

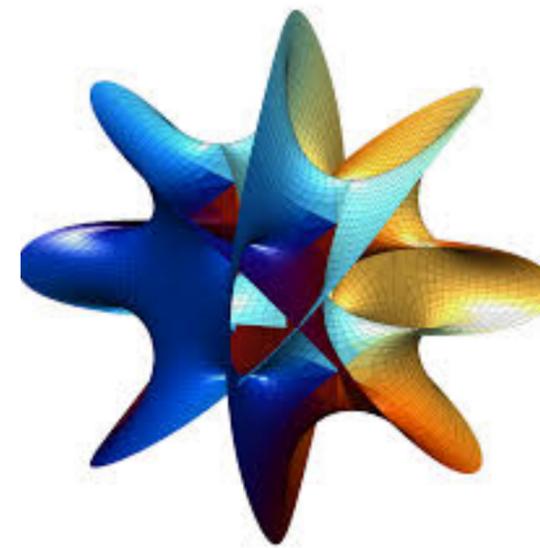
# Twisted Holography & Koszul Duality

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**in collaboration w/ Kevin Costello**

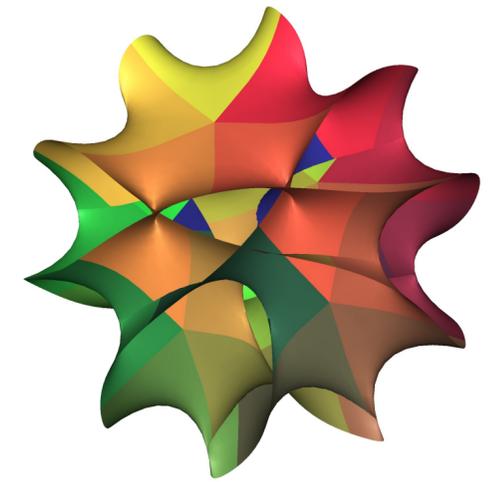
*String dualities beget novel mathematics*



**IIA**



(homological)  
mirror symmetry

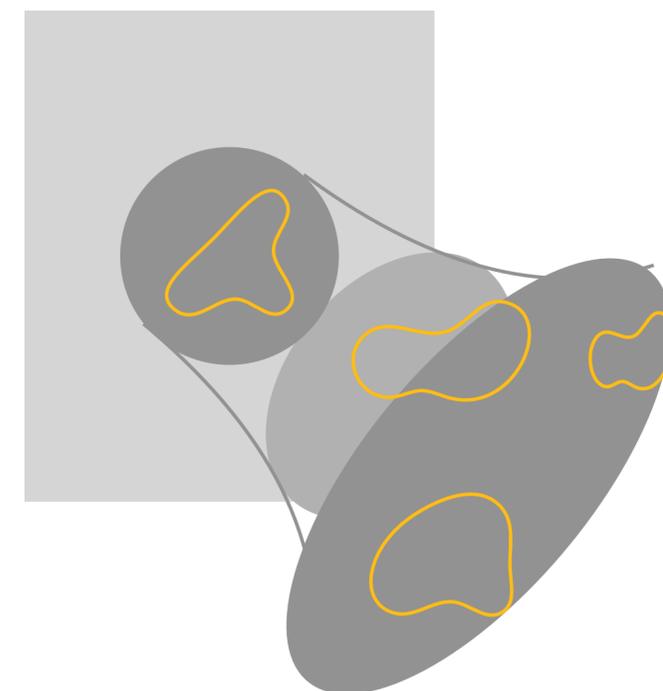
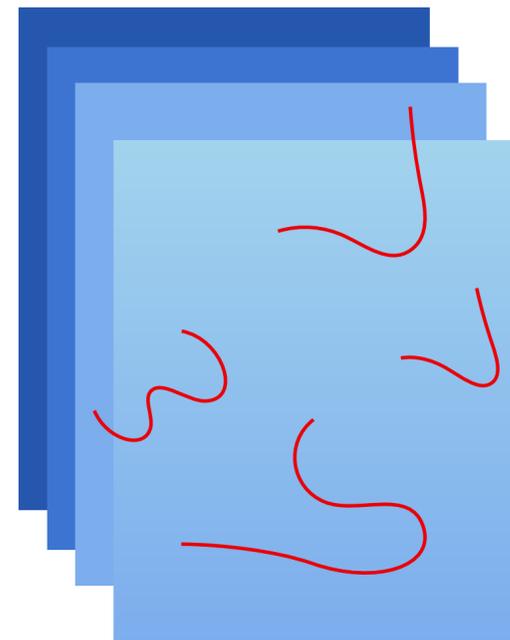


