline{\widehat{\rho}_P}) := \sum_{\emptyset\subseteq T\subseteq P}(-1)^{|T|}\dim(\mathcal{H}_T)^{\alpha}\left[\mathrm{Tr}(\widehat{\rho}_T)^q\right]^r$$

$$\xrightarrow{\ \alpha \to 0,\ \times\frac{1}{r(q-1)}\ }$$

Tsallis/Rényi Deformed Mutual Information:
$$I_{q,r}(\widehat{\boldsymbol{\rho}}_P) = \sum_{T\subseteq P}(-1)^{|T|-1}S^{\mathrm{TR}}_{q,r}(\widehat{\rho}_T)$$

$$\xrightarrow{\ q \to 1\ }$$

Mutual Information:
$$I(\underline{\widehat{\rho}_P}) = \sum_{T\subseteq P}(-1)^{|T|-1}S^{\mathrm{vN}}(\widehat{\rho}_T) \in \mathbb{R}$$

$$\xrightarrow{\ q \to 0,\ \alpha,\ r \in \mathbb{Z}\ }$$

$$\sum_{\emptyset\subseteq T\subseteq P}(-1)^{|T|}\dim(\mathcal{H}_T)^{\alpha}\,\mathrm{rank}(\widehat{\rho}_T)^r \in \mathbb{Z}$$

The GNS Construction Assigns Vector Spaces to States

Recall the GNS construction:

$$\rho : R \to \mathbb{C} \xmapsto{\mathtt{GNS}_R} L^2_\rho[R/\mathfrak{I}_\rho]$$

where: $\mathfrak{I}_\rho = \{r \in R : \rho(r^* r) = 0\} \leq R$.

The GNS Construction Assigns Vector Spaces to States

Recall the GNS construction:

$$\rho : R \to \mathbb{C} \xmapsto{\texttt{GNS}_R} L^2_\rho[R/\mathfrak{I}_\rho]$$

where: $\mathfrak{I}_\rho = \{r \in R : \rho(r^* r) = 0\} \leq R$.

- $R = \mathrm{Fun}_{\mathbb{C}}(\Omega) \Rightarrow \texttt{GNS}(\rho) \cong \mathrm{Fun}(\Omega_{\mu \neq 0})$
- In finite dimensions: $\texttt{GNS}(\rho) \cong \mathcal{H} \otimes \mathrm{Image}(\rho)^\vee$. So $\dim_{\mathbb{C}} \texttt{GNS}(\rho) = n \operatorname{rank}(\rho) = n^1 \operatorname{Tr}[\widehat{\rho}^0]^1 = \dim_{1,0,1}(\widehat{\rho})$.

The GNS Functor[2]

$$\mathtt{GNS} : \underbrace{\textbf{State}^{\mathrm{op}}}_{\substack{\text{category of} \\ \text{unipartite states}}} \longrightarrow \textbf{Rep}$$

[2] Related to a refinement of independent work by Arthur J. Parzygnat: 1609.08975.

The GNS Functor[2]

$$\mathtt{GNS} : \underbrace{\textbf{State}^{\mathrm{op}}}_{\substack{\text{category of} \\ \text{unipartite states}}} \longrightarrow \textbf{Rep}$$

	State	**Rep**
Objects	(R, ρ)	Algebras and "left modules" $(R, {}_R M)$
Morphisms	(pre)duals of algebra maps playing nicely with states "partial traces"	Algebra maps + intertwiners playing nicely together
(co)products	Coproduct: Classical sum $(A, \rho) \boxplus (B, \varphi) = (A \times B, \rho \times \varphi)$	Products $(A, M) \times (B, N) = (\mathrm{A} \times B, M \times N)$

[2]Related to a refinement of independent work by Arthur J. Parzygnat: 1609.08975.

The GNS Functor[2]

$$\texttt{GNS} : \underbrace{\textbf{State}^{\mathrm{op}}}_{\substack{\text{category of} \\ \text{unipartite states}}} \longrightarrow \textbf{Rep}$$

	State	**Rep**
Objects	(R, ρ)	Algebras and "left modules" $(R, {}_RM)$
Morphisms	(pre)duals of algebra maps playing nicely with states "partial traces"	Algebra maps + intertwiners playing nicely together
(co)products	Coproduct: Classical sum $(A, \rho) \boxplus (B, \varphi) = (A \times B, \rho \times \varphi)$	Products $(A, M) \times (B, N) = (\mathrm{A} \times B, M \times N)$

$\texttt{GNS}(\rho \to \varphi) =$ "Radon-Nikodym Derivative/Relative Modular flow"

$$\texttt{GNS}(\boxplus) = \times$$

[2]Related to a refinement of independent work by Arthur J. Parzygnat: 1609.08975.

(Non-Comm.) Geometry from a Multipartite State

A multipartite state over a finite set P is a functor

$$\underline{\rho} : \textbf{Subsets}(P)^{\mathrm{op}} \longrightarrow \textbf{State}$$

(Non-Comm.) Geometry from a Multipartite State

A multipartite state over a finite set P is a functor

$$\begin{aligned} \underline{\rho} : \textbf{Subsets}(P)^{\mathrm{op}} &\longrightarrow \textbf{State} \\ T &\longmapsto (R_T, \rho_T) \end{aligned}$$

(Non-Comm.) Geometry from a Multipartite State

A multipartite state over a finite set P is a functor

$$\begin{aligned}
\underline{\rho} : \mathbf{Subsets}(P)^{\mathrm{op}} &\longrightarrow \mathbf{State} \\
T &\longmapsto (R_T, \rho_T) \\
(T \subseteq U) &\longmapsto \underbrace{[(-) \otimes 1_{U \setminus T} \colon R_T \to R_U]^\wedge}_{\text{“partial trace over } U\setminus T\text{”}} : \rho_U \longrightarrow \rho_T
\end{aligned}$$

