Volumes of Supermoduli Spaces

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I am going to explain how to generalize Maryam Mirzakhani's work on the volume of the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g to the corresponding moduli space $\mathfrak{M}_g$ of super Riemann surfaces of genus g, following D. Stanford and EW, "JT Gravity and the Ensembles of Random Matrix Theory," arXiv:1907.03363. I will not try to explain the context for this calculation, which involved the topics mentioned in the title of the paper. (This was explained in a "Western Hemisphere Seminar" that you can find at `http://web.math.ucsb.edu/~drm/WHCGP/`; for a written version see EW, "Volumes and Random Matrices," arXiv:2004.05183.)

Related work:

Y. Huang, R. C. Penner, and A. M. Zeitlin “Super McShane Identity” arXiv:1907.09978

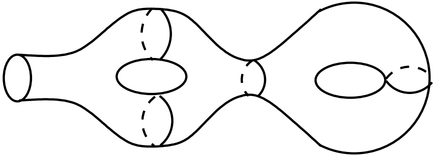
on the supermoduli space identities, and

P. Norbury, “A New Cohomology Class on the Moduli Space of Curves” arXiv:1712.03662, “Enumerative Geometry Via The Moduli Space of Super Riemann Surfaces” arXiv:2005.04378

on the relevant spectral curve, and a proof that the generating function of supervolumes satisfies a KdV equation.

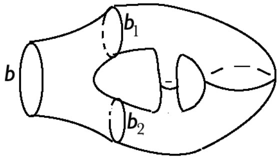
I am going to begin by reviewing the work of Mirzakhani. First of all, she studied Riemann surfaces as *hyperbolic* two-manifolds, in other words every Riemann surface Y in this discussion comes with a metric of constant curvature $R = -2$. Equivalently, Y is the quotient of the complex upper half-space $\mathcal{H}$ by a discrete group $\Gamma \subset \mathrm{PSL}(2, \mathbb{Z})$. In Mirzakhani's work, it is important to allow Y to have boundaries, which are always taken to be geodesics. The moduli space $\mathcal{M}_{g,\vec{b}}$ of Riemann surfaces of genus g with geodesic boundaries of lengths $\vec{b} = (b_1, b_2, \cdots, b_n)$ has a Weil-Petersson volume $V_{g,\vec{b}}$ defined in the same way as for $\mathcal{M}_g$, and she computes this volume.

A key tool is that a hyperbolic Riemann surface Y (possibly with geodesic boundaries) can be built by gluing together three-holed spheres (with geodesic boundaries).



The subtlety comes from the fact that there are many ways to do this.

If the decomposition in three-holed spheres with geodesic boundary were unique, we would get a simple recursion relation for volumes. For instance this picture



would lead to

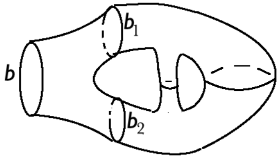
$$V_{2,b} \stackrel{?}{=} \int_0^\infty b_1 \mathrm{d}b_1 \, b_2 \mathrm{d}b_2 \, V_{1,b_1,b_2}.$$

This is wrong because there is no modular-invariant way to pick a particular three-holed sphere $\Lambda \subset Y$ with the given boundary. There are infinitely many choices and the right hand side of the formula involves an infinite over-counting.

A similar formula without integrating over b_1, b_2,

$$\mathcal{V}_{2,b} \stackrel{?}{=} b_1 \mathrm{d}b_1 \, b_2 \mathrm{d}b_2 \, \mathcal{V}_{1,b_1,b_2},$$

is actually correct if we interpret $\mathcal{V}_{2,b}$ and $\mathcal{V}_{1,b_1,b_2}$ as Weil-Petersson *volume forms* (rather than integrated volumes) This formula cannot be integrated to give the volumes because it does not properly take the mapping class group into account:



To belabor this point: $\mathcal{M}_g$ or $\mathcal{M}_{g,b}$ is a moduli space of flat $\mathrm{PSL}(2,\mathbb{R})$ connections *divided by the mapping class group*. If we replace $\mathrm{PSL}(2,\mathbb{R})$ by a compact structure group such as $\mathrm{SU}(2)$, then one defines the moduli space of flat connections without dividing by the mapping class group, and their volumes do obey a simple identity analogous to the naive

$$V_{2,b} \stackrel{?}{=} \int_0^\infty b_1 \mathrm{d}b_1 \, b_2 \mathrm{d}b_2 \, V_{1,b_1,b_2}.$$

For $\mathrm{PSL}(2,\mathbb{R})$, the need to divide by the mapping class group means that there is no such simple identity. This is the difficulty that Mirzakhani overcame.

To explain the basic idea: Consider any choice of a three-holed sphere $\Lambda \subset Y$ with γ – of length b – as one of its boundaries, and let b_1 and b_2 be the other boundary lengths. Suppose there were a function $f(b, b_1, b_2)$ such

$$1 = \sum_{\Lambda} f(b, b_1, b_2).$$

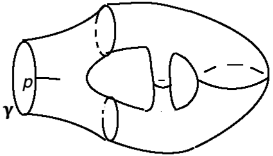
Then the naive identity could be corrected by simply inserting a factor of f on the right hand side:

$$V_{2,b} = \int_0^\infty b_1 \mathrm{d}b_1 \, b_2 \mathrm{d}b_2 \, f(b, b_1, b_2) V_{1,b_1,b_2}.$$

We would be counting all the possible Λ's, but weighting them with the factor $f(b, b_1, b_2)$, which adds to 1. This would compensate for the fact that the smaller surface has a smaller mapping class group than the larger one.

L. McShane had found an identity of roughly the necessary form in a particular case (a hyperbolic once-punctured torus) and Mirzakhani generalized it to the context of hyperbolic surfaces with geodesic boundaries.

To explain how: Pick one boundary γ, and a point p in it, and let ℓ_p be the geodesic orthogonal to γ at p:



If ℓ_p is continued it might

0) go on forever without intersecting itself or returning to γ;

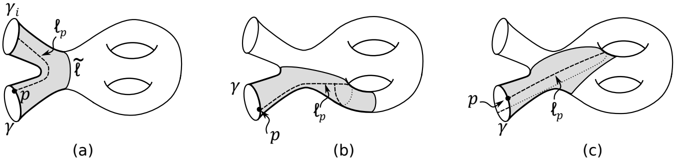
1) return to γ

2) leave Y by a different boundary (not possible in the case drawn)

3) intersect itself.

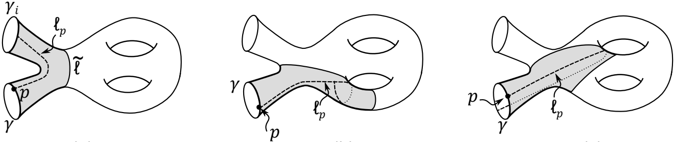
If it does not do 0), then it does one of 1), 2), or 3) first.

A theorem of Birman and Series implies that the probability of outcome 0) is 0, so with probability 1, the outcome will be 1), 2), or 3). This figure illustrates the three possibilities:



In each case, a three-holed sphere with geodesic boundaries containing γ *and* ℓ_p *is naturally determined.* (The union of γ, ℓ_p, and γ_i - if present - can be thickened slightly to give what topologically is a three-holed sphere. By minimizing the lengths of the boundaries in their homotopy classes, one gets a three-holed sphere with geodesic boundaries.)

Let Υ be the set of all three-holed spheres $\Lambda \subset Y$ with boundary γ and two internal geodesics, and let Υ_i be the set of Λ's whose boundary is γ, an internal geodesic, and another boundary γ_i of Y. For $\Lambda \subset \Upsilon$, let $\mathcal{A}_\Lambda$ be the subset of γ where the picture looks like b) or c) and for $\Lambda \subset \Upsilon_i$, let $\mathcal{B}_\Lambda$ be the subset where the picture looks like a).

