4d Chern-Simons Theory as a 3d Toda Theory, and a 3d-2d Correspondence

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Scope of Presentation

- Introduction/Summary
- Topological-Holomorphic "GL"-twist of 5d MSYM
- 3d Toda Theory
- Open Topological Sigma Model on Bogomolny Moduli Space
- 3d-2d Correspondence
- Conclusion and Future Work

Introduction/Summary

- The AGT correspondence relates four-dimensional $\mathcal{N} = 2$ gauge theory and two-dimensional Toda theory on a Riemann surface, $\tilde{\Sigma}$.

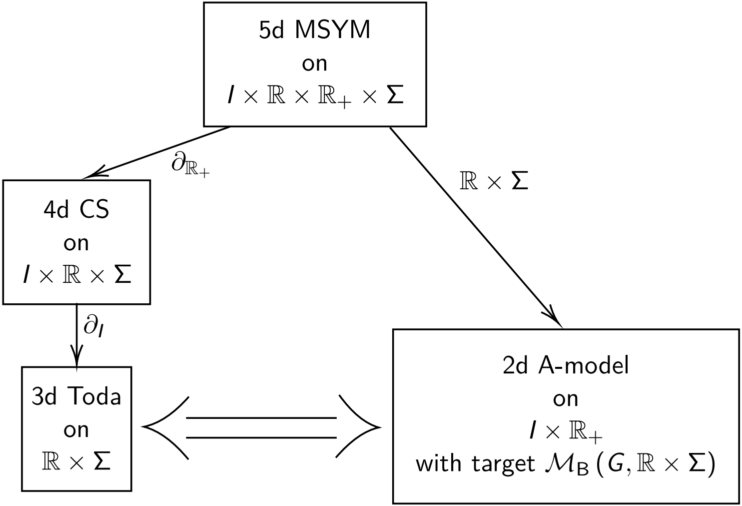
- Among other approaches, it is known that the AGT correspondence can be understood in terms of the relationship between Hitchin's moduli space, $\mathcal{M}(G, \tilde{\Sigma})$, and W-algebra representations.[1]

1. N. Nekrasov and E. Witten. "The omega deformation, branes, integrability and Liouville theory". In: *Journal of High Energy Physics* 2010.9 (2010), p. 92.

- In this work, we present an analogous relationship, i.e., between the moduli space of solutions to the **Bogomolny equations**, and representations of a novel W-algebra arising from a **three-dimensional analogue of Toda theory**.

- This **3d Toda theory** is a boundary dual of **4d Chern-Simons theory**.

- By embedding 4d CS in partially-twisted five-dimensional $\mathcal{N} = 2$ SYM on a manifold with corners, we shall show that **3d Toda theory is dual to an open topological sigma model on the Bogomolny moduli space**.


5d MSYM
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$I \times \mathbb{R} \times \mathbb{R}_+ \times \Sigma$
$\partial_{\mathbb{R}_+}$
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2d A-model
on
$I \times \mathbb{R}_+$
with target $\mathcal{M}_B(G, \mathbb{R} \times \Sigma)$

– The equivalence of the $\mathcal{Q}$-cohomology of physical states, will be shown to imply that **modules of 3d W-algebras** defined on $\mathbb{R} \times \Sigma$ are modules for the **quantized algebra of certain holomorphic functions** on $\mathcal{M}_{\mathrm{B}}(G, \mathbb{R} \times \Sigma)$.

