On 5d and 4d SCFTs from canonical singularities

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What this talk is about:

superconformal quantum field theories and **singularities.**

We will be interested in QFTs with 8 supercharges in $d < 6$.

- They generally have a Coulomb branch (CB):

$$\text{SCFT} \; : \; \langle \mathcal{E} \rangle = u \neq 0 \qquad \longrightarrow \qquad U(1)^r$$

  and a Higgs branch (HB). We will want to study both.

- They can be *'engineered'* in Type IIA/IIB string theory or in M-theory.

Let $\mathbf{X}$ be a $\dim_{\mathbb{C}}$-3 affine variety with a single, isolated canonical singularity.

[Reid, 1980]

In the first part of this talk, we will look at the 5d $\mathcal{N} = 1$ SCFTs:

$$\mathcal{T}_{\mathbf{X}}^{5\mathrm{d}} \qquad \leftrightarrow \qquad \text{M-theory on } \mathbb{R}^{1,4} \times \mathbf{X}$$

[Witten, 1996; Morrison, Seiberg, 1996; Morrison, Seiberg, Intriligator 1996;...]

[Hayashi, Lawrie, Morrison, Schafer-Nameki, 2014; Del Zotto, Heckman, Morrison, 2017; Jefferson, Kim, Vafa, Zafrir, 2017; Jefferson, Katz, Kim, Vafa, 2018; Apruzzi, Lin, Mayrhofer, 2018; Bardwaj, Jefferson, 2018; Apruzzi, Lawrie, Lin, Schafer-Nameki, Wang, 2019;...]

In the second part, we will also look at the 4d $\mathcal{N} = 2$ SCFTs:

$$\mathscr{T}_{\mathbf{X}}^{4\mathrm{d}} \qquad \leftrightarrow \qquad \text{type IIB on } \mathbb{R}^{1,3} \times \mathbf{X}$$

[Katz, Klemm, Vafa, 1996; Shapere, Vafa, 1999;...]

[Cecotti, Neitzke, Vafa, 2010; Cecotti, Del Zotto, 2011, 2012; Xie, 2012; Del Zotto, Vafa, Xie, 2015; Xie, Yau, 2015; Wang, Xie, Yau, Yau, 2016; ...]

General questions for today:

- Given $\mathbf{X}$, *what do we know* about $\mathcal{T}_{\mathbf{X}}^{\mathrm{5d}}$ *and* $\mathscr{T}_{\mathbf{X}}^{\mathrm{4d}}$?

$$\begin{tikzcd} & X \arrow[dl, bend right, "M"'] \arrow[dr, bend left, "IIB"] & \\ T_X^{5D} \arrow[rr, leftrightarrow, bend right, "3d"'] & & \mathscr{T}_X^{4D} \end{tikzcd}$$

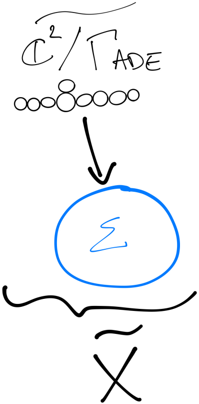
- How are $\mathcal{T}_{\mathbf{X}}^{\mathrm{5d}}$ and $\mathscr{T}_{\mathbf{X}}^{\mathrm{4d}}$ related, if at all?

The view from M-theory (5d)

Geometric engineering in IIA

First, a bit of history. *Geometric engineering* of 4d $\mathcal{N} = 2$ *gauge theories:*

[Katz, Klemm, Vafa, 1996]



In type-IIA string theory:

- D2-branes wrapping fiber curves give W-bosons, with mass t_f.
- Need to decouple the D2-branes on Σ, of size t_b. 'Geometric engineering limit' (4d limit):

$$t_f = \epsilon\,, \qquad \Lambda^{2N} = e^{-\frac{1}{g^2}} = \frac{e^{-t_b}}{\epsilon^{2N}} \text{ fixed}\,,\ \epsilon \to 0$$

(here for $SU(N)$)

- Gauge-theory instantons = worldsheet instantons. $\mathcal{F} = \mathcal{F}_{\rm pert} + \sum_n c_n e^{-n t_f}$.