(Non-Comm.) Geometry from a Multipartite State

A multipartite state over a finite set P is a functor

$$\begin{aligned}
\underline{\rho} : \mathbf{Subsets}(P)^{\mathrm{op}} &\longrightarrow \mathbf{State} \\
T &\longmapsto (R_T, \rho_T) \\
(T \subseteq U) &\longmapsto \underbrace{[(-) \otimes 1_{U \setminus T} : R_T \to R_U]^{\wedge}}_{\text{“partial trace over } U \setminus T\text{”}} : \rho_U \longrightarrow \rho_T
\end{aligned}$$

(Non-Comm.) Geometry from a Multipartite State

A multipartite state over a finite set P is a functor

$$
\begin{aligned}
\underline{\rho} : \textbf{Subsets}(P)^{\mathrm{op}} &\longrightarrow \textbf{State} \\
T &\longmapsto (R_T, \rho_T) \\
(T \subseteq U) &\longmapsto \underbrace{[(-) \otimes 1_{U \setminus T} \colon R_T \to R_U]^{\wedge}}_{\text{“partial trace over } U \setminus T\text{”}} : \rho_U \longrightarrow \rho_T
\end{aligned}
$$

Can make this covariant using complementation on sets, then use Čech theory to construct a “simplicial state”

$$
\underbrace{\rho_\emptyset}_{(\mathbb{C},\rho(1))} \longleftarrow \underset{|T|=1}{\boxplus}\, \rho_T \leftleftarrows \underset{|T|=2}{\boxplus}\, \rho_T \; \substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow} \cdots \; \underbrace{\substack{\longleftarrow \\ \longleftarrow \\ \vdots \\ \longleftarrow}}_{N-1 \text{ arrows}} \; \underset{|T|=N-1}{\boxplus}\, \rho_T \; \underbrace{\substack{\longleftarrow \\ \longleftarrow \\ \vdots \\ \longleftarrow}}_{\substack{N \text{ arrows} \\ \text{“partial traces”}}} \; \rho_P
$$

(Non-Comm.) Geometry from a Multipartite State

$$\mathrm{Geom}(\underline{\rho_P}) = \underbrace{\rho_\emptyset}_{(\mathbb{C},\rho(1))} \longleftarrow \bigboxplus_{|T|=1} \rho_T \leftleftarrows \cdots \underbrace{\begin{array}{c}\longleftarrow\\ \longleftarrow\\ \vdots\\ \longleftarrow\end{array}}_{N-1\text{ arrows}} \bigboxplus_{|T|=N-1} \rho_T \underbrace{\begin{array}{c}\longleftarrow\\ \longleftarrow\\ \vdots\\ \longleftarrow\end{array}}_{\substack{N\text{ arrows}\\ \text{“partial traces”}}} \rho_P$$

$$\mathrm{Geom}^k(\underline{\rho_P}) = \bigboxplus_{|T|=k+1} \rho_T$$

(Non-Comm.) Geometry from a Multipartite State

$$\mathrm{Geom}(\underline{\rho_P}) = \underbrace{\rho_\emptyset}_{(\mathbb{C},\rho(1))} \longleftarrow \mathop{\boxplus}\limits_{|T|=1} \rho_T \leftleftarrows \cdots \underbrace{\begin{matrix}\longleftarrow\\ \longleftarrow\\ \vdots\\ \longleftarrow\end{matrix}}_{N-1 \text{ arrows}} \mathop{\boxplus}\limits_{|T|=N-1} \rho_T \underbrace{\begin{matrix}\longleftarrow\\ \longleftarrow\\ \vdots\\ \longleftarrow\end{matrix}}_{\substack{N \text{ arrows} \\ \text{“partial traces”}}} \rho_P$$

$$\mathrm{Geom}^k(\underline{\rho_P}) = \mathop{\boxplus}\limits_{|T|=k+1} \rho_T$$

$$\chi[\mathrm{Geom}^k(\underline{\widehat{\boldsymbol{\rho_P}}})] = \sum_{k=-1}^{N-1} (-1)^k \dim \left[\mathop{\boxplus}\limits_{|T|=k+1} \widehat{\rho}_T \right] = -\mathfrak{X}(\underline{\widehat{\rho}_P})$$

(Non-Comm.) Geometry from a Multipartite State

$$\rho_\emptyset \longleftarrow \boxplus_{|T|=1} \rho_T \leftleftarrows \boxplus_{|T|=2} \rho_T \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; \cdots \;\substack{\longleftarrow\\ \longleftarrow\\ \vdots\\ \longleftarrow}\; \boxplus_{|T|=N-1} \rho_T \;\substack{\longleftarrow\\ \longleftarrow\\ \vdots\\ \longleftarrow}\; \rho_P$$

$\Big\downarrow$ GNS

$$\underbrace{\mathtt{GNS}(\rho_\emptyset)}_{{}_{\mathbb{C}}\mathbb{C}} \longrightarrow \prod_{|T|=1} \mathtt{GNS}(\rho_T) \rightrightarrows \prod_{|T|=2} \mathtt{GNS}(\rho_T) \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \cdots \;\substack{\longrightarrow\\ \longrightarrow\\ \vdots\\ \longrightarrow}\; \prod_{|T|=N-1} \mathtt{GNS}(\rho_T) \;\substack{\longrightarrow\\ \longrightarrow\\ \vdots\\ \longrightarrow}\; \mathtt{GNS}(\rho_P)$$