This talk is based on

- M. Ashwinkumar, K.-S. Png, M.-C. Tan, *4d Chern-Simons Theory as a 3d Toda Theory, and a 3d-2d Correspondence*, To appear

Topological-Holomorphic "GL"-twist of 5d Maximally Supersymmetric Yang-Mills Theory

Partial Twist

Five dimensional maximally supersymmetric Yang-Mills involves fields which transform as reps. of $SO_{\mathcal{M}}(5) \times SO_R(5)$:

$$\begin{aligned} A_M &: (\mathbf{5}, \mathbf{1}) \\ \varphi_{\widehat{M}} &: (\mathbf{1}, \mathbf{5}) \\ \rho_{A\widehat{A}} &: (\mathbf{4}, \mathbf{4}) \end{aligned} \tag{2.2}$$

with the classical action:

$$S = -\frac{1}{g_5^2}\int_{\mathcal{M}} d^5x \, \mathrm{Tr}\Big(\frac{1}{4}F_{MN}F^{MN} + \frac{1}{2}D_M\varphi_{\widehat{M}}D^M\varphi^{\widehat{M}} + \frac{1}{4}[\varphi_{\widehat{M}},\varphi_{\widehat{N}}][\varphi^{\widehat{M}},\varphi^{\widehat{N}}] \\ + i\rho^{A\widehat{A}}(\Gamma^M)_A{}^B D_M\rho_{B\widehat{A}} + \rho^{A\widehat{A}}(\Gamma^{\widehat{M}})_{\widehat{A}}{}^{\widehat{B}}[\varphi_{\widehat{M}},\rho_{A\widehat{B}}]\Big).$$

We shall choose $G = SU(N)$ as our gauge group in what follows.

We shall take the 5d worldvolume to be $\mathcal{M} = V \times \Sigma$, whereby we wish to realize a topological twist of the D4-brane worldvolume theory along V.

This amounts to redefining the $SO_V(3)$ rotation group to be the diagonal subgroup

$$SO_V(3)' \subset SO_V(3) \times SO_R(3),$$

where $SO_R(3) \subset SO_R(5)$ rotates $\{\varphi_{\hat{1}}, \varphi_{\hat{2}}, \varphi_{\hat{3}}\}$.

As a result, the scalar fields $\{\varphi_{\hat{1}}, \varphi_{\hat{2}}, \varphi_{\hat{3}}\}$ now transform as the components $\{\varphi_1, \varphi_2, \varphi_3\}$ of a one-form on V.

The twist results in four supercharges that are scalar along V. We wish to pick a supercharge, $\mathcal{Q}$, **that is scalar along** V, w.r.t. which we shall eventually localize the theory.

We shall choose a certain linear combination of two supercharges to be $\mathcal{Q}$.

The twisted supersymmetry transformations generated by $\mathcal{Q}$ are

$$\begin{aligned}
\delta_t A_\alpha &= i\psi_\alpha + it\widetilde{\psi}_\alpha & \delta_t \eta &= t\left(F_{45} + D_\alpha \varphi^\alpha\right) + [\bar{\sigma}, \sigma] \\
\delta_t \varphi_\alpha &= it\psi_\alpha - i\widetilde{\psi}_\alpha & \delta_t \widetilde{\eta} &= -\left(F_{45} + D_\alpha \varphi^\alpha\right) + t[\bar{\sigma}, \sigma] \\
\delta_t A_4 &= i\Upsilon + it\widetilde{\Upsilon} & \delta_t \psi_\alpha &= D_\alpha \sigma + t[\varphi_\alpha, \sigma] \\
\delta_t A_5 &= it\Upsilon - i\widetilde{\Upsilon} & \delta_t \widetilde{\psi}_\alpha &= tD_\alpha \sigma - [\varphi_\alpha, \sigma] \\
\delta_t \sigma &= 0 & \delta_t \Upsilon &= D_4 \sigma + tD_5 \sigma \\
\delta_t \bar{\sigma} &= i\eta + it\widetilde{\eta} & \delta_t \widetilde{\Upsilon} &= tD_4 \sigma - D_5 \sigma
\end{aligned} \tag{2.3}$$

$$\delta_t \chi_\alpha = \frac{1}{2}\left[F_{\alpha 4} + D_5 \varphi_\alpha + \frac{1}{2}\varepsilon_{\alpha\beta\gamma}\left(F^{\beta\gamma} - \left[\varphi^\beta, \varphi^\gamma\right]\right)\right] + \frac{1}{2}t\left[F_{\alpha 5} - D_4 \varphi_\alpha + \varepsilon_{\alpha\beta\gamma} D^\beta \varphi^\gamma\right]$$