Geometric engineering in IIB

Local mirror CY_3 in IIB: [Katz, Mayr, Vafa, 1996]

$$\widehat{\mathbf{X}}^\vee \;:\; u^2 + v^2 + P(x,y) = 0$$

resums the instantons. $P(x,y)=0$ is the SW curve of the 4d $\mathcal{N}=2$ theory (after taking the 'geometric engineering'/4d limit). The BPS states are D3-branes wrapped on 3-cycles. Central charge:

$$Z_\gamma = \int_{S^3_\gamma} \Omega_3 = \int_\gamma \lambda_{\mathrm{SW}}$$

We can take further limits. For instance, there is the Argyres-Douglas limit to:

$$u^2 + v^2 + x^2 + y^N = 0$$

This directly engineers the A_N AD theory in IIB. [Shapere, Vafa, 1999]

Geometric engineering in M-theory

The geometric engineering/4d limit is *not necessary* to have a QFT interpretation. The same local CY$_3$ $\widetilde{\mathbf{X}}$ in IIA has an interpretation as a 5d SCFT on a circle:

[Nekrasov, 1996; Lawrence, Nekrasov, 1997]

$$D_{S^1}\mathcal{T}^{\rm 5d}_{\mathbf{X}} \qquad \leftrightarrow \qquad \text{IIA on } \mathbb{R}^{1,3}\times \mathbf{X}$$

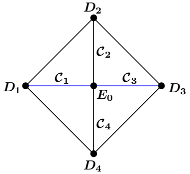
The 5d SCFT, $\mathcal{T}^{\rm 5d}_{\mathbf{X}}$, may be *defined* as the low-energy limit of M-theory on $\mathbf{X}$:

$$\mathcal{T}^{\rm 5d}_{\mathbf{X}} \qquad \leftrightarrow \qquad \text{M-theory on } \mathbb{R}^{1,4}\times \mathbf{X}$$

These are the 5d fixed points discovered by [Seiberg, 1996]:

- M2-branes on fiber give W-bosons
- M2-branes on base give instanton particle
- 5d prepotential computed classically from $\widetilde{\mathbf{X}}$

$$\mathcal{F}_{\rm 5d} = \frac{1}{6} S\cdot S\cdot S$$



$$E_1 \;:\; \mathcal{O}(-2,-2)\to \mathbb{P}^1\times\mathbb{P}^1$$

General features of the five-dimensional SCFT:

- Real Coulomb branch:

$$\mathcal{M}_C \cong \mathbb{R}^r / W$$

- No marginal parameters. There can be deformations (mass terms)[1], including:

$$\delta\mathscr{L} = \mu_0 \mathcal{O} \qquad \xrightarrow{\text{RG}} \qquad \delta\mathscr{L} = \mu_0 \left(\mathrm{tr} F \wedge *F + \cdots \right) \;, \qquad \mu_0 = \frac{1}{g_{5d}^2}$$

  That is, 5d $\mathcal{N} = 1$ gauge theory can arise as *effective field theory in the IR.*

- Non-trivial Higgs branch, in general:

$$\mathcal{M}_H = \text{hyperKähler cone of real dimension } 4d_H$$

  Note: 'quantum HB' of SCFT—i.e. larger than the HB of IR gauge theory.

[1] [Cordova, Dumitrescu, Intriligator, 2016]

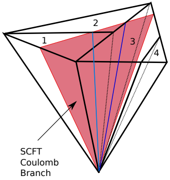
CB and HB of $\mathcal{T}_{\mathbf{X}}^{\mathrm{5d}}$ from M-theory

5d Coulomb branch: *crepant resolutions*

$$\pi \ : \ \widetilde{\mathbf{X}} \to \mathbf{X}$$

ECB of real dimension $r + f$.

$$\pi^{-1}(0) = \bigcup_{a=1}^{r} S_a \ , \quad \rho(\mathbf{X}) = f$$



Extended Kähler cone of $\mathbf{X}$ is ECB

Classical geometry!