(Non-Comm.) Geometry from a Multipartite State

$$\rho_\emptyset \longleftarrow \mathop{\boxplus}_{|T|=1} \rho_T \leftleftarrows \mathop{\boxplus}_{|T|=2} \rho_T \mathrel{\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}} \cdots \mathrel{\substack{\longleftarrow\\ \longleftarrow\\ \vdots\\ \longleftarrow}} \mathop{\boxplus}_{|T|=N-1} \rho_T \mathrel{\substack{\longleftarrow\\ \longleftarrow\\ \vdots\\ \longleftarrow}} \rho_P$$

GNS

$$\underbrace{\mathtt{GNS}(\rho_\emptyset)}_{{}_{\mathbb{C}}\mathbb{C}} \longrightarrow \prod_{|T|=1} \mathtt{GNS}(\rho_T) \rightrightarrows \prod_{|T|=2} \mathtt{GNS}(\rho_T) \mathrel{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}} \cdots \mathrel{\substack{\longrightarrow\\ \longrightarrow\\ \vdots\\ \longrightarrow}} \prod_{|T|=N-1} \mathtt{GNS}(\rho_T) \mathrel{\substack{\longrightarrow\\ \longrightarrow\\ \vdots\\ \longrightarrow}} \mathtt{GNS}(\rho_P)$$

Forget Algebra
+Alternating sum
of arrows

$$0 \to \mathbb{C} \xrightarrow{d^{-1}} \prod_{|T|=1} \mathtt{GNS}(\rho_T) \xrightarrow{d^0} \prod_{|T|=2} \mathtt{GNS}(\rho_T) \xrightarrow{d^1} \cdots \xrightarrow{d^{N-2}} \prod_{|T|=N-1} \mathtt{GNS}(\rho_T) \xrightarrow{d^{N-1}} \mathtt{GNS}(\rho_P) \to 0$$

Cohomology from a Multipartite State

$$0 \to \mathbb{C} \xrightarrow{d^{-1}} \prod_{|T|=1} \mathtt{GNS}(\rho_T) \xrightarrow{d^0} \prod_{|T|=2} \mathtt{GNS}(\rho_T) \xrightarrow{d^1} \cdots \xrightarrow{d^{N-2}} \prod_{|T|=N-1} \mathtt{GNS}(\rho_T) \xrightarrow{d^{N-1}} \mathtt{GNS}(\rho_P) \to 0$$

For a bipartite state:

$$0 \to \mathbb{C} \xrightarrow{\lambda \mapsto \lambda(1,1)} \underbrace{\mathtt{GNS}(\rho_{\mathsf{A}}) \times \mathtt{GNS}(\rho_{\mathsf{B}})}_{\text{degree } 0} \xrightarrow{(a,b) \mapsto [1\otimes b - a \otimes 1]} \underbrace{\mathtt{GNS}(\rho_{\mathsf{AB}})}_{\text{degree } 1} \to 0$$

$$H^0[\underline{\rho}_{\mathsf{AB}}] = \{(a,b) : 0 = \rho_{\mathsf{AB}}[x(a \otimes 1 - 1 \otimes b)] \text{ for all } x \in R_{\mathsf{A}} \times R_{\mathsf{B}}\}$$

Cohomology from a Multipartite State

$$0 \to \mathbb{C} \xrightarrow{d^{-1}} \prod_{|T|=1} \mathtt{GNS}(\rho_T) \xrightarrow{d^0} \prod_{|T|=2} \mathtt{GNS}(\rho_T) \xrightarrow{d^1} \cdots \xrightarrow{d^{N-2}} \prod_{|T|=N-1} \mathtt{GNS}(\rho_T) \xrightarrow{d^{N-1}} \mathtt{GNS}(\rho_P) \to 0$$

For a bipartite state:

$$0 \to \mathbb{C} \xrightarrow{\lambda \mapsto \lambda(1,1)} \underbrace{\mathtt{GNS}(\rho_{\mathrm{A}}) \times \mathtt{GNS}(\rho_{\mathrm{B}})}_{\text{degree } 0} \xrightarrow{(a,b) \mapsto [1 \otimes b - a \otimes 1]} \underbrace{\mathtt{GNS}(\rho_{\mathrm{AB}})}_{\text{degree } 1} \to 0$$

$$H^0[\underline{\rho}_{\mathrm{AB}}] = \{(a,b) : 0 = \rho_{\mathrm{AB}}[x(a \otimes 1 - 1 \otimes b)] \text{ for all } x \in R_{\mathrm{A}} \times R_{\mathrm{B}}\}$$

H^0 for a pure bipartite state is given in terms of the Schmidt decomposition. Let S be the Schmidt rank.

$$\dim H^0 = S^2 - 1$$

Cohomology from a Multipartite State

$$0 \to \mathbb{C} \xrightarrow{d^{-1}} \prod_{|T|=1} \mathtt{GNS}(\rho_T) \xrightarrow{d^0} \prod_{|T|=2} \mathtt{GNS}(\rho_T) \xrightarrow{d^1} \cdots \xrightarrow{d^{N-2}} \prod_{|T|=N-1} \mathtt{GNS}(\rho_T) \xrightarrow{d^{N-1}} \mathtt{GNS}(\rho_P) \to 0$$

For a bipartite state:

$$0 \to \mathbb{C} \xrightarrow{\lambda \mapsto \lambda(1,1)} \underbrace{\mathtt{GNS}(\rho_{\mathrm{A}}) \times \mathtt{GNS}(\rho_{\mathrm{B}})}_{\text{degree } 0} \xrightarrow{(a,b) \mapsto [1 \otimes b - a \otimes 1]} \underbrace{\mathtt{GNS}(\rho_{\mathrm{AB}})}_{\text{degree } 1} \to 0$$

$$H^0[\underline{\rho}_{\mathrm{AB}}] = \{(a,b) : 0 = \rho_{\mathrm{AB}}[x(a \otimes 1 - 1 \otimes b)] \text{ for all } x \in R_{\mathrm{A}} \times R_{\mathrm{B}}\}$$

H^0 for a pure bipartite state is given in terms of the Schmidt decomposition. Let S be the Schmidt rank.

$$\dim H^0 = S^2 - 1$$

$$\dim H^1 = \underbrace{(\dim \mathcal{H}_{\mathrm{A}} - S)(\dim \mathcal{H}_{\mathrm{B}} - S)}_{\text{“measure of maximal entanglement”}}$$

Simplicial Complexes for Measures on a Finite Set

For multipartite measures on a finite set, the GNS representation can be made into another commutative W^*-algebra.