**IIB**

**Top down models of the AdS/CFT from brane constructions (open/closed duality)**

towards worldsheet derivations: [Berkovits, Ooguri, Vafa, Gopakumar...]

- mathematical underpinnings? derivation?
  - connections w/ recent emphasis on operator algebras
  - riches from large-N topological string dualities
- why not hope for even more w/ dynamical gravity?



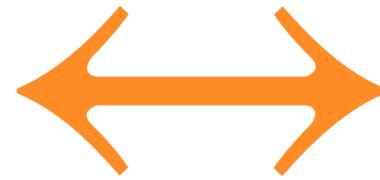
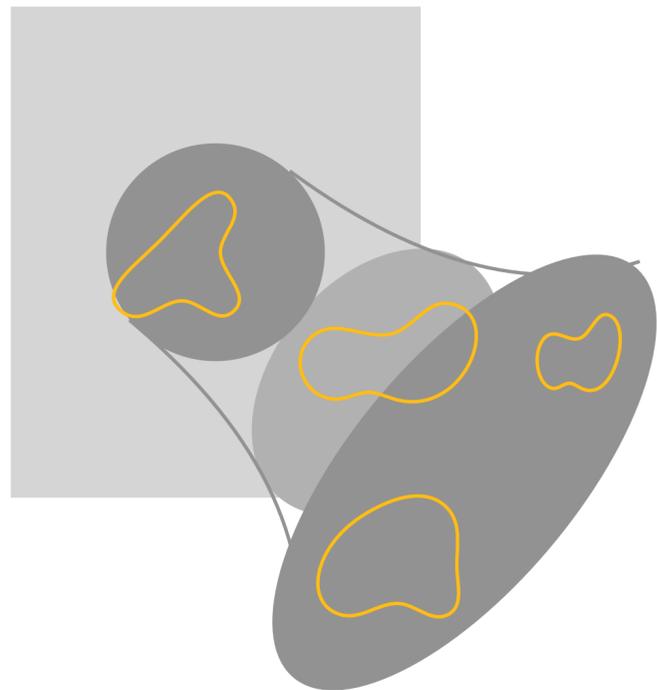
[Maldacena]

Today we study a **twisted** top-down dual pair  
& connect it to **generalizations of Koszul Duality**

## IIB supergravity

$$AdS_3 \times S^3 \times T^4$$

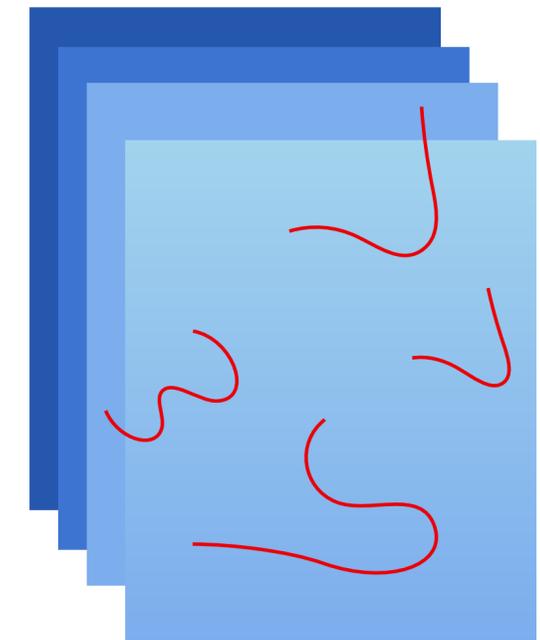
$$(g_s \rightarrow 0)$$



## 2d CFT

$\sigma$ -model w/ target

$$\text{Hilb}^N(T^4) \quad (N \rightarrow \infty)$$



- technically simple
- well-studied, good testbed of ideas
- physically rich
- twisted version is a nice toy model