$$\delta_t \widetilde{\chi}_\alpha = \frac{1}{2}t\left[F_{\alpha 4} + D_5 \varphi_\alpha - \frac{1}{2}\varepsilon_{\alpha\beta\gamma}\left(F^{\beta\gamma} - \left[\varphi^\beta, \varphi^\gamma\right]\right)\right] - \frac{1}{2}\left[F_{\alpha 5} - D_4 \varphi_\alpha - \varepsilon_{\alpha\beta\gamma} D^\beta \varphi^\gamma\right]$$

i.e., we have $\mathcal{Q} = \mathcal{Q}_L + t\mathcal{Q}_R$, $t \in \mathbb{CP}^1$.[2]

2. M. Ashwinkumar and M.-C. Tan. "Unifying lattice models, links and quantum geometric Langlands via branes in string theory". In: *arXiv preprint arXiv:1910.01134* (2019).

Such a twist has also been discussed conceptually by Elliot and Pestun.[3]

The transformations take a form very similar to those of GL-twisted $\mathcal{N} = 4$ SYM, as considered by Kapustin and Witten.[4]

In fact, taking $\Sigma = \mathbb{R} \times S^1$ or T^2, whereby the x^5 direction is S^1, we can dimensionally reduce along the latter to obtain precisely the transformations of Kapustin and Witten via $A_5 \to \varphi_4$, $\chi_\alpha \to \chi^+_{\alpha 4}$, $\widetilde{\chi}_\alpha \to \chi^-_{\alpha 4}$, $\Upsilon \to \psi_4$, $\widetilde{\Upsilon} \to \widetilde{\psi}_4$.

3. C. Elliott and V. Pestun. "Multiplicative Hitchin systems and supersymmetric gauge theory". In: *Selecta Mathematica* 25.4 (2019), p. 64.
4. A. Kapustin and E. Witten. "Electric-magnetic duality and the geometric Langlands program". In: *Communications in Number Theory and Physics* 1.1 (2007), pp. 1–236.

In what follows, we shall be interested in $V = I \times \mathbb{R} \times \mathbb{R}_+$ and $\Sigma = \mathbb{CP}^1$, $\mathbb{C}^\times$, or $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. This is a manifold with two corners.

The action of 5d "GL"-twisted theory can then be written as

$$S = \{\mathcal{Q}, \widetilde{V}\} + \frac{\widetilde{\Psi}}{4\pi} \int_{I \times \mathbb{R} \times \mathbb{R}_+ \times \Sigma} dz \wedge \mathrm{Tr}\left(\mathcal{F} \wedge \mathcal{F}\right). \qquad (2.4)$$

Here, $\widetilde{\Psi}$ is a real parameter (as we have selected $t = -1$ for our purpose).

$\mathcal{F}$ is the curvature of the complex connection involving $\mathcal{A}_\alpha = A_\alpha + i\varphi_\alpha$ (where $\alpha = 1, 2, 3$), and $\mathcal{A}_{\overline{z}} = \frac{1}{2}(A_4 + iA_5)$, where z (i.e., $x^4 + ix^5$) and $\overline{z}$ are local complex coordinates on Σ.

Boundary conditions

At the origin of $\mathbb{R}_+$ ($x^3 = 0$), we pick NS5-type boundary conditions to maintain supersymmetry, where we require that

$$\mathcal{Q}(\mathcal{A}_{\widetilde{\alpha}}) = 0, \tag{2.5}$$

(where $\widetilde{\alpha} = 1, 2$) and

$$\mathcal{Q}(\mathcal{A}_{\overline{z}}) = 0, \tag{2.6}$$

where $\mathcal{A}_{\widetilde{\alpha}}$ and $\mathcal{A}_{\overline{z}}$ obey Neumann boundary conditions.