We can take spec to recover a set from each algebra;

The result is a simplicial set/simplicial complex.

Simplicial Complexes for Measures on a Finite Set

For multipartite measures on a finite set, the GNS representation can be made into another commutative W^*-algebra.

We can take spec to recover a set from each algebra;

The result is a simplicial set/simplicial complex.

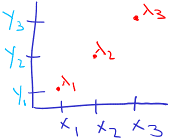
The state index is a deformation the Euler characteristic of this complex that takes into account the “sizes” of each simplex (given by the measure).

Multipartite Measures and Commutative Geometry

Multipartite Measures $\longleftrightarrow$ Commutative Geometry

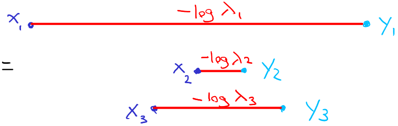
encoding non-local correlations

$\mu : X \times Y \to \mathbb{R}_{\geq 0}$



$\lambda_1 + \lambda_2 + \lambda_3 = 1$

$G_\mu =$

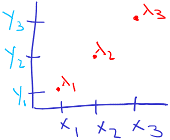


Multipartite Measures and Commutative Geometry

Multipartite Measures $\longleftrightarrow$ Commutative Geometry

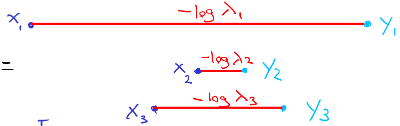
encoding non-local correlations

$\mu : X \times Y \to \mathbb{R}_{\geq 0}$



$\lambda_1 + \lambda_2 + \lambda_3 = 1$

$G_\mu =$



From 0-cells

From 1-cells

$$\chi_q(G_\mu) = 2(\lambda_1^q + \lambda_2^q + \lambda_3^q) - (\lambda_1^q + \lambda_2^q + \lambda_3^q)$$

$$= \lambda_1^q + \lambda_2^q + \lambda_3^q$$

Summary

- Mutual information (and its deformations) of a multipartite state emerge naturally from the Euler characteristic (the "state index") of some canonically associated non-commutative space.

Summary

- Mutual information (and its deformations) of a multipartite state emerge naturally from the Euler characteristic (the "state index") of some canonically associated non-commutative space.

- In the world of probability measures this space is a simplicial complex.

Summary

- Mutual information (and its deformations) of a multipartite state emerge naturally from the Euler characteristic (the "state index") of some canonically associated non-commutative space.
- In the world of probability measures this space is a simplicial complex.
- The precise operators/random variables capturing non-local correlations are captured by cohomology.

Summary

- Mutual information (and its deformations) of a multipartite state emerge naturally from the Euler characteristic (the "state index") of some canonically associated non-commutative space.
- In the world of probability measures this space is a simplicial complex.
- The precise operators/random variables capturing non-local correlations are captured by cohomology.
- Cohomology can detect how things are glued together, Euler characteristics only count how many things are glued together:

Summary

- Mutual information (and its deformations) of a multipartite state emerge naturally from the Euler characteristic (the "state index") of some canonically associated non-commutative space.

- In the world of probability measures this space is a simplicial complex.

- The precise operators/random variables capturing non-local correlations are captured by cohomology.

- Cohomology can detect how things are glued together, Euler characteristics only count how many things are glued together: cohomology is a finer invariant than mutual information!

Other Comments

- There is a G-equivariant generalization of the state index: "G-equivariant mutual info/entropy"?
- The category of states is a small part of the story between equivalence of 2-categories through the Kasparov construction:

$$\left\{\begin{matrix} C^*/W \text{ Algebras} \\ \text{and Completely Positive maps} \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} C^*/W \text{ Algebras} \\ \text{and (pointed) Hilbert Bimodules} \end{matrix}\right\}$$

Other Comments

- There is a G-equivariant generalization of the state index: "G-equivariant mutual info/entropy"?
- The category of states is a small part of the story between equivalence of 2-categories through the Kasparov construction:

$$\left\{ \substack{C^*/W \text{ Algebras} \\ \text{and Completely Positive maps}} \right\} \longleftrightarrow \left\{ \substack{C^*/W \text{ Algebras} \\ \text{and (pointed) Hilbert Bimodules}} \right\}$$

- As a substory of this equivalence, in work[3] with Roman Geiko and Greg Moore we are exploring an equivalence of categories:

$$\left\{ \text{Matrix Product States} \right\} \longleftrightarrow \left\{ \text{Completely Positive Maps} \right\}$$

In order to provide insight into how open-closed 2D topological field theories emerge as EFTs of 1D lattice systems).

[3]Inspired largely by work of Verstraete and Kapustin-Turzillo-Yau

A Sample of Future Directions

- Do things become nice for states arising from holography? Quantum Code states (c.f. Pastawksi-Yoshida-Harlow-Preskill)?
- Would cohomology class representatives encoding multipartite non-local correlations be useful for quantum information theorists?