Our twist: cohomology of  $\tilde{Q} = Q + d_{BRST}$  s.t.  $\partial_{\bar{z}_j}$  are Q-trivial

A holomorphic twist of 2d CFT  $\rightarrow$  vertex algebra, chiral de Rham [Witten, Kapustin, MSV]

[Costello-Li] Conj: A twist of IIB supergravity on CY 5-fold can be defined  $\rightarrow$  Kodaira-Spencer (B-model)

Derivations in parallel

Physical vs. Twisted

[BCOV]

see also [Costello-Gaiotto]

- IIB string theory on  $\mathbb{R}^6 \times T^4$
- $N_1$  D1-branes on  $(\mathbb{R}^2 \subset \mathbb{R}^6) \times \{\text{pts in } T^4\}$
- $N_5$  D5-branes on  $(\mathbb{R}^2 \subset \mathbb{R}^6) \times T^4$
- Dimensionally reduce on  $T^4 \rightarrow$  6d theory +  $N := N_1 N_5$  "strings"
- D1-D5 branes source RR flux that deforms the metric of  $\mathbb{R}^6 \setminus \mathbb{R}^2$  to that of  $AdS_3 \times S^3$
- Judicious near-horizon/decoupling limits
- Kodaira-Spencer theory on  $\mathbb{C}^3 \times T^4$
- $N_1$  B1-branes on  $(\mathbb{C} \subset \mathbb{C}^3) \times \{\text{pts in } T^4\}$
- $N_5$  B5-branes on  $(\mathbb{C} \subset \mathbb{C}^3) \times T^4$
- Dimensionally reduce on  $T^4 \rightarrow$  6d theory +  $N$  "strings"
- B1-B5 branes source a field that deforms the cplx str. of  $\mathbb{C}^3 \setminus \mathbb{C} \times \mathbb{C}^{0|4}$  to that of the "superconifold" (cf.  $SL(2, \mathbb{C})$ )
- Limits are trivial

# Slogans & Upshots, pt 1

1. Kodaira-Spencer theory (*deformations of cplx structures*) on  $X^0 \rightarrow \mathbb{C}^{0|4}$

$$u_i, w_i \in \mathbb{C}^4 \quad X^0 \subset \mathbb{C}^{4|4}$$

$$F \in H^2(T^4, \mathbb{C})$$

$$\eta_a \in \mathbb{C}^{0|4}$$

$$u_2 w_1 - u_1 w_2 = F^{ab} \eta_a \eta_b$$

In terms of flat space:

$$N \xrightarrow{\quad} \mathbb{C}_z \times \mathbb{C}_{w_1, w_2}^2 \times T^4 \quad \eta_a \leftrightarrow dz_a$$

- KS theory of *polyvector fields* (satisfying eom, constraints)

$$PV^{i,j}(X) = \Omega^{(0,j)}(X, \wedge^i TX) \stackrel{\vee}{\simeq} \Omega_X^{5-i,j}$$

- RR flux descends to Beltrami differential
- In new cplx structure,  $u_i, w_j$  are holomorphic fns