5d Higgs branch: *deformation* to smooth variety:

$$\widehat{\mathbf{X}}$$

We have 3-cycles:

$$H_3(\widehat{\mathbf{X}}, \mathbb{Z}) = \mathbb{Z}^{\mu}$$

Examples:

- toric deformations [Altmann, 1992]
- hypersurface (see below)

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = \epsilon$$

M2-brane instantons affect HB metric

$\rightarrow$ Hard problem...

Isolated hypersurface singularities

Take $\mathbf{X}$ an hypersurface in $\mathbb{C}^4$, with an isolated singularity at the origin:

$$F(x_1, x_2, x_3, x_4) = 0 \ .$$

Quasi-homogeneous condition: $x_i \to \lambda^{q_i} x_i$, $F(\lambda^q x) = \lambda F(x)$.
Canonical singularity condition: [Gukov, Vafa, Witten, 1999; Shapere, Vafa, 1999]

$$\sum_{i=1}^{4} q_i > 1$$

Complete classification. [Yau, Yu, 2003; Xie, Yau, 2015]
Examples:

- $F = x_1^2 + x_2^2 + x_3^2 + x_4^2$ (conifold) gives free hyper in 5d.
- $F = x_1^3 + x_2^3 + x_3^3 + x_4^3$ (local dP_6) gives rank-1 E_6 5d SCFT.
- $F = x_1^2 + x_2^4 + x_3^4 + x_4^4$ (local dP_7) gives rank-1 E_7 5d SCFT.
- $F = x_1^2 + x_2^3 + x_3^6 + x_4^6$ (local dP_8) gives rank-1 E_8 5d SCFT.
- $\cdots$

Minor ring, vanishing cohomology and Higgs branch

Deformations $\widehat{\mathbf{X}}$ of $\mathbf{X}$. Homology $H_3(\widehat{\mathbf{X}}, \mathbb{Z}) = \mathbb{Z}^\mu$, 'bouquet' of 3-spheres.

- Deformations are generators of the Milnor ring:

$$F(x) + \sum_{l=1}^{\mu} t_l x^{\mathfrak{m}_l} = 0 \ , \qquad x^{\mathfrak{m}_l} \in \mathcal{M}(F) = \mathbb{C}[x_1, x_2, x_3, x_4]/(dF)$$

The Milnor ring is graded by q charges: spectrum of the singularity:

$$\mathcal{S}_{\mathbf{X}} = \left\{ \ell_l \equiv \sum_{i=1}^{4} q_i(\mathfrak{m}_{l,i} + 1) - 1 \right\}_{l=1}^{\mu} \subset (0, 2) \ , \qquad \ell_1 \leq \ell_2 \leq \cdots \leq \ell_\mu$$

- Monodromy operators acting on $H_3(\widehat{\mathbf{X}})$ (Picard-Lefschetz theory).
- 'Classical monodromy operator', M, determines weight filtration and mixed Hodge structure:

$H^{1,2}(\widehat{\mathbf{X}})$: $x^{\mathfrak{m}_l}$ such that $\ell_l < 1$, $H^{2,2}(\widehat{\mathbf{X}})$: $x^{\mathfrak{m}_l}$ such that $\ell_l = 1$,

$H^{2,1}(\widehat{\mathbf{X}})$: $x^{\mathfrak{m}_l}$ such that $\ell_l > 1$

Let us define the integers:

$$\widehat{r} = \dim H^{1,2}(\widehat{\mathbf{X}}) \;, \qquad f = \dim H^{2,2}(\widehat{\mathbf{X}})$$

Compare to f as defined above. There are f conifold-type transitions:

$$\mathcal{C}_\alpha \subset \widetilde{\mathbf{X}} \qquad \longleftrightarrow \qquad S^3_\alpha \subset \widehat{\mathbf{X}} \;, \qquad \alpha = 1, \cdots, f$$

The quaternionic dimension of the Higgs branch is given by the number of *dynamical hypers* in M-theory, that arise from elements of the vanishing cohomology with $\ell_l \leq 1$: see [Gukov, Vafa, Witten, 1999]

$$\boxed{d_H = \widehat{r} + f}$$

From this, of course, we only get the HB dimension, not the metric, and not even the algebraic structure. Here, we will propose a way to obtain the latter.