A Sample of Future Directions

- Do things become nice for states arising from holography? Quantum Code states (c.f. Pastawksi-Yoshida-Harlow-Preskill)?
- Would cohomology class representatives encoding multipartite non-local correlations be useful for quantum information theorists?
- Generalization of the geometric interpretation of strong subaddivity using the Ryu-Takayanagi formula (Headrick-Takayanagi) beyond holographic states?

A Sample of Future Directions

- Do things become nice for states arising from holography? Quantum Code states (c.f. Pastawksi-Yoshida-Harlow-Preskill)?
- Would cohomology class representatives encoding multipartite non-local correlations be useful for quantum information theorists?
- Generalization of the geometric interpretation of strong subaddivity using the Ryu-Takayanagi formula (Headrick-Takayanagi) beyond holographic states?
- Link invariants.
- Generalizations of Poincaré polynomials to holomorphic functions: much finer numerical invariants than mutual information!

A Sample of Future Directions

- Do things become nice for states arising from holography? Quantum Code states (c.f. Pastawksi-Yoshida-Harlow-Preskill)?
- Would cohomology class representatives encoding multipartite non-local correlations be useful for quantum information theorists?
- Generalization of the geometric interpretation of strong subaddivity using the Ryu-Takayanagi formula (Headrick-Takayanagi) beyond holographic states?
- Link invariants.
- Generalizations of Poincaré polynomials to holomorphic functions: much finer numerical invariants than mutual information!
- Full understanding of the infinite dimensional story and its connections to (relative) modular flow/Tomita-Takesaki theory, non-commutative L^p-spaces, etc.

A Sample of Future Directions

- Do things become nice for states arising from holography? Quantum Code states (c.f. Pastawksi-Yoshida-Harlow-Preskill)?
- Would cohomology class representatives encoding multipartite non-local correlations be useful for quantum information theorists?
- Generalization of the geometric interpretation of strong subaddivity using the Ryu-Takayanagi formula (Headrick-Takayanagi) beyond holographic states?
- Link invariants.
- Generalizations of Poincaré polynomials to holomorphic functions: much finer numerical invariants than mutual information!
- Full understanding of the infinite dimensional story and its connections to (relative) modular flow/Tomita-Takesaki theory, non-commutative L^p-spaces, etc.
- I'm looking for other applications!

Software

Software computing cohomology/Poincaré polynomials is available at github.com/tmainero.

Phase 1

Insert N-partite Density State $\widehat{\rho}_{\{1,\cdots,N\}} \in \mathrm{Dens}(\bigotimes_{i=1}^{N} \mathcal{H}_i)$

$\rightarrow$

Phase 2

?

$\rightarrow$

Phase 3

Profit:
Degree $N-1$ polynomials with positive integer coefficients

(Bonus Slide!) Commutative Geometry from a Measure

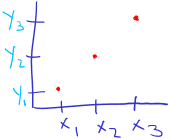
Multipartite Measures

$\longleftrightarrow$

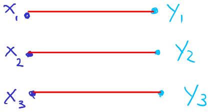
Commutative Geometry

encoding non-local correlations

$\mu : X \times Y \to \mathbb{R}_{\geq 0}$



$G_\mu =$

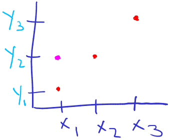


(Bonus Slide!) Commutative Geometry from a Measure

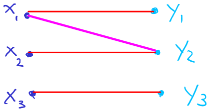
Multipartite Measures $\longleftrightarrow$ Commutative Geometry

encoding non-local correlations

$\mu : X \times Y \to \mathbb{R}_{\geq 0}$



$G_\mu =$

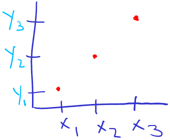


(Bonus Slide!) Commutative Geometry from a Measure

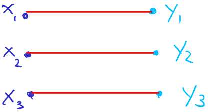
Multipartite Measures $\longleftrightarrow$ Commutative Geometry

encoding non-local correlations

$\mu : X \times Y \to \mathbb{R}_{\geq 0}$



$G_\mu =$

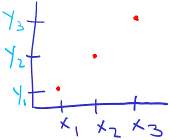


$H^0(G_\mu; \mathbb{C}) \cong \mathbb{C}^3$

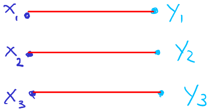
(Bonus Slide!) Commutative Geometry from a Measure

Multipartite Measures $\longleftrightarrow$ Commutative Geometry encoding non-local correlations

$\mu : X \times Y \to \mathbb{R}_{\geq 0}$



$G_\mu =$



$$H^0(G_\mu; \mathbb{C}) = \mathbb{C}\langle (1_{x_1}, 1_{y_1}), (1_{x_2}, 1_{y_2}), (1_{x_3}, 1_{y_3}) \rangle$$

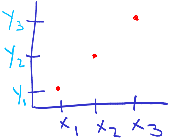
1_z = indicator function on pt. z

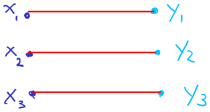
= Pairs of "non-locally", maximally correlated random variables

(Bonus Slide!) Commutative Geometry from a Measure

Multipartite Measures $\longleftrightarrow$ Commutative Geometry encoding non-local correlations

$$\mu : X \times Y \to \mathbb{R}_{\geq 0}$$



$G_\mu =$ 

$$H^0(G_\mu;\mathbb{C}) = \mathbb{C}\langle (1_{x_1}, 1_{y_1}), (1_{x_2}, 1_{y_2}), (1_{x_3}, 1_{y_3}) \rangle$$