$$u_1 = w_1 z - F^{ab} \eta_a \eta_b \frac{\bar{w}_2}{|w|^2}$$

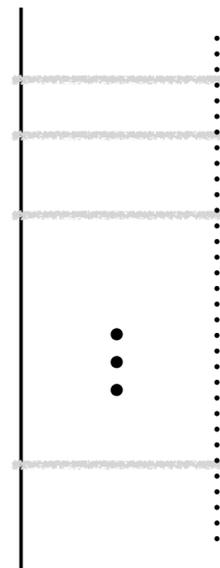
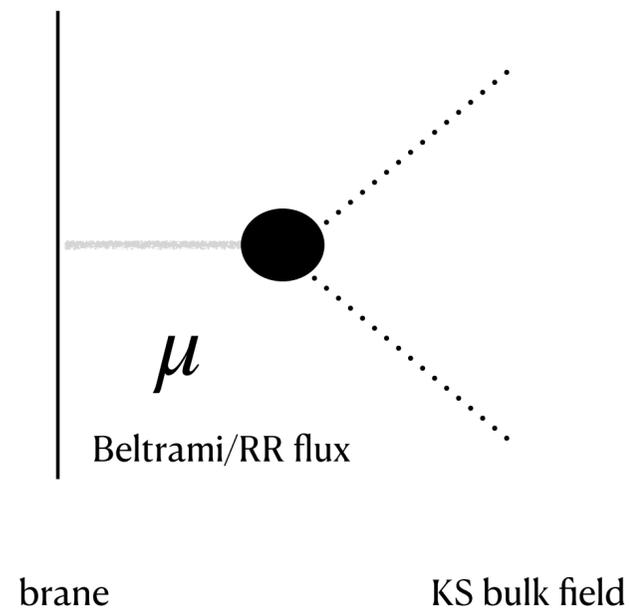
$$u_2 = w_2 z + F^{ab} \eta_a \eta_b \frac{\bar{w}_1}{|w|^2}$$

## Slogans & Upshots, pt 2

2. Since  $F^3 = 0$  in  $H^*(T^4, \mathbb{C}) = \mathbb{C}[\eta_a]$ , backreaction only corrects flat space  $(\mathbb{C}^3 \setminus \mathbb{C}) \times \mathbb{C}^{0|4}$  answers ( $F = 0$ ) by *a finite number of terms*

Contrast with top. string on deformed conifold:

[Gopakumar-Vafa, Costello-Gaiotto]



propagator in backreacted geometry

$$u_1 w_2 - u_2 w_1 = N$$

$$AdS_3 \times S^3 \simeq SL(2, \mathbb{C})$$

$\cap$

$$AdS_5 \times S^5$$

# Slogans & Upshots, pt 3

## Single-particle gravitational states

after [Costello-Gaiotto]

- Compactify  $\overline{X^0}$ , w/ boundary  $\simeq \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{C}^{0|4}$
- Impose vacuum boundary conditions
- $\alpha, \gamma, \mu_0, \mu_2$  satisfy e.o.m. + b.c. except at  $z = 0$

### Takeaways from the analysis

- $\alpha, \gamma, \mu_0, \mu_2$  are highest weight states for  $\mathfrak{sl}_2, SU(2)_R$
- descendants also included
- each occur with multiplicity  $H^*(T^4)$
- form *short*  $\mathfrak{psu}(1,1|2)$  multiplets [dB, D-M-W]
- manifestly  $Spin(5) \curvearrowright H^*(T^4)$  invariant
- can take multiparticle combinations

KS field	Statistics	$SU(2)_R$ Highest weight	$L_0$ Highest weight
$\alpha \sim \delta_{z=0}^{(l)} n^{-k} + \dots$	Fermionic	$(k-1)/2$ $k \geq 1$	$(k+1)/2$
$\gamma \sim \delta_{z=0}^{(l)} n^{-k} + \dots$	Fermionic	$(k-1)/2$ $k \geq 1$	$(k+1)/2$
$\mu_0 \sim d \log ndzn^{-k} \delta_{z=0}^{(l)} + \dots$	Bosonic	$k/2$ $k \geq 1$	$k/2$
$\mu_2 \sim d \log ndwn^{-k-2} \delta_{z=0}^{(l)}$	Bosonic	$(k-2)/2$ $k \geq 2$	$(k+2)/2$

comports with untwisted supergravity analysis\* 

## Slogans & Upshots, pt 4

Can prove that the Lie algebra of (vacuum-preserving) gauge symmetries is

$$\text{Vect}_0(X^0/\mathbb{C}^{0|4}) \oplus (\mathbb{C}^2 \otimes \Pi\mathcal{O}(X^0))$$

1. hol'c divergence-free vector fields (symmetries of superconifold)
2. fermionic hol'c functions

Holographic duality suggests the following natural conjecture:

**Conj:** The Lie algebra  $\text{Vect}_0(X^0/\mathbb{C}^{0|4})$  acts on the chiral algebra of  $\text{Sym}^{N \rightarrow \infty}(T^4)$  by “global symmetries”, i.e. symmetries which preserve the vacuum at  $0, \infty$