At infinity along $\mathbb{R}_+$, the boundary conditions are taken to be x^3-independent $\mathcal{Q}$-invariant configurations that involve $\mathcal{A}_{\widetilde{\alpha}}$ and $\mathcal{A}_{\overline{z}}$.

At the boundaries of I, we can have Nahm pole-type boundary conditions, defined in terms of the complex gauge fields, where the latter are $\mathcal{Q}$-invariant.

With $x^2 = \tau$ parametrizing $\mathbb{R}$, we first define the convenient coordinates

$$x^{\pm} = \tau \pm i\overline{z}, \tag{2.7}$$

and

$$\mathcal{A}_{\pm} = \frac{1}{2}\left(\mathcal{A}_{\tau} \mp i\mathcal{A}_{\overline{z}}\right). \tag{2.8}$$

With $x^1 = \sigma$ parametrizing $I = [0, \pi]$, the Nahm pole-type boundary conditions are defined as follows. As $\sigma \to 0$,

$$\mathcal{A} \to \frac{id\sigma}{\sigma} H + \frac{dx^+}{\sigma} T_+. \tag{2.9}$$

Here, a homomorphism $\rho : \mathfrak{su}(2) \to \mathfrak{su}(N)$ has been chosen such that the image, T_+, of the raising operator of $\mathfrak{su}(2)$ is a maximal length Jordan block, i.e.,

$$T_+ = i \begin{pmatrix} 0 & \mu_1 & 0 & \cdots & 0 \\ 0 & 0 & \mu_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_{N-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{2.10}$$

Analogously, as $\sigma \to \pi$,

$$\mathcal{A} \to \frac{id\sigma}{\sigma - \pi} H + \frac{dx^{-}}{\sigma - \pi} T_{-}, \tag{2.11}$$

where

$$T_{-} = -i \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \nu_1 & 0 & \cdots & 0 & 0 \\ 0 & \nu_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \nu_{N-1} & 0 \end{pmatrix}. \tag{2.12}$$

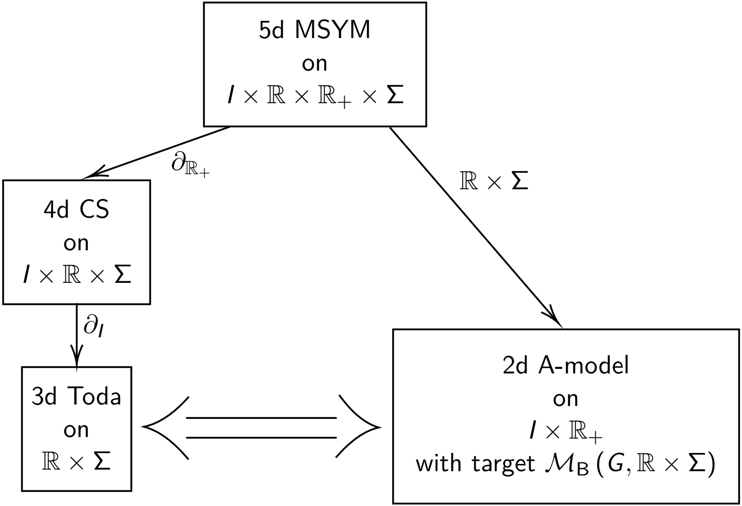
Localization to 4d Chern-Simons theory

The path integral of this theory localizes to solutions of $\mathcal{Q}$-invariant configurations, whereby it reduces to the **path integral of 4d Chern-Simons theory**, i.e.,

$$\int_{\widetilde{\Gamma}} D\mathcal{A} \exp\left(\frac{i}{2\pi\hbar}\int_{I\times\mathbb{R}\times\Sigma} dz \wedge \mathrm{Tr}\left(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right)\right), \tag{2.13}$$

where $\widetilde{\Gamma}$ is the integration cycle defined by the $\mathcal{Q}$-invariant localization equations, and $\frac{i}{\hbar} = \frac{\widetilde{\Psi}}{2}$.