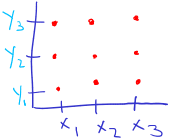
$$\tilde{H}^0(G_\mu;\mathbb{C}) = H^0(G_\mu;\mathbb{C}) \,/\, \mathbb{C}\langle \underbrace{(\textstyle\sum_i 1_{x_i}, \sum_i 1_{y_i})}_{\text{Pairs of Constant random Vars.}} \rangle$$

non-trivial" "non-local" maximal Correlations

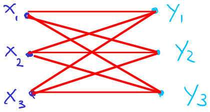
(Bonus Slide!) Commutative Geometry from a Measure

Multipartite Measures $\longleftrightarrow$ Commutative Geometry encoding non-local correlations

$\mu : X \times Y \to \mathbb{R}_{\geq 0}$



$G_\mu =$



$$H^0(G_\mu;\mathbb{C}) \cong \mathbb{C}$$

$$\tilde{H}^0(G_\mu;\mathbb{C}) = 0$$

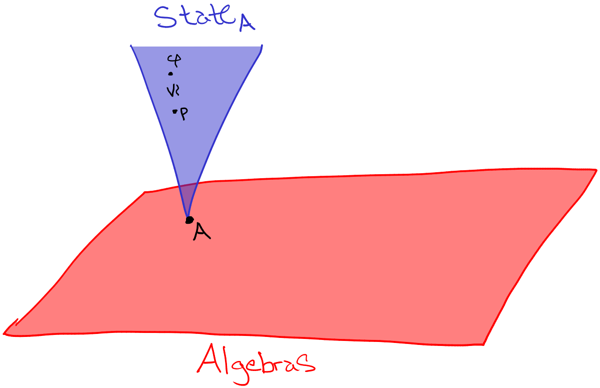
(Bonus Slide!) The GNS Functor on an Algebra

The GNS representation for states on an algebra A is a functor:

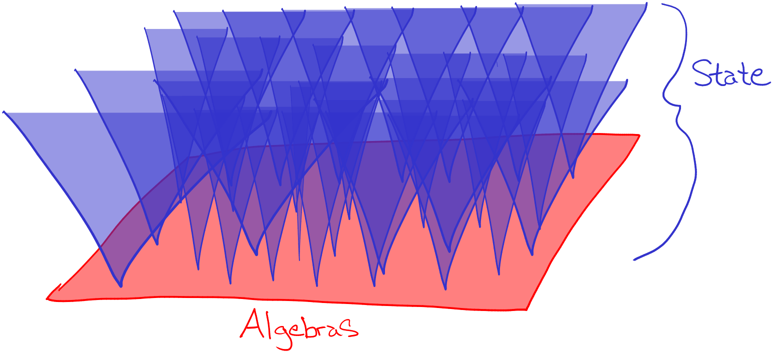
$$\mathtt{GNS}_A : \mathbf{State}_A \to \mathbf{Rep}_A$$

	$\mathbf{State}_A$	$\mathbf{Rep}_A$
Objects	Positive linear funls $\rho : R \longrightarrow \mathbb{C}$	$*$-representations of A
Morphisms	$\rho \longrightarrow \varphi$ $\Updownarrow$ $\rho \leq C\varphi$ for some $C > 0$	(bounded) intertwiners

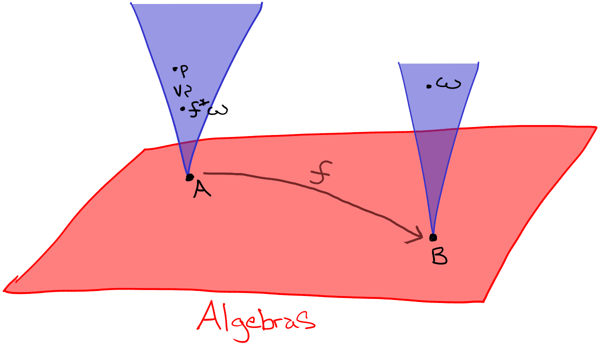
(Bonus Slide!) The Category of States



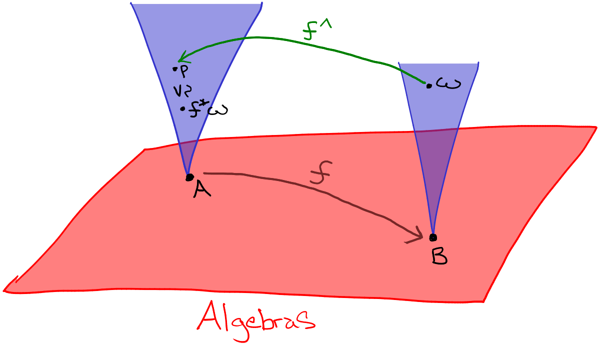
(Bonus Slide!) The Category of States



(Bonus Slide!) The Category of States



(Bonus Slide!) The Category of States



(Bonus Slide!) The Category of States (Again)

$$\mathtt{GNS} : \textbf{State}^{\mathrm{op}} \longrightarrow \textbf{Rep}$$

	State	**Rep**
Objects	(R, ρ)	Algebras and "left modules" $(R, {}_RM)$
Morphisms	"preduals" of algebra maps playing nicely with states "partial traces"	Algebra maps + intertwiners playing nicely together
(co)products	Classical sum $(A,\rho) \boxplus (B,\varphi) = (A \times B, \rho \times \varphi)$	Products $(A, M) \times (B, N) = (A \times B, M \times N)$

$\mathtt{GNS}(\rho \to \varphi) =$ "Radon-Nikodym Derivative/Relative Modular flow"

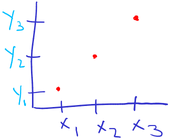
$$\mathtt{GNS}(\boxplus) = \times$$

(Bonus Slide!) Comm. Geometry From a Measure (Long)