*Could the ground ring of  $AdS_3 \times S^3 \times T^4$  string be related to the algebra of functions on  $X^0$ ?*

See also: [Witten, Rastelli-Wijnholt]

## A survey of the situation thus far:

- Twisted supergravity, in the sense of Costello-Li, yields, as the bulk gravity theory, Kodaira-Spencer theory on the superconifold when one follows a twisted derivation of AdS/CFT.
- One can use algebro-geometric methods to enumerate single-particle states.
- These form short representations of  $\mathfrak{psu}(1,1|2)$ , as expected from the untwisted duality.
- The symmetry algebra can be determined and given a very explicit geometric description, and is conjecturally isomorphic to the global symmetry algebra of the chiral algebra of the symmetric orbifold CFT in the large-N limit.
- More generally, the full large-N vertex algebra (not just the global symmetry subalgebra), can in principle be extracted from Witten diagram computations in KS theory, and matched to 2 & 3-pt functions in the twisted CFT/vertex algebra ([Costello-NMP, WIP]).
- This procedure is rather tedious, even in the holomorphic theory.
- A nice, precise toy example of AdS/CFT amenable to rigorous formulation, whose untwisted counterpart is well-studied and relatively simple due to low dimensionality.

# What's Koszul got to do with it?

- Holography beyond leading order in “planar limit”,  $1/N$  corrections
- Is there some structure that governs (twisted) dual pairs more generally?
- Definition of holographic duality?

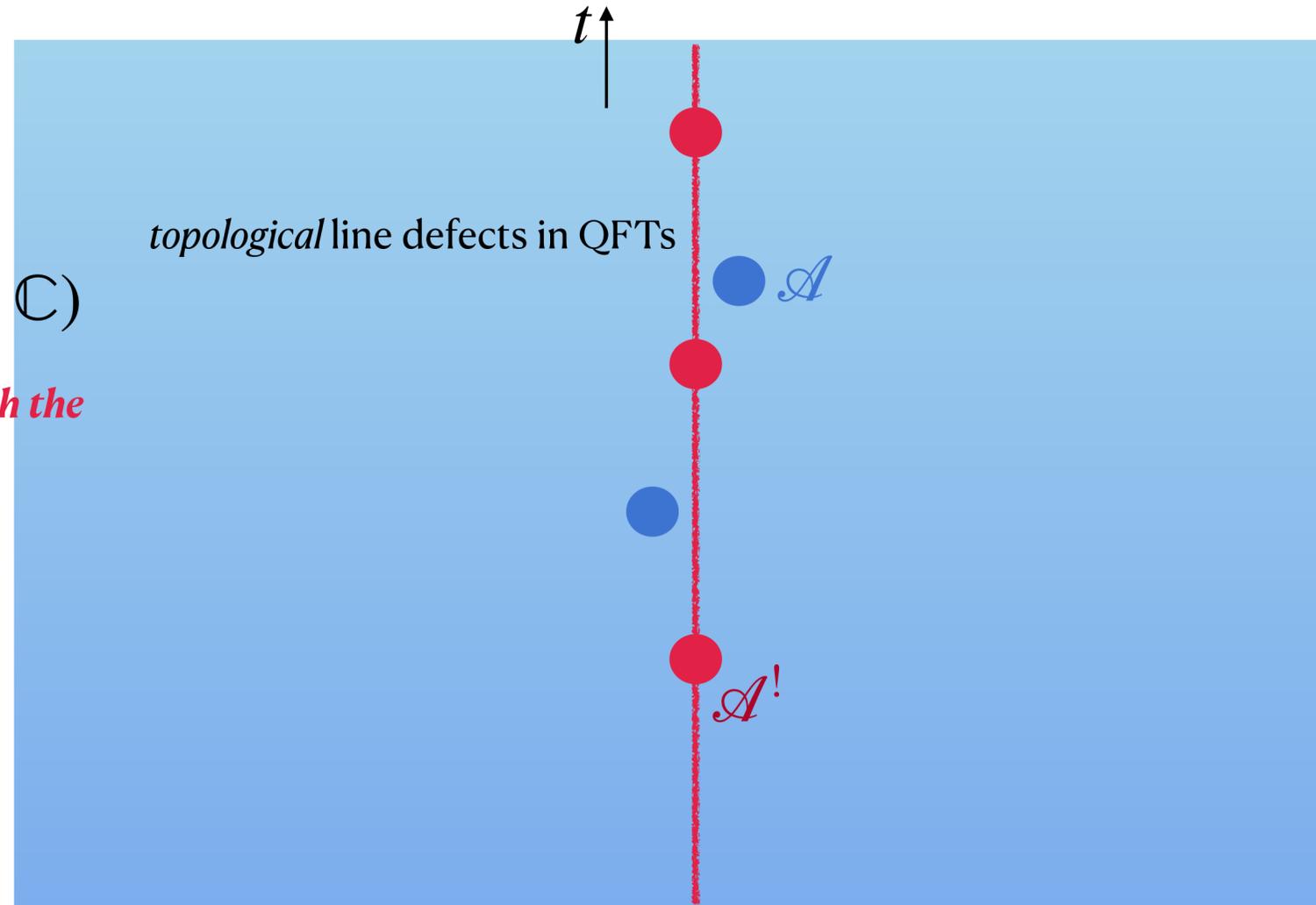
The shape of things to come:

$$\mathcal{A}^! := \text{End}_{\mathcal{A}}(\mathbb{C}) = \text{Ext}_{\mathcal{A}}^*(\mathbb{C}, \mathbb{C})$$

*symmetries of the 1d module commuting with the action of  $\mathcal{A}$*

Complexes!

$$\tilde{Q} = d_{BRST} + Q$$



topological line defects in QFTs

$$\{d_{BRST}, Q\} = d_t$$

Koszul dual

(dg-associative,  $A_\infty$ )

algebras

$\mathcal{A}, \mathcal{A}^!$

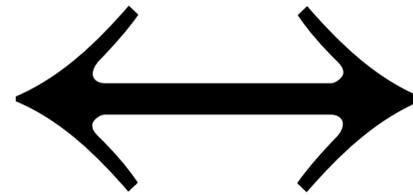
Morita equivalences



$\mathcal{A}'$  looks like it governs some kind of **universal** topological line defect for  $\mathcal{A}$

see also [Gaiotto-Oh, Oh-Zhou, Gaiotto-Abajian]

An anomaly-free coupling of top.  
QM w/ algebra  $\mathcal{B}$  to bulk QFT



A choice of  
 $\text{Hom}(\mathcal{A}', \mathcal{B})$

$\mathcal{A}'$  governs symmetries a defect theory needs to have to couple nonanomalously  
to the original theory along a line

A path integral argument:  $d_{BRST}\mathcal{O}^{(0)} = d_t\mathcal{O}^{(1)} + [\mathcal{O}^{(1)}, \mathcal{O}^{(0)}] \rightarrow d_{BRST}\mathcal{O}^{(1)} + \frac{1}{2}[\mathcal{O}^{(1)}, \mathcal{O}^{(1)}] = 0$   
(master eqn to Maurer-Cartan)

$$\boxed{\text{MC}(\mathcal{A} \otimes \mathcal{B}) \simeq \text{Hom}(\mathcal{A}', \mathcal{B})}$$

Smells like a Fourier transform...

See related work on Koszul duality for

boundary conditions defined by transverse polarizations:

[Bullimore-Dimofte-Gaiotto-Hilburn]

[Ben-Zvi-Dimofte-Neitzke]

[Dimofte's String-Math 2017 talk]

appendix of [Costello-NMP]

# Koszul duality for (dg) vertex algebras

special cases: [Si Li]

topological  $\rightarrow$  holomorphic

associative algebra  $\rightarrow$  vertex algebra

commutator  $\rightarrow$  OPE

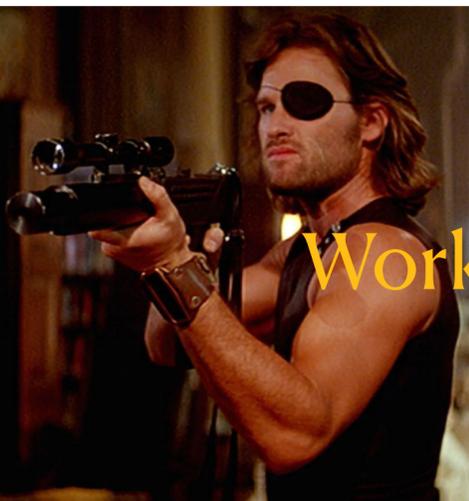
$$\mathbb{R}_t \rightarrow \mathbb{C}_z$$