This integration cycle is a **Lefschetz thimble** that ensures the convergence of the path integral.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Three-dimensional Toda Theory

3d Toda Theory from 4d Chern-Simons

A 3d analogue of WZW theory dual to 4d CS theory can be obtained when the the topological directions have a boundary.[5]

Something similar can be done in our current set up, where there are now two boundaries on the ends of I.

Let us focus on the boundary at $\sigma \to 0$, since similar arguments can be applied to the other end of the interval (i.e. $\sigma \to \pi$).

5. M. Ashwinkumar. "Integrable lattice models and holography". In: *arXiv preprint arXiv:2003.08931* (2020).

Via the boundary condition $\mathcal{A}_- = 0$, the action for $\sigma \to 0$ is that of a 3d analogue of the 2d chiral WZW model

$$\begin{aligned}\lim_{\sigma\to 0} S_{\text{4d CS}}[\mathcal{A}] = S_{\text{3d WZW}}[g] \\ = \frac{1}{2\pi\hbar}\int_{\mathbb{R}\times\Sigma} dz \wedge dx^+ \wedge dx^- \,\mathrm{Tr}\,\left(\partial_+ g\, g^{-1}\partial_- g\, g^{-1}\right) \\ - \frac{1}{6\pi\hbar}\int_{I\times\mathbb{R}\times\Sigma} dz \wedge \mathrm{Tr}\,\left(dg\, g^{-1}\wedge dg\, g^{-1}\wedge dg\, g^{-1}\right)\end{aligned} \tag{3.2}$$

The ensuing conserved currents then take the form

$$J_+ = g^{-1}\,\partial_+ g \tag{3.3a}$$

$$J_- = \partial_- g\, g^{-1}. \tag{3.3b}$$

Similarly, at the boundary where $\sigma \to \pi$, one finds the currents

$$J'_+ = -g^{-1}\,\partial_+ g \tag{3.4a}$$

$$J'_- = -\partial_- g\, g^{-1}, \tag{3.4b}$$

We project the currents onto Cartan directions, as well as positive and negative roots of the (complex) Lie algebra

$$J = J^{-i}R_{-i} + J^{+i}R_{+i} + J^{0i}R_{0i}, \tag{3.5}$$

where $i = 1, \cdots, N-1$. Then, the rest of the Nahm pole-type boundary conditions result in current constraints

$$J_{+}^{+i} = \mu_i, \qquad J_{+}^{0i} = 0 \tag{3.6a}$$
$$J_{-}^{\prime -i} = -\nu_i \qquad J_{-}^{\prime 0i} = 0 \tag{3.6b}$$

Next, we use a Gauss decomposition

$$g = e^{X_i R_i^+} e^{\phi_i R_i^0} e^{Y_i R_i^-}, \tag{3.7}$$

which allows the current constraints to be rewritten as

$$\partial_+ \partial_- \phi_i + 2\mu_i \nu_i e^{C_{ij}\phi^j} = 0 \tag{3.8}$$

The corresponding action that gives these constraints as an EOM takes the form of a 3d analogue of 2d analytically-continued Toda theory

$$\boxed{S_{3\text{d Toda}}[\phi] = \frac{1}{2\pi\hbar} \int_{\mathbb{R}\times\Sigma} dz\, dx^+ dx^- \left(C_{ij} \partial_+ \phi^i \partial_- \phi^j - 4 \sum_i \mu_i \nu_i e^{C_{ij}\phi^j} \right)} \tag{3.9}$$

3d W-algebras

Writing $\sum_j C^{-1}_{ij} = \gamma_i$, the 3d Toda theory is invariant under the **infinitesimal coordinate transformation** $x^+ \to x^+ + \varepsilon(x^+)$ together with the transformation of ϕ_i as

$$\phi_i(z, x^+, x^-) \to \phi_i(z, x^+, x^-) - \varepsilon \partial_+ \phi_i - \gamma_i \partial_+ \varepsilon. \tag{3.10}$$

The corresponding Noether current takes the form

$$\Theta = \frac{1}{\hbar} C_{ij} \partial_+ \phi^i \partial_+ \phi^j - \frac{2}{\hbar} C_{ij} \gamma^i \partial_+ \partial_+ \phi^j, \tag{3.11}$$

where $\partial_- \Theta = 0$.