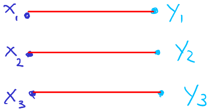
Multipartite Measures $\longleftrightarrow$ Commutative Geometry

encoding non-local correlations

$\mu : X \times Y \to \mathbb{R}_{\geq 0}$



$G_\mu =$

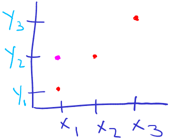


(Bonus Slide!) Comm. Geometry From a Measure (Long)

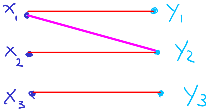
Multipartite Measures $\longleftrightarrow$ Commutative Geometry

encoding non-local correlations

$\mu : X \times Y \to \mathbb{R}_{\geq 0}$



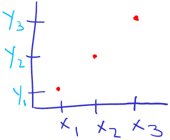
$G_\mu =$



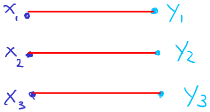
(Bonus Slide!) Comm. Geometry From a Measure (Long)

Multipartite Measures $\longleftrightarrow$ Commutative Geometry encoding non-local correlations

$$\mu : X \times Y \to \mathbb{R}_{\geq 0}$$



$G_\mu =$

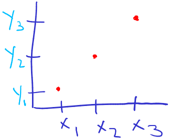


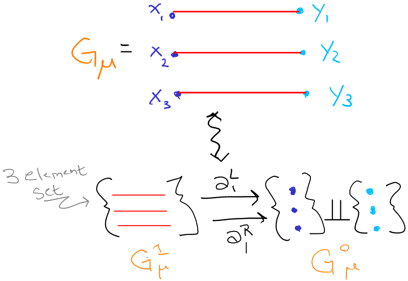
$$H^0(G_\mu; \mathbb{C}) \cong \mathbb{C}^3$$

(Bonus Slide!) Comm. Geometry From a Measure (Long)

Multipartite Measures $\longleftrightarrow$ Commutative Geometry encoding non-local correlations

$\mu : X \times Y \to \mathbb{R}_{\geq 0}$



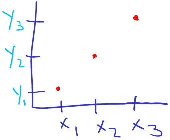


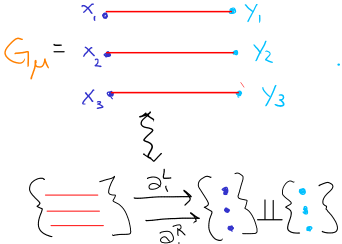
(Bonus Slide!) Comm. Geometry From a Measure (Long)

Multipartite Measures $\longleftrightarrow$ Commutative Geometry

encoding non-local correlations

$\mu : X \times Y \to \mathbb{R}_{\geq 0}$





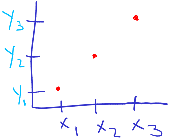
$\partial_1^L(x_i \text{———} y_i) = x_i$

(Bonus Slide!) Comm. Geometry From a Measure (Long)

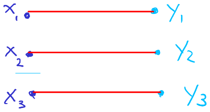
Multipartite Measures $\longleftrightarrow$ Commutative Geometry

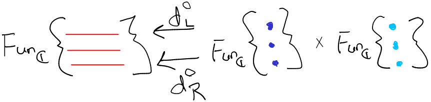
encoding non-local correlations

$\mu : X \times Y \to \mathbb{R}_{\geq 0}$



$G_\mu =$



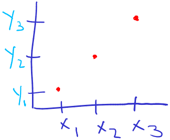


(Bonus Slide!) Comm. Geometry From a Measure (Long)

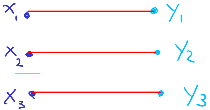
Multipartite Measures $\longleftrightarrow$ Commutative Geometry

encoding non-local correlations

$\mu : X \times Y \to \mathbb{R}_{\geq 0}$



$G_\mu =$



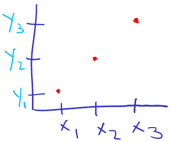
$$\mathrm{Fun}_{\mathbb{C}}\{\vdots\} = \mathbb{C}\langle 1_x, 1_y, 1_z \rangle$$
$$= \mathrm{Rand}(X)$$

$$\mathrm{Fun}_{\mathbb{C}}\{\equiv\} \underset{d_R^0}{\overset{d_L^0}{\leftleftarrows}} \mathrm{Fun}_{\mathbb{C}}\{\vdots\} \times \mathrm{Fun}_{\mathbb{C}}\{\vdots\}$$

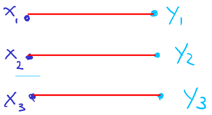
(Bonus Slide!) Comm. Geometry From a Measure (Long)

Multipartite Measures $\longleftrightarrow$ Commutative Geometry encoding non-local correlations

$$\mu : X \times Y \to \mathbb{R}_{\geq 0}$$



$G_\mu =$



$$\mathrm{Fun}_{\mathbb{C}}\left\{\equiv\right\} = \mathbb{C}\langle\{\mathbb{1}_{(x_i,y_i)}\}_i\rangle \leq \mathbb{C}\langle\{\mathbb{1}_{(x_i,y_j)}\}_{i,j}\rangle = \mathrm{Rand}(X \times Y)$$

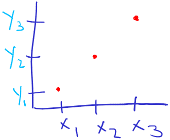
$$\mathrm{Fun}_{\mathbb{C}}\left\{\equiv\right\} \overset{d_L^0}{\underset{d_R^0}{\longleftarrow}} \mathrm{Fun}_{\mathbb{C}}\left\{\vdots\right\} \times \mathrm{Fun}_{\mathbb{C}}\left\{\vdots\right\}$$