- **Define:**  $\mathcal{A}'$  is the **universal vertex algebra** that can be coupled to the theory as algebra of operators of a defect wrapping  $\mathbb{C}$ .
- Any coupling of  $\mathcal{B}$  to  $\mathcal{A}$  is the same as giving an algebra homomorphism  $\text{Hom}(\mathcal{A}', \mathcal{B})$

D-branes as defects in flat space:



(backreaction will be back! later)



# Escape from Abstraction: Compute the chiral algebra

Work with the simplest algebra!...e.g. bulk algebra with (a complex of) free ghosts

$$\text{e.g. } \frac{1}{r!s!} \int_{\mathbb{C}^{1|4}} J[r, s](\eta_a) \partial_{w_1}^r \partial_{w_2}^s \mu_0(\eta_a) dz d^4 \eta_a$$

$J[n, 0] \leftrightarrow$  chiral primary

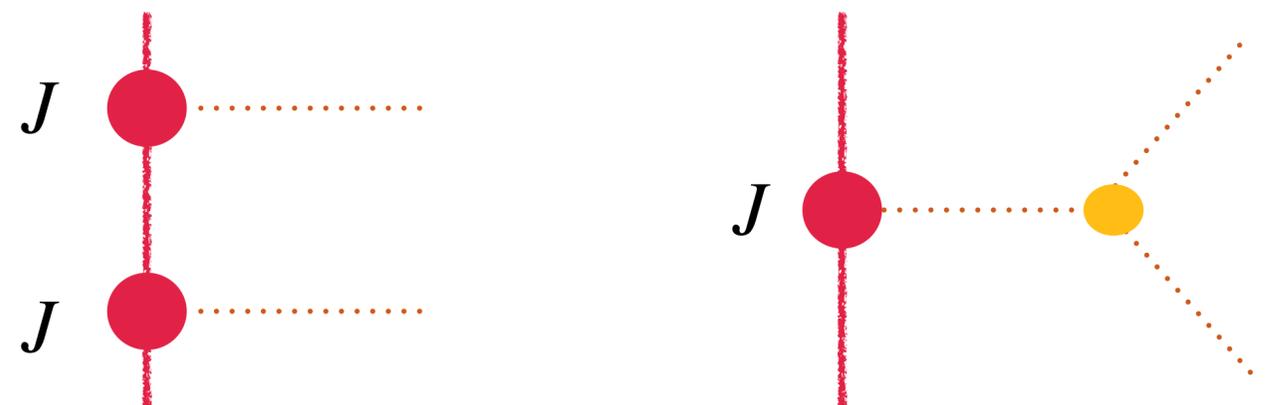
Write down Feynman diagrams, demand their BRST-variation cancels

tree level  $\leftrightarrow (N \rightarrow \infty)$ ; quantum corrections *deform* OPE

$$J[r, s](0)J[k, l](z) \sim \frac{1}{z}(rl - ks)J[r + k - 1, l + s - 1](z)$$

$$\mathcal{L}^{1|4}_{w_\infty} : \mathbb{C}^x \times \mathbb{C}^{0|4} \rightarrow w_\infty$$

hol'c vector fields on  $\mathbb{C}^2_{w_1, w_2}$



see also: [Costello-Witten-Yamazaki]

Koszul & holography: [Costello, Costello-Li]

Related explorations: [Ishtiaque-Moosavian-Zhou, Gaiotto-Oh, Oh-Zhou, Mezei-Pufu-Wang, Gaiotto-Abajian]