We focus on the simplest case where $G = SL(2,\mathbb{C})$, and $[C_{ij}] = 2$. Take x^- to be the temporal direction, and write $x^+ = \xi$.

We use an equivalent gauged 3d WZW model, where Θ takes the form

$$\Theta(z,\xi) = T(z,\xi) - \partial_\xi J_3(z,\xi), \tag{3.13}$$

where $T(z,\xi) = -J_a(z,\xi)J^a(z,\xi)$.

Using the symplectic form on phase space, we compute the PB to obtain a 3d analogue of the classical Virasoro algebra

$$\Big[\Theta(z',\xi'),\Theta(z,\xi)\Big]_{\text{PB}} = \left(\partial_\xi\Theta(z,\xi)\delta(\xi-\xi') + 2\Theta(z,\xi)\delta'(\xi-\xi') - \frac{1}{2}\delta'''(\xi-\xi')\right)\delta(z-z'). \tag{3.14}$$

Via Moyal quantization[6], we then expect (3.14) to be corrected to

$$\boxed{\begin{aligned}[\Theta(z',\xi'),\Theta(z,\xi)] = \Big(&\partial_\xi\Theta(z,\xi)\delta(\xi-\xi') + 2\Theta(z,\xi)\delta'(\xi-\xi') \\ &+ \left(k-\frac{1}{2}\right)\delta'''(\xi-\xi') + k\delta'(\xi-\xi')\Big)\,\delta(z-z')\end{aligned}} \tag{3.15}$$

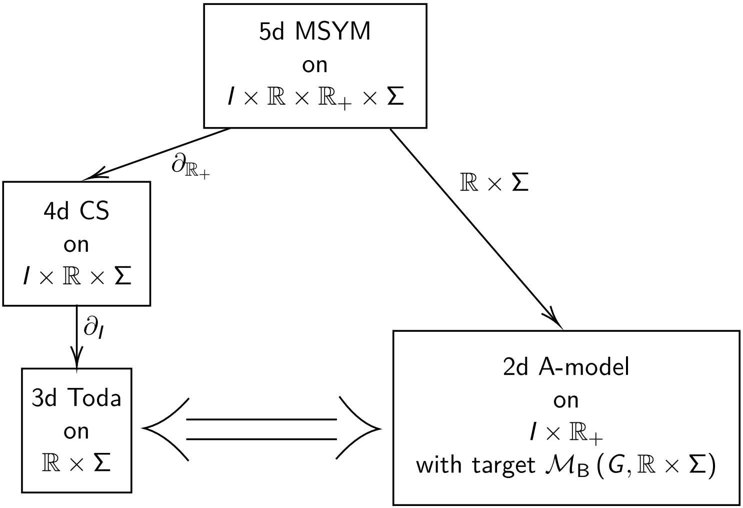
where $k \in \mathbb{R}$ describes the quantum deformation of the Poisson bracket expression. This is a **3d analogue of the quantum Virasoro algebra**.

6. G. Jorjadze and G. Weigt. "Poisson structure and Moyal quantisation of the Liouville theory". In: *Nuclear Physics B* 619.1-3 (2001), pp. 232–256.

Analogous results for higher spin currents in higher-rank 3d Toda theory can be obtained via a generalized Sugawara construction of W-algebra currents.

We will then obtain 3d W-algebras similar to their 2d counterparts, but with generators having **holomorphic dependence** on Σ, and $\delta(z - z')$ appearing as an overall factor on the RHS.