Coulomb branch from resolution of hypersurfaces

Given an hypersurface $F(x) = 0$, we can study its *crepant resolutions* systematically:

$$\pi^{-1}(0) = \bigcup_{a=1}^{r} S_a \ , \qquad \mathbb{P}^1 \to S_a \to \Sigma_{g_a}$$

- The cohomology of $\widetilde{\mathbf{X}}$ was studied by [Caibar, 1999]. One finds:

$$H_1(\widetilde{\mathbf{X}}, \mathbb{Z}) = 0 \ , \quad H_2(\widetilde{\mathbf{X}}, \mathbb{Z}) = r + f \ , \quad H_3(\widetilde{\mathbf{X}}, \mathbb{Z}) = b_3 \ , \quad H_4(\widetilde{\mathbf{X}}, \mathbb{Z}) = r$$

  with $b_3 = 2\sum_{a=1}^{r} g_a$.
- In general, we can have residual terminal singularities:

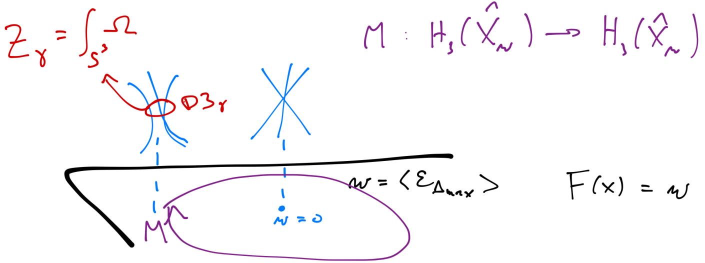
$$\mathbf{X}_{\rm IR}^1, \cdots, \mathbf{X}_{\rm IR}^k \qquad \text{at } \ p_1, \cdots, p_k \in \widetilde{\mathbf{X}}$$

- Physics interpretation: $\frac{1}{2}b_3$ free hypers and k *'IR SCFTs'* on the 5d CB...

The view from Type IIB

Hypersurface singularities: local CY_3 as SW geometry

Local CY_3 manifolds in IIB provide nice examples of ‘Seiberg-Witten curves’ which are not curves. [Katz, Mayr, Vafa, 1997]



Argyres-Douglas-like theories can be engineered at canonical singularities defined by quasi-homogeneous polynomials:

$$F(\lambda^q x) = \lambda F(x) \ , \qquad q_i = \frac{\alpha_i}{m_0} \ , \qquad M^{m_0} = \mathbf{1}$$

For instance:

- Ordinary AD theory of type $G = ADE$: [Shapere, Vafa, 1999]

$$F = x_1^2 + x_2^2 + F_G(x_3, x_4) \ , \qquad F_{A_k}(x,y) = x^2 + y^{k+1} \ , \qquad F_{E_6} = x^3 + y^4$$
$$F_{D_k} = x^{k-1} + xy^2 \qquad F_{E_7} = x^3 + xy^3$$
$$F_{E_8} = x^3 + y^5$$

- The CNV $[G, G']$ theories: [Cecotti, Neitzke, Vafa, 2010]

$$F = F_G(x_1, x_2) + F_{G'}(x_3, x_4)$$

- Or any other! e.g. the Laufer singularity ($\mu = 11$, $f = 1$, $\widehat{r} = 5$; $m_0 = 18$):

$$F = x_1^2 + x_2^3 + x_3^2 x_4 + x_2 x_4^3 = 0 \ , \ \Delta = \left\{ \frac{5}{4}, \frac{11}{8}, \frac{3}{2}, \frac{7}{4}, \frac{9}{4} \right\} \ , \quad a = \frac{137}{48} \ , \ c = \frac{139}{48}$$