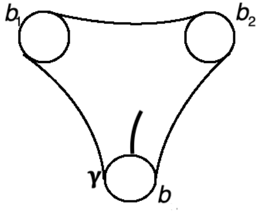


(a) (b) (c)

Let $\mu(\mathcal{A}_\Lambda)$ and $\mu(\mathcal{B}_\Lambda)$ be the measures of $\mathcal{A}_\Lambda$ and $\mathcal{B}_\Lambda$. Since γ has total length b, we get a sum rule:k

$$b = \sum_{\Lambda \subset \Upsilon} \mu(\mathcal{A}_\Lambda) + \sum_{\Lambda \subset \Upsilon_i} \mu(\mathcal{B}_\Lambda),$$

For a given Λ, the quantities $\mu(\mathcal{A}_\Lambda)$ and $\mu(\mathcal{B}_\Lambda)$ can be computed by just studying geodesics in Λ, so they only depend on the boundary lengths b, b_1, and b_2 of Λ:



Hence these are explicitly computable functions of b, b_1, b_2.

So the sum rule

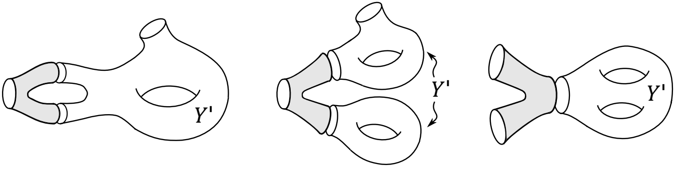
$$b = \sum_{\Lambda \subset \Upsilon} \mu(\mathcal{A}_\Lambda) + \sum_{\Lambda \subset \Upsilon} \mu(\mathcal{B}_\Lambda)$$

takes an explicit form with $\mu(\mathcal{A}_\Lambda)$ and $\mu(\mathcal{B}_\Lambda)$ being explicit functions of the boundary lengths of Λ. Using this sum rule, Mirzakhani gets a corrected version of the naive identity. In detail

$$\begin{aligned} bV_{g,b,B} = \frac{1}{2} \int_0^\infty b' \mathrm{d}b' \, b'' \mathrm{d}b'' \, D(b, b', b'') \Bigg(& V_{g-1,b',b'',B} \\ & + \sum_{\text{stable}} V_{h_1,b',B_1} V_{h_2,b'',B_2} \Bigg) \\ + \sum_{k=1}^{|B|} \int_0^\infty b' \mathrm{d}b' \left(b - T(b, b', b_k) \right) V_{g,b',B \setminus b_k}. & \qquad (1) \end{aligned}$$

with explicit functions D, T.

The three terms on the right hand side of Mirzakhani's recursion relation correspond to the three topologically distinct ways to make a surface Y by gluing a three-holed sphere Λ onto a simpler surface Y':



Mirzakhani's recursion relation is quite useful. She used it to prove many old and new facts about volumes and intersection numbers on moduli space. Her recursion relation was interpreted by Eynard and Orantin in terms of "topological recursion" for a particular spectral curve, and this was the starting point of work of Saad, Shenker, and Stanford on "JT Gravity as a Matrix Model." That paper in turn was the starting point for my work with Stanford (however, as I remarked at the beginning, today I will only describe the "geometrical" part of that work, not the part related to random matrices and quantum gravity).

Now let us go over to super Riemann surfaces.