Choosing x^- instead of x^+ to be the temporal direction leads to **another copy** of the W-algebra, which is x^--dependent.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Open Topological Sigma Model on Bogomolny Moduli Space

2d A-model from 5d "GL"-twisted Theory

Using the topological-holomorphic symmetry of the 5d "GL"-twisted theory on $I \times \mathbb{R} \times \mathbb{R}_+ \times \Sigma$, we can scale down the three-manifold $\mathbb{R} \times \Sigma$ while simultaneously blowing up $I \times \mathbb{R}_+$.

To ensure the action remains finite, one requires

$$\boxed{\begin{aligned} F_{\tau\bar{z}} - iD_{\bar{z}}\varphi_\tau &= 0 \\ D_\tau\varphi^\tau - 2iF_{z\bar{z}} &= 0, \end{aligned}} \tag{4.1}$$

which are the **Bogomolny equations** on $\mathbb{R} \times \Sigma$.

This results in a **sigma model** on $I \times \mathbb{R}_+$, with target the moduli space, $\mathcal{M}_{\mathrm{B}}\left(G, \mathbb{R} \times \Sigma\right)$, of solutions to these equations, with metric

$$\widetilde{G} = \frac{1}{g_5^2} \int_{\mathbb{R}\times\Sigma} d^3x \, \mathrm{Tr}\left(\delta A^p \otimes \delta A_p + \delta\varphi^\tau \otimes \delta\varphi_\tau\right), \qquad (4.2)$$

where A_p is the gauge field defined along $\mathbb{R} \times \Sigma$.

To be precise, we obtain an **A-model** in symplectic structure

$$\omega_K^{\mathrm{B}} \propto \int_{\mathbb{R}\times\Sigma} d^3x \; \mathrm{Tr}(\delta\varphi_\tau \wedge \delta A_4 - \delta A_5 \wedge \delta A_\tau) \qquad (4.3)$$

(the pullback of ω_K^{B} to $I \times \mathbb{R}_+$ arises from non-$\mathcal{Q}$-exact terms in the 5d action).

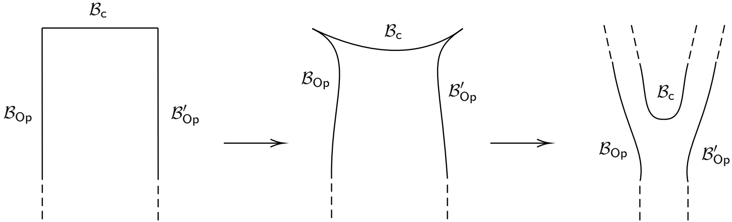
Branes

The NS5-type boundary condition (b.c.) (of the 5d gauge theory) gives rise to a space-filling **coisotropic** A-brane, $\mathcal{B}_{\mathrm{c}}$, in $\mathcal{M}_{\mathrm{B}}(G, \mathbb{R} \times \Sigma)$. Defining complex/symplectic structures analogous to those on Hitchin moduli space, this is a (B, A, A) brane.

The Nahm pole-type b.c. of the 5d gauge theory give rise to **Lagrangian** branes $\mathcal{B}_{\mathrm{Op}}$ and $\mathcal{B}'_{\mathrm{Op}}$ in $\mathcal{M}_{\mathrm{B}}(G, \mathbb{R} \times \Sigma)$. These are analogues of branes of opers in Hitchin moduli space, and are both (A, B, A) branes.

Physical States

To compute physical states of the A-model we perform the following topological deformation

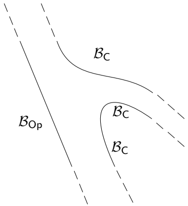


If we were to compute the physical states of $(\mathcal{B}_{\mathrm{Op}}, \mathcal{B}'_{\mathrm{Op}})$ strings, we will only find one state, as $\mathcal{B}_{\mathrm{Op}}$ and $\mathcal{B}'_{\mathrm{Op}}$ only intersect at one point in $\mathcal{M}_{\mathrm{B}}(G, \mathbb{R} \times \Sigma)$.