Full classification (into XIX families). [Yau, Yu, 2005; Xie, Yau, 2015]

CB and HB for $\mathscr{T}_{\mathbf{X}}^{\text{4d}}$ from IIB

4d Coulomb branch: $\widehat{\mathbf{X}}$

ECB: $H^{1,2}(\widehat{\mathbf{X}}) \oplus H^{2,2}(\widehat{\mathbf{X}})$

CB spectrum from spectrum:

$$\Delta_l = \frac{\sum_{i=1}^4 q_i - \ell_l}{\sum_{i=1}^4 q_i - 1}$$

determines a, c

[Shapere, Tachikawa, 2008]

Classical geometry!

5d Higgs branch: *crepant resolutions* $\widetilde{\mathbf{X}}$

- One hypermultiplet for each 2-cycle. Gives 4d HB dimension:

$$\boxed{\widehat{d}_H = r + f}$$

- b_3 free vectors from 3-cycles.
- $\mathbf{X}_{\text{IR}}$: residual IR SCFT!

D1/D3-brane instantons affect HB metric

Matches known structure of HB of AD theories

see *e.g.* [Beem, Meneghelli, Rastelli, 2019]

Anomaly matching on the Higgs branch / relation between $\widehat{\mathbf{X}}$

$$Tr(U(1)_r) = 24(c-a) = \widehat{d}_H - \frac{1}{2}b_3 + 24(c^{\text{IR}} - a^{\text{IR}})$$

Non-trivial check on the overall picture!

One-form symmetries of $\mathscr{T}_{\mathbf{X}}^{4\mathrm{d}}$

Another general aspect of $\mathscr{T}_{\mathbf{X}}^{4\mathrm{d}}$ that can be read off from Type IIB setup: **one-form symmetries.**[2,3]

Charged line defects visible on the CB. They are D3-branes on non-compact 3-cycles that wrap torsion 2-cycles on the boundary five-manifold:

$$\partial\widehat{\mathbf{X}} \cong L_5(\mathbf{X}) \ , \qquad \mathrm{Tor}\, H_2(L_5(\mathbf{X}), \mathbb{Z}) = \mathfrak{f} \oplus \mathfrak{f}$$

This torsion group is related to the intersection pairing of 3-cycles in $\mathbf{X}$. Choose sub-lattice $\mathfrak{f}$: *one-form symmetry* $\mathfrak{f}$.

For instance, for $[G, G'] = [E_k, A_l]$:

$\Gamma^{(1)} = \mathfrak{f}$	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}	$\cdots$
E_6	0	0	$\mathbb{Z}_3$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_3$	0	0	0	0	0	
E_7	0	0	0	0	$\mathbb{Z}_3$	0	0	$\mathbb{Z}_2^3$	0	0	$\mathbb{Z}_3$	0	
E_8	0	0	0	0	$\mathbb{Z}_5$	0	0	0	$\mathbb{Z}_3^2$	0	$\mathbb{Z}_5$	0	

[2] [Aharony, Seiberg, Tachikawa, 2013; Gaiotto, Kapustin, Seiberg, Willett, 2014]

[3] [Garcia Etxebarria, Heidenreich, Regalado, 2019; Morrison, Schafer-Nameki, Willett, 2020; Albertini, Del Zotto, Garcia Etxebarria, Hosseini, 2020; see also: Del Zotto, Garcia Etxebarria, Hosseini, 2020]

The view from 3d

Relating $\mathcal{T}^{5\mathrm{d}}_{\mathbf{X}}$ and $\mathscr{T}^{4\mathrm{d}}_{\mathbf{X}}$