From a holomorphic point of view, a super Riemann surface $\widehat{Y}$ is a complex supermanifold of dimension $1|1$, meaning that locally it can be parametrized by an even variable z and an odd variable θ, with some additional structure: a distribution $\mathcal{D}$ of rank $0|1$ that is "completely unintegrable." One can always pick local coordinates $z|\theta$ such that $\mathcal{D}$ is generated by

$$D_\theta = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}.$$

"Complete unintegrability" means that $D_\theta^2 = \partial/\partial z$ is everywhere linearly independent of D_θ. This definition, which leads to a theory in surprisingly close parallel to the classical theory of Riemann surfaces, takes some getting used to.

However, today we are studying super Riemann surfaces from a hyperbolic point of view. For this, we just replace $\mathrm{PSL}(2,\mathbb{R})$ by $\mathrm{OSp}(1|2)$. Recalling that $\mathrm{SL}(2,\mathbb{R})$ is the automorphism group of a symplectic form $\mathrm{d}u \wedge \mathrm{d}v$ on $\mathbb{R}^2$, in a similar way $\mathrm{OSp}(1|2)$ is the automorphism group of a symplectic form $\mathrm{d}u \wedge \mathrm{d}v - \mathrm{d}\vartheta^2$ on $\mathbb{R}^{2|1}$. The usual upper half plane is $\mathcal{H} = \mathrm{PSL}(2,\mathbb{R})/\mathrm{U}(1) = \mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ (where one $\mathrm{U}(1)$ is a double cover of the other) and the superanalog of the upper half plane is $\widehat{\mathcal{H}} = \mathrm{OSp}(1|2)/\mathrm{U}(1)$ (with $\mathrm{U}(1) \subset \mathrm{SL}(2,\mathbb{R}) \subset \mathrm{OSp}(1|2)$).

A hyperbolic Riemann surface is $Y = \mathcal{H}/\Gamma$, for a discrete group $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ and similarly a hyperbolic super Riemann surface is $\widehat{Y} = \widehat{\mathcal{H}}/\Gamma$, for a discrete group $\Gamma \subset \mathrm{OSp}(1|2)$. (See for example Baranov, Frolov and Schwarz (1987), Crane and Rabin (1988), Penner and Zeitlin (2015).) In classical Riemann surface theory, lifting Γ from $\mathrm{PSL}(2, \mathbb{R})$ to $\mathrm{SL}(2, \mathbb{R})$ means endowing Y with a spin structure. In the super case, there is no way to avoid the lift, since the maximal bosonic subgroup of $\mathrm{OSp}(1|2)$ is $\mathrm{SL}(2, \mathbb{R})$, not $\mathrm{PSL}(2, \mathbb{R})$, so a super Riemann surface always comes with a spin structure.

A Weil-Petersson symplectic form on the moduli space $\mathfrak{M}_g$ of super Riemann surfaces (and various generalizations for super Riemann surfaces with punctures and/or boundaries) can be defined by imitating any of the classical definitions. A (compact) symplectic supermanifold has a volume, so we can define the volume $\widehat{V}_g$ of $\mathfrak{M}_g$ and the goal is to find a Mirzakhani-style method to calculate it. At the end, I will explain a formula for $\widehat{V}_g$ in terms of ordinary bosonic geometry.

Let me describe the super upper half-plane $\widehat{\mathcal{H}}$ a little more concretely. We recall that $\mathrm{OSp}(1|2)$ was defined to act on $\mathbb{R}^{2|1}$, with coordinates $u, v|\theta$, preserving a certain symplectic form $\widehat{\omega} = \mathrm{d}u\mathrm{d}v - \mathrm{d}\vartheta^2$. Therefore it acts on the projective space $\mathbb{RP}^{1|1}$, which has $u, v|\theta$ as homogenous coordinates. $\mathbb{RP}^{1|1}$ is a super-circle. It is convenient to introduce affine coordinates $z = v/u$, $\theta = \vartheta/u$. Then $\mathrm{OSp}(1|2)$ acts on $z|\theta$ by a superanalog of fractional linear transformations.