To find more states, we have to consider the $(\mathcal{B}_{\mathrm{c}}, \mathcal{B}_{\mathrm{Op}})$ and $(\mathcal{B}_{\mathrm{c}}, \mathcal{B}'_{\mathrm{Op}})$ strings that arise from the corners on $I \times \mathbb{R}_+$. Their physical states can be computed to be the following spaces of J-holomorphic sections:

$$\begin{aligned}\mathcal{H}_{(\mathcal{B}_{\mathrm{c}}, \mathcal{B}_{\mathrm{Op}})} &= H^0(\mathcal{B}_{\mathrm{Op}}, K^{1/2}_{\mathcal{B}_{\mathrm{Op}}}) \\ \mathcal{H}_{(\mathcal{B}_{\mathrm{c}}, \mathcal{B}'_{\mathrm{Op}})} &= H^0(\mathcal{B}'_{\mathrm{Op}}, K^{1/2}_{\mathcal{B}'_{\mathrm{Op}}}).\end{aligned} \tag{4.4}$$

Moreover, the algebra of $(\mathcal{B}_{\mathsf{c}}, \mathcal{B}_{\mathsf{c}})$ strings acts on the space of states of $(\mathcal{B}_{\mathsf{c}}, \mathcal{B}_{\mathsf{Op}})$ and $(\mathcal{B}_{\mathsf{c}}, \mathcal{B}'_{\mathsf{Op}})$ strings, by attaching to the appropriate ends of the latter.



This is just the **quantized algebra** of J-**holomorphic functions** on $\mathcal{M}_{\mathsf{B}}(G, \mathbb{R} \times \Sigma)$.

Therefore the space of states of the $(\mathcal{B}_{\mathsf{c}}, \mathcal{B}_{\mathsf{Op}})$ and $(\mathcal{B}_{\mathsf{c}}, \mathcal{B}'_{\mathsf{Op}})$ strings are modules for the quantized algebra of J-holomorphic functions on $\mathcal{M}_{\mathsf{B}}(G, \mathbb{R} \times \Sigma)$.

3d-2d Correspondence

The 5d "GL"-twisted gauge theory on $I \times \mathbb{R} \times \mathbb{R}_+ \times \Sigma$ has given rise to two effective descriptions.

The $\mathcal{Q}$-cohomology of states of the 5d theory ought to remain invariant in reducing to these effective descriptions. This implies a **duality** between the two theories:

3d Toda theory on $\mathbb{R} \times \Sigma$	$\Leftrightarrow$	2d A-model on $I \times \mathbb{R}_+$ with target $\mathcal{M}_{\mathrm{B}}(G, \mathbb{R} \times \Sigma)$

Mathematically, this implies that

Modules of 3d W-algebras defined on $\mathbb{R} \times \Sigma$	$\Leftrightarrow$	$H^0(\mathcal{B}_{\mathrm{Op}}, K^{1/2}_{\mathcal{B}_{\mathrm{Op}}}) \otimes H^0(\mathcal{B}'_{\mathrm{Op}}, K^{1/2}_{\mathcal{B}'_{\mathrm{Op}}})$

Moreover, we find that

Modules of 3d W-algebras defined on $\mathbb{R} \times \Sigma$ are modules for the quantized algebra of J-holomorphic functions on $\mathcal{M}_{\mathrm{B}}(G, \mathbb{R} \times \Sigma)$.

Conclusion and Future Directions

- We have shown that 3d Toda theory and a 2d A-model on Bogomolny moduli space are dual to each other.
- We have found that modules of 3d W-algebras are modules for quantized algebras of holomorphic functions on the Bogomolny moduli space.
- The crucial ingredient is the fact that the 5d $\mathcal{N} = 2$ SYM theory admits a partial twist that is topological-holomorphic, and analogous to the GL-twist of 4d $\mathcal{N} = 4$ SYM.

- Future work involves defining a GL-type partial twist (in two directions) for maximally supersymmetric Yang-Mills theory in 6d, whereby one expects to relate **4d W-algebras** with the quantized algebra of holomorphic functions on the **moduli space of instantons**.

Thank you for your attention!