Reduce $\mathcal{T}^{5\mathrm{d}}_{\mathbf{X}}$ on a T^2: reduce $\mathscr{T}^{4\mathrm{d}}_{\mathbf{X}}$ on S^1:

$$T^{5D}_X \;/\; \text{M-theory}$$

$$\big\downarrow S^1_M$$

$$D_{S^1}\widetilde{T}^{5D}_X \;/\; \text{IIA} \qquad\qquad \widetilde{T}^{4D}_X \;/\; \text{IIB}$$

$$\big\downarrow S^1_\beta \qquad\qquad\qquad \big\downarrow S^1_{\beta^{-1}}$$

$$D_{T^2}T^{5D}_X \underset{3D}{\cong} EQ^5[X] \qquad\qquad D_{S^1}\widetilde{T}^{4D}_X \underset{3D}{\cong} EQ^4[X]$$

T-duality

They both are **3d $\mathcal{N}=4$ SQFT** (KK theories). Below KK scale, we define the electric quiverines $\mathrm{EQ}^{(5)}$ and $\mathrm{EQ}^{(4)}$. Type IIA and IIB related by T-duality.

Magnetic quivers (and quiverines)

The T-duality essentially realizes 3d $\mathcal{N}=4$ mirror symmetry. [4]
More precisely, let us denote by MQ ('magnetic quiverine') the mirror of an electric quiverine:

$$\mathsf{MQ}^{(5)} \overset{\text{3d mirror}}{\Longleftrightarrow} \mathsf{EQ}^{(5)}~, \qquad \mathsf{MQ}^{(4)} \overset{\text{3d mirror}}{\Longleftrightarrow} \mathsf{EQ}^{(4)}~.$$

We then propose the relation:

$$\mathsf{MQ}^{(5)} \cong \mathsf{EQ}^{(4)}/U(1)^f~, \qquad \mathsf{MQ}^{(4)} \cong \mathsf{EQ}^{(5)}/U(1)^f~.$$

Here, we have an S-type gauging of $U(1)^f$, which is reversible in 3d: [5]

$$\mathsf{EQ}^{(5)} \cong \mathsf{MQ}^{(4)}/U(1)^f_T~, \qquad \mathsf{EQ}^{(4)} \cong \mathsf{MQ}^{(5)}/U(1)^f_T~.$$

[4] [Intriligator, Seiberg, 1996; Hori, Ooguri, Vafa, 1997]
[5] [Kapustin, Strassler, 1999; Witten, 2003]

3d mirror symmetry and the Higgs branch

$$\mathsf{MQ}^{(5)} \cong \mathsf{EQ}^{(4)}/U(1)^f \ , \qquad \mathsf{MQ}^{(4)} \cong \mathsf{EQ}^{(5)}/U(1)^f \ .$$

Key point: In 3d $\mathcal{N}=4$, **the CB and HB are both hyper-Kähler cones.**

	CB	HB
$\mathsf{EQ}^{(5)}$	r	$d_H = \widehat{r} + f$
$\mathsf{MQ}^{(5)}$	$d_H = \widehat{r} + f$	r
$\mathsf{EQ}^{(4)}$	$\widehat{r}$	$\widehat{d}_H = r + f$
$\mathsf{MQ}^{(4)}$	$\widehat{d}_H = r + f$	$\widehat{r}$

Power of the **'magnetic quiver'** concept:[6] [Ferlito, Hanany, Mekareeya, Zafrir, 2018] We can compute the 5d or 4d Higgs branch as a 3d Coulomb branch:

$$\mathcal{M}_H[\mathcal{T}_{\mathbf{X}}^{\rm 5d}] = \mathrm{CB}[\mathsf{MQ}^{(5)}] \ , \qquad \mathcal{M}_H[\mathscr{T}_{\mathbf{X}}^{\rm 4d}] = \mathrm{CB}[\mathsf{MQ}^{(4)}]$$