## Return of the Backreaction

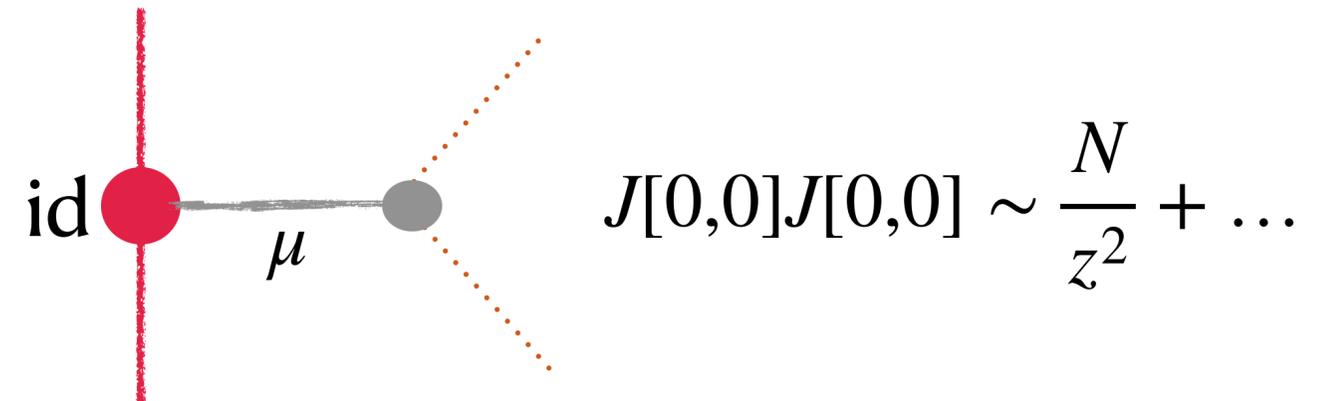
Modify the recipe:

$$\text{Bulk action } S \mapsto S + S_{BR} \quad \delta_{BRST} S_{BR} \neq 0 \quad \text{only on } \mathbb{C}_Z$$

- **w/out backreaction:** defect OPEs from demanding coupling has vanishing anomaly
- **w/ backreaction:** defect OPEs from coupling w/ nonzero anomaly that cancels that of  $S_{BR}$

$F^3 = 0$  : finite # of new diagrams at each order in  $1/N$

3 additional classes of diagrams of  $N \rightarrow \infty$ , e.g:



Proof of all-orders results using homological algebra techniques?

*Curved* Koszul duality for dg-vertex algebras

## To summarize:

- (Curved) Koszul duality for vertex algebras arises by considering anomaly-free bulk/defect couplings
- One can compute the defect algebra from the bulk algebra (or vice versa): diagrammatically, order-by-order. Defect operators *must* furnish a representation of  $\mathcal{A}'$ .
- In our AdS/CFT context, incorporating the backreaction at infinite  $N$  can recover the global symmetry algebra we saw before (incl.  $\text{Vect}_0(X^0/\mathbb{C}^{0|4})$ ); in particular, you can easily see  $\mathfrak{psu}(1, 1 | 2)$  in the low-lying modes!
- Moving beyond the global subalgebra produces  $\mathcal{N} = 4$  SCA in low-lying modes
- Moving beyond  $N \rightarrow \infty$ : computing some (finitely many!) new diagrams [\[Costello-NMP, WIP\]](#)
- **Directly obtain (universal parts of) boundary chiral algebra from the bulk theory**
- **(Dual to some 2-to-n scattering processes in KS theory)**

## Much more to determine!

- Relationship with worldsheet analyses? cf. [Eberhardt-Gaberdiel-Gopakumar]
- Recover chiral primary 3-pt functions from Kodaira-Spencer calculations? cf. [Li-Troost]
- BPS black hole solutions from superconifold quotients?
- How far can one feasibly push  $1/N$  computations? Proof?
- Twisted holography + integrability?
- More, higher-dimensional examples? Extended operators?
- More computable Koszul dual pairs...Any factorization algebra pairs computable?
- Holomorphic-topological twists...  $E_d$ -algebra actions
- Truncations in other examples? ([Costello-NMP, WIP] w/ K3)
- Towards proving AdS/CFT



*A*

*A!*

**Thank you!**

*Jekyll & Hyde*