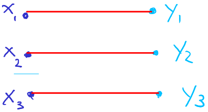
(Bonus Slide!) Comm. Geometry From a Measure (Long)

Multipartite Measures $\longleftrightarrow$ Commutative Geometry

encoding non-local correlations

$\mu : X \times Y \to \mathbb{R}_{\geq 0}$



$G_\mu =$ 

$$\mathrm{Fun}_{\mathbb{C}}\{\equiv\} = \mathbb{C}\langle \{ \mathbb{1}_{(x_i, y_i)} \}_i \rangle$$

$$= \mathrm{Rand}^{+}_{\mu}(X \times Y) \leq \mathrm{Rand}(X \times Y)$$

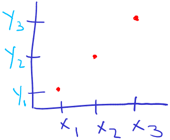
$$\mathrm{Fun}_{\mathbb{C}}\{\equiv\} \overset{d_L^\circ}{\underset{d_R^\circ}{\leftleftarrows}} \mathrm{Fun}_{\mathbb{C}}\{\vdots\} \times \mathrm{Fun}_{\mathbb{C}}\{\vdots\}$$

(Bonus Slide!) Comm. Geometry From a Measure (Long)

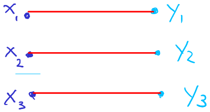
Multipartite Measures $\longleftrightarrow$ Commutative Geometry

encoding non-local correlations

$\mu : X \times Y \to \mathbb{R}_{\geq 0}$



$G_\mu =$



$$\mathrm{Fun}_{\mathbb{C}}\{\equiv\} = \mathbb{C}\langle\{\mathbb{1}_{(x_i,y_i)}\}_i\rangle = \mathrm{Rand}^+_\mu(X\times Y) \leq \mathrm{Rand}(X\times Y)$$

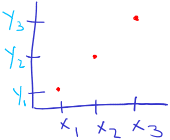
$$\mathrm{Fun}_{\mathbb{C}}\{\equiv\} \overset{d^0_L}{\underset{d^0_R}{\leftleftarrows}} \mathrm{Fun}_{\mathbb{C}}\{\vdots\} \times \mathrm{Fun}_{\mathbb{C}}\{\vdots\}$$

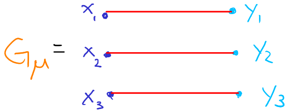
(Bonus Slide!) Comm. Geometry From a Measure (Long)

Multipartite Measures $\longleftrightarrow$ Commutative Geometry

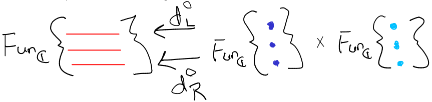
encoding non-local correlations

$\mu : X \times Y \to \mathbb{R}_{\geq 0}$





$\mathrm{Rand}^+_\mu(X\times Y) \leftleftarrows \mathrm{Rand}^+_\mu(X) \times \mathrm{Rand}^+_\mu(Y)$

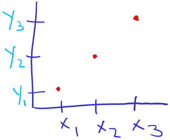


(Bonus Slide!) Comm. Geometry From a Measure (Long)

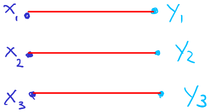
Multipartite Measures $\longleftrightarrow$ Commutative Geometry

encoding non-local correlations

$\mu : X \times Y \to \mathbb{R}_{\geq 0}$



$G_\mu =$



$$0 \leftarrow \mathrm{Rand}^+_\mu(X \times Y) \leftarrow \mathrm{Rand}^+_\mu(X) \times \mathrm{Rand}^+_\mu(Y) \leftarrow 0$$

$$0 \leftarrow \mathrm{Fun}_{\mathbb{C}}\{\equiv\} \xleftarrow{d^0_L - d^0_R} \mathrm{Fun}_{\mathbb{C}}\{\vdots\} \times \mathrm{Fun}_{\mathbb{C}}\{\vdots\} \leftarrow 0$$

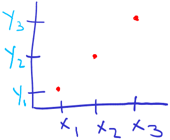
$$H^0(G_\mu) = \ker(d^0_L - d^0_R)$$

(Bonus Slide!) Comm. Geometry From a Measure (Long)

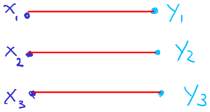
Multipartite Measures $\longleftrightarrow$ Commutative Geometry

encoding non-local correlations

$\mu : X \times Y \to \mathbb{R}_{\geq 0}$



$G_\mu =$



$$H^0(G_\mu;\mathbb{C}) = \mathbb{C}\langle (1_{x_1},1_{y_1}), (1_{x_2},1_{y_2}), (1_{x_3},1_{y_3})\rangle$$

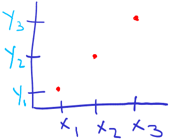
1_z = indicator function on pt. z

= Pairs of "non-locally", maximally correlated random variables

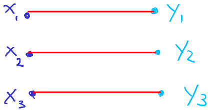
(Bonus Slide!) Comm. Geometry From a Measure (Long)

Multipartite Measures $\longleftrightarrow$ Commutative Geometry encoding non-local correlations

$\mu : X \times Y \to \mathbb{R}_{\geq 0}$



$G_\mu =$



$$H^0(G_\mu;\mathbb{C}) = \mathbb{C}\langle (1_{x_1},1_{y_1}),(1_{x_2},1_{y_2}),(1_{x_3},1_{y_3})\rangle$$

$$\tilde{H}^0(G_\mu;\mathbb{C}) = H^0(G_\mu;\mathbb{C}) \,/\, \mathbb{C}\langle \underbrace{(\textstyle\sum_i 1_{x_i}, \sum_i 1_{y_i})}_{\text{Pairs of constant random vars.}} \rangle$$

"non-trivial" "non-local" maximal correlations