Coulomb-branch monopole operators correspond to M2/D-brane insantons!
Can be resummed more efficiently. [Cremonesi, Hanany, Zaffaroni, 2013]

[6] [Hanany, Witten, 1996; ...; Bourget, Cabrera, Grimminger, Hanany, Sperling, Zajac, Zhong, 2019]

Examples and curiosities

Example 1: The rank-N E_8 5d SCFT

Take the singularity:

$$\mathbf{X}_{E_8} \ : \ F = x_1^2 + x_2^3 + x_3^6 + x_4^{6N} = 0$$

r	f	d_H	$\widehat{r}$	$\widehat{d}_H$	$\Delta\mathcal{A}_r$	b_3	$\mathfrak{f}$
N	8	$30N-1$	$30N-9$	$N+8$	$-N+1$	$2N-2$	$\mathbb{Z}_N$

Crepant resolution (rank $r = N$):

$$(x_1^{(3)}\ , x_2^{(2)}\ , x_3^{(1)}\ , x_4^{(1)}\ ; \delta_1)\ ,$$
$$(x_1^{(3)}\ , x_2^{(2)}\ , x_3^{(1)}\ , \delta_i^{(1)}\ ; \delta_{i+1})\ , \qquad \text{for } i = 1, \cdots, N-1\ .$$

Exceptional divisors:

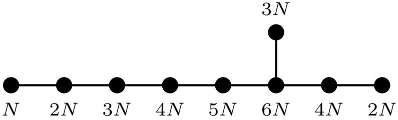
$$\mathbb{P}^1 \to S_i \to T^2\ ,\ i = 1, \cdots, N-1\ , \qquad S_N = dP_8$$

From the intersection numbers: 5d gauge-theory: [Intriligator, Morrison, Seiberg, 1997]

$$Sp(N) + 7\boldsymbol{F} + \boldsymbol{AS}$$

'UV fixed point' is a rank-N generalization of the E_1 rank-1 5d SCFT.

Deformation of the singularity: From the spectrum, which satisfies $\Delta \in \mathbb{Z}$, one finds that $\mathscr{T}_{\mathbf{X}}^{4\mathrm{d}}$ is a Lagrangian 4d SCFT described by the $SU(d_k N)$ quiver: [7]



In this case, this is also the electric quiver, $\mathsf{EQ}^{(4)}$. By gauging the $U(1)^8$ flavor symmetry, we get the $\mathsf{MQ}^{(5)}$, which is the same quiver but with $U(d_k N)$ gauge groups. Remark: HB of $\mathscr{T}_{\mathbf{X}}^{4\mathrm{d}}$: clear interpretation of 3-cycles in $\widetilde{\mathbf{X}}$ as free vectors.

[7] [Del Zotto, Vafa, Xie, 2015]

Example 2: A rank-zero $\mathcal{T}_{\mathbf{X}}^{\rm 5d}$

Take the following singularity, which gives AD$[A_2, D_4]$ in IIB:

$$F = x_1^2 + x_2^3 + x_3^3 + x_4^3 = 0$$

Terminal singularity, no crepant resolution: $\mathcal{M}_H[\mathscr{T}_{\mathbf{X}}^{\rm 4d}] \cong \{pt\}$.

r	f	d_H	$\widehat{r}$	$\widehat{d_H}$	$\Delta\mathcal{A}_r$	μ	b_3	$\mathfrak{f}$	Δ	a	c
0	0	4	4	0	0	8	0	$\mathbb{Z}_2$	$\{2, \frac{4}{3}, \frac{4}{3}, \frac{4}{3}\}$	2	2

This can be described by the following conformal gauging:

$$\text{AD}[A_2, D_4] = \quad \begin{array}{ccccc} & & A_3 & & \\ & & | & & \\ A_3 & — & SU(2) & — & A_3 \end{array} \quad \text{or} \quad \begin{array}{ccccc} & & A_3 & & \\ & & | & & \\ A_3 & — & SO(3)_\pm & — & A_3 \end{array}$$

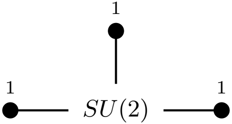
In IIB, this is ‘just’ a nice 4d SCFT. What is the view from M-theory?