So far $z|\theta$ are naturally real. $\mathrm{OSp}(1|2)$ obviously still acts on $z|\theta$ if we complexify them. If we complexify $z|\theta$ but with $\mathrm{Im}\, z > 0$, we get the super upper half-plane $\widehat{\mathcal{H}}$. It has a super Riemann surface structure that is invariant under $\mathrm{OSp}(1|2)$ (and is inherited from the symplectic structure we started with on $\mathbb{R}^{2|1}$), so every quotient $\widehat{Y} = \widehat{\mathcal{H}}/\Gamma$ is also a super Riemann surface.

What do we mean by a geodesic in $\widehat{\mathcal{H}}$? By setting $\theta = 0$, we get an embedding $\mathcal{H} \subset \widehat{\mathcal{H}}$. An ordinary geodesic $\ell \in \mathcal{H}$ is an example of what we call a geodesic in $\widehat{\mathcal{H}}$. It is a submanifold of dimension $1|0$, of course. Acting with $\mathrm{OSp}(1|2)$, we can transform ℓ to another submanifold of dimension $1|0$, which we also call a geodesic. These are the geodesics in $\widehat{\mathcal{H}}$.

If $\ell \in \widehat{\mathcal{H}}$ is a geodesic, and p is a point in ℓ, then there is no problem with the notion of the geodesic $\ell_{\perp}$ that is orthogonal to ℓ at p. We just transform to the case that $\ell \subset \mathcal{H} \subset \widehat{\mathcal{H}}$ and in that frame, we define $\ell_{\perp} \subset \mathcal{H}$ to be the geodesic orthogonal to ℓ at p in the classical sense.

The notion of a "super Riemann surface $\widehat{Y}$ with geodesic boundary" requires some discussion. $\widehat{Y}$ has real dimension $2|2$, so its boundary, which should be of codimension $1|0$, should have dimension $1|2$. But a geodesic has dimension $1|0$. I think the best explanation is that a geodesic (or any generic submanifold of dimension $1|0$) has a canonical "thickening," of dimension $1|2$, obtained by displacing it in the direction of the unintegrable distribution. It is really this thickening that is the boundary of "a super Riemann surface with geodesic boundary," but that phrase is a convenient shorthand.

Just as in the classical case, super Riemann surfaces with geodesic boundary can be built by gluing together three-holed spheres with geodesic boundary. Once we get this far, it is not too hard to see that all previous statements have superanalogs. (See also Y. Huang, R. C. Penner, and A. M. Zeitlin "Super McShane Identity" arXiv:1907.09978 for the generalization of the original McShane identity.) For example, if γ is one of the geodesic boundaries of $\widehat{Y}$, and p is a point in γ, then the geodesic $\gamma_\perp$ orthogonal to γ at p has the same options as before: 0) go on forever without intersecting itself or returning to γ;

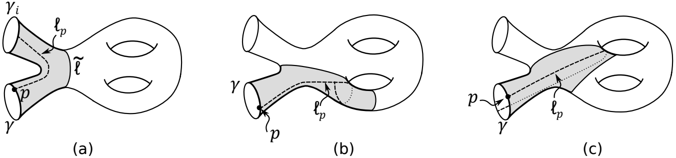
1) return to γ

2) leave $\widehat{Y}$ by a different boundary

3) intersect itself.

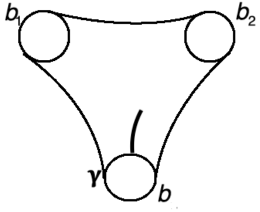
The measure for option 0) is again 0, so we really only have to consider 1), 2), 3).

The three alternatives lead to the same three pictures as before, in each case picking out a preferred three-holed sphere containing γ:



Again we get a sum rule in which γ, of total measure b, is decomposed (after throwing away a set of measure 0) as a disjoint union of subsets that correspond to one of the three-holed spheres that can appear in the picture.