It is known that

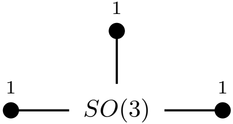
$$D_{S^1}\mathsf{AD}[A_1, A_3] \overset{\text{3d}}{\cong} U(1)+2\boldsymbol{F}$$

We then infer (here, $f=0$ so no S-gauging necessary):

$\mathsf{EQ}^{(4)} = \mathsf{MQ}^{(5)} =$



or



Structure of the 5d HB ring from Hilbert series (monopole formula):

[Cremonesi, Hanany, Zaffaroni, 2013]

$$\mathrm{HS}_{SU(2)} = \frac{1+28t^2+70t^4+28t^6+t^8}{(1-t^2)^8}\,, \qquad \mathrm{HS}_{SO(3)} = \frac{1}{(1-t)^8}\,,$$

Higgs branch $\mathcal{M}_H[\mathcal{T}^{\text{5d}}_{\mathbf{X}}] \cong \mathbb{H}^4/\mathbb{Z}_2$ (MLO of $Sp(4)$) or $\mathbb{H}^4$.

Tentatively (conjecture): $\mathcal{T}^{\text{5d}}_{\mathbf{X}}$ consists of 5d hypermultiplets (of $/\mathbb{Z}^2$)?
This generalize to any AD / CNV theory with $r=0$.

Example 3: A new rank-one $\mathcal{T}_{\mathbf{X}}^{\rm 5d}$?

Take the following singularity, which gives AD$[D_4, E_8]$ in IIB:

$$x_1^3 + x_2^3 + x_3^3 + x_4^5 = 0$$

Rank-one resolution with residual terminal singularity $\mathbf{X}_{\rm IR}$ which gives $[A_2, D_4]$.

r	f	d_H	$\widehat{r}$	$\widehat{d}_H$	$\Delta\mathcal{A}_r$	μ	b_3	$\mathfrak{f}$	Δ	a	c
1	0	16	16	1	-1	32	2	$\mathbb{Z}_5$	$\{\Delta_l\}$	16	16

$\Delta = \left\{\frac{4}{3}, \frac{4}{3}, \frac{4}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, 2, \frac{7}{3}, \frac{7}{3}, \frac{7}{3}, 3, \frac{10}{3}, \frac{10}{3}, \frac{10}{3}, 4, 5\right\}$.

- In M-theory: **new rank-one $\mathcal{T}_{\mathbf{X}}^{\rm 5d}$.** There is an additional massless sector on the 5d CB. Hundreds of examples like this.
- Here, $\mathcal{T}_{\mathbf{X}}^{\rm 5d}$ has a $\mathbb{Z}_5$ 0-form or 3-form symmetry. (Could it be a $\mathbb{Z}_5$ discrete gauging of known rank-0 theory?)
- Physics to be explored further. *Stay tuned...*

Conclusions and outlook

Conclusions:

- We revisited the geometric engineering of $\mathcal{T}_{\mathbf{X}}^{5\mathrm{d}}$ and $\mathscr{T}_{\mathbf{X}}^{4\mathrm{d}}$ at hypersurface singularities. Classical mathematics, new physics.
- Analysis captures essential features of both the CB and HB.
- To explore further the HB physics, we reduced to 3d. Then, the CB and HB of the two theories become related by 3d $\mathcal{N}=4$ mirror symmetry.
  3d monopoles play the role of M2/D-brane instantons.
- Many more 5d SCFTs appear in this analysis compared to previous studies.

Outlook/questions:

- Are they *really* all new theories? What are the rules of the game?
- Can we understand the rank-0 5d SCFTs in QFT language?
- Role of Ricci-flat conical metric (in general, $\nexists$) in all this?