To determine the measure of the piece corresponding to a particular three-holed sphere with boundary lengths b, b_1, b_2 is again a universal computation that only depends on those boundary lengths:

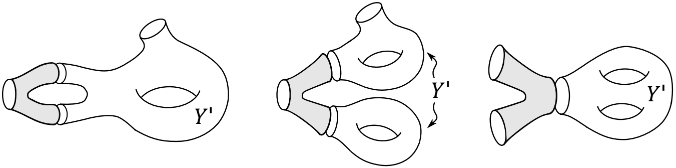


After doing this computation, we again get a Mirzakhani-style recursion relation.

The recursion relation has the same general form as before:

$$b\widehat{V}_{g,b,B} = \frac{1}{2}\int_0^\infty b'\mathrm{d}b'\ b''\mathrm{d}b''\ \widehat{D}(b,b',b'')\Bigg(V_{g-1,b',b'',B} + \sum_{\text{stable}} V_{h_1,b',B_1}V_{h_2,b'',B_2}\Bigg) - 2\sum_{k=1}^{|B|}\int_0^\infty b'\mathrm{d}b'\ \widehat{T}(b,b',b_k)\,V_{g,b',B\setminus b_k}.$$

The terms again correspond to the following three options for building $\widehat{Y}$ by gluing a three-holed sphere onto some $\widehat{Y}'$:



Mirzakhani's recursion relation was interpreted by Eynard and Orantin in terms of "topological recursion" for a spectral curve

$$y^2 = -\sinh^2 2\pi\sqrt{E},$$

whose relation to JT gravity was the starting point for Saad, Shenker, and Stanford. Stanford and I showed that the super version of the recursion relation is related in a similar way to a spectral curve

$$y^2 = -\frac{\cosh^2 2\pi\sqrt{E}}{E},$$

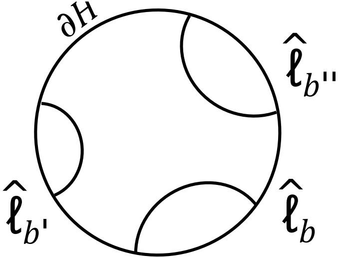
which has a similar relation to JT supergravity. (This spectral curve had been introduced by Norbury in arXiv:1712.03662 with a different motivation, based on its analogy with the "sine" spectral curve.) Putting these facts together, volumes of supermoduli spaces have a relationship to random matrix theory that is analogous to what holds for volumes of ordinary moduli spaces.

There are two facts I would still like to explain:

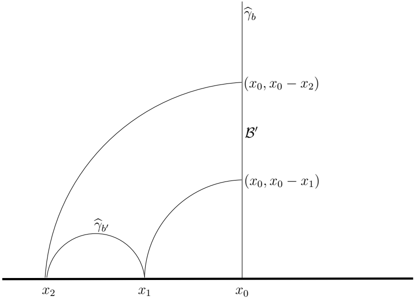
1) How to compute the universal functions that appear in the recursion relations.

2) What the volumes of supermoduli spaces mean in terms of ordinary geometry.

On the first point, if Λ is a hyperbolic three-holed sphere with geodesic boundaries, then its universal cover is a region in the upper half plane bounded by three geodesics. In the disc model of $\mathcal{H}$, the picture looks like this:

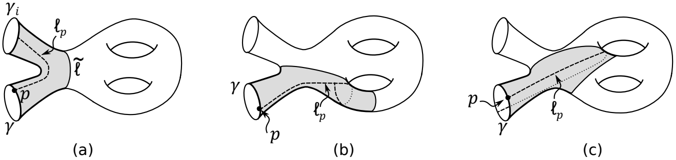


In the upper half plane model, we can take one of the two bounding geodesics to be a vertical line $\widehat{\gamma}_b$ and then the other two are semi-circles with center on the boundary, such as $\widehat{\gamma}_{b'}$:



The interval $\mathcal{B}'$ consists of points $p \in \widehat{\gamma}_b$ such that the orthogonal geodesic to $\widehat{\gamma}_b$ at p does meet $\widehat{\gamma}_{b'}$. Thus the length of $\mathcal{B}'$ is one of Maryam Mirzakhani's universal functions. The same picture makes sense in the super Riemann surface case.

The lesson here is that for a single three-holed sphere, the description of what happens to an orthogonal geodesic is simple. The complication of the original problem is that there are (countably) infinitely many choices of what the three-holed sphere would be:



On the second point, I claim that there is a useful general formula for the volume of a symplectic supermanifold. In general, let $\widehat{M}$ be a smooth supermanifold with "reduced space" M (M is defined by setting odd variables to zero, in other words by reducing functions on M modulo nilpotents). M is canonically embedded in $\widehat{M}$, with a normal bundle $V \to M$ that is "completely fermionic." As a smooth supermanifold, $\widehat{M}$ is just the total space of $V \to M$. (This description is highly non-unique and in general would not exist holomorphically. Supermoduli space $\mathfrak{M}$ is an example in which there is no holomorphic "splitting" or "projection" (Donagi and Witten 1993).) Let $\pi : \widehat{M} = V \to M$ be the projection.

Now suppose that $\widehat{M}$ is a smooth supermanifold with symplectic form $\widehat{\omega}$. Restricting $\widehat{\omega}$ to M, we learn that M is an ordinary symplectic manifold with symplectic form $\omega = \widehat{\omega}|_M$. Moreover, evaluating $\widehat{\omega}$ along M as a bilinear form on the fibers of $V \to M$, we get a (symmetric) nondegenerate quadratic form $V \otimes V \to \mathbb{R}$. So the structure group of V reduces to the orthogonal group.

Since the fibers of $\pi : \widehat{M} \to M$ are “topologically trivial” (being purely fermionic) one has

$$\widehat{\omega} = \pi^*(\omega) + \mathrm{d}\Psi$$

for a 1-form Ψ. The volume of $\widehat{M}$ does not depend on Ψ (as long as Ψ is sufficiently generic so that the volume is defined). Any choice of connection A on the orthogonal vector bundle $V \to M$ gives a convenient choice of Ψ, which one can write as

$$\Psi = \sum_i \theta_i D\theta_i,$$

where θ_i are odd coordinates associated to a local orthonormal basis of $V \to M$ and $D = \mathrm{d} + [A, \cdot]$ is the gauge-covariant exterior derivative.

We therefore can compute the volume of $\widehat{M}$ with this choice of Ψ and we get

$$\mathrm{Vol}(\widehat{M}) = \int_M \chi(V) e^{\omega},$$

where $\chi(V)$ is the Euler class of V.

In the example studied in this talk, $\widehat{M} = \mathfrak{M}_{g,\vec{b}}$ is the moduli space of super Riemann surfaces $\widehat{Y}$ with geodesic boundaries, and $M = \mathcal{M}_{g,\vec{b},\mathrm{spin}}$ is the moduli space of ordinary Riemann surfaces Y with geodesic boundary and a spin structure. After shrinking the geodesic boundaries to punctures (which does not affect the topological invariant $\chi(V)$), V can be identified as the vector bundle over M whose fiber at the point corresponding to some Y is $H^1(Y, K^{-1/2})$. This is the definition used by Norbury. So the supervolumes are quantities that can be defined purely in terms of the ordinary moduli spaces $\mathcal{M}_{g,\vec{b},\mathrm{spin}}$.

In short, I have reviewed Maryam Mirzakhani's recursion relation that determines the volumes of the moduli spaces of Riemann surfaces, and explained that the necessary ingredients have analogs for the moduli spaces of super Riemann surfaces. I gave some idea of how to compute the functions that come into the recursion relation. And finally, I explained an interpretation of the volumes of supermoduli spaces purely in terms of bosonic geometry.