One-loop Integrations with Hypergeometric Functions

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Aim

Purpose

- To handle arbitrary combination of mass parameters in d-dimensional 1-loop calculations (massive and massless cases).
- Tensor integral in Feynman parameter representation.

  $\Leftarrow$ differentiation in terms of mass parameters.
- To obtain numerically stable expressions.

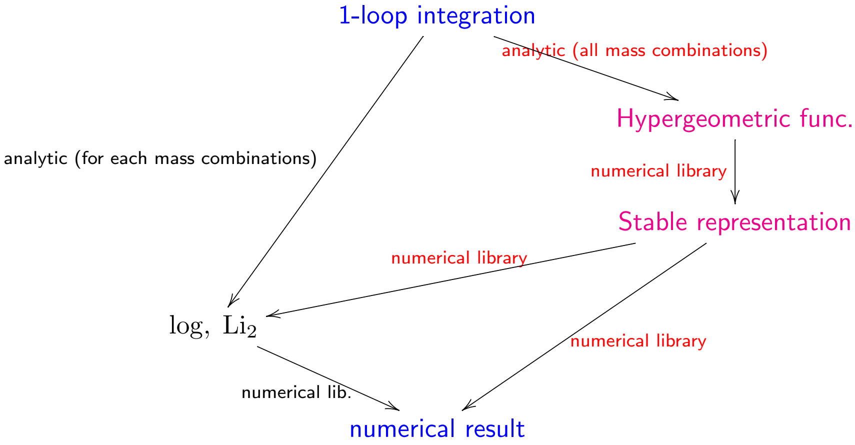
Hypergeometric functions will be useful.

- Regge (1969, a class of generalized Hypergeometric equations)
- Tarasov et al., Davydychev, Kalmykov, ... (1-, 2-loop, ...)
- Duplančić and Nižić, Kurihara (1-loop, for massless QCD with IR)

This work:

- New analytic expression for general box integration using extended hypergeometric functions.

Plan


1-loop integration
analytic (all mass combinations)
Hypergeometric func.
numerical library
Stable representation
analytic (for each mass combinations)
numerical library
log, Li_2
numerical library
numerical lib.
numerical result

Attention!

- Results depends of the order of limiting process.

$$\frac{(p^2)^\epsilon - 1}{\epsilon} \to \begin{cases} -1/\epsilon & (\epsilon > 0, \quad p^2 \to 0) \\ \log p^2 & (p^2 \neq 0, \quad \epsilon \to +0) \end{cases}$$

$\Rightarrow$ different expressions among massive and massless cases around 4-dim.

- We take limits in the following order:
  1. Differentiation in terms of m_j for **tensor** integration.
  2. If particles are massless (and on-shell), take the limit $m_j \to 0$ (and $p_k^2 \to 0$) **before** $d \to 4$.
  3. $d \to 4$, or **expansion** in terms of $(d-4)$, before the numerical calculation.

- Poles of ϵ may be produced by an integration.

$\Rightarrow$ not always possible to expand the integrand.

- Numerical instability: $\infty - \infty \to$ finite, $\quad 0/0 \to$ finite.

  Analytic properties of the expressions are important not only for physical behavior but also for numerical stabilities.

2-point function

- 2-point scalar integration for general parameters.

$$I_2^{(\alpha)} = \int_0^1 dx \ \mathcal{D}^\alpha,$$
$$\mathcal{D} = -p^2 x(1-x) + m_1^2 x + m_2^2 (1-x) - i\varepsilon$$

Scalar integration: $\alpha = +0$ in 4-dim.

- Factorize with the roots $x = \gamma_\pm$ of $\mathcal{D} = 0$:

$$I_2^{(\alpha)} = \mathcal{D}(0)^\alpha \int_0^1 \left(1 - \frac{x}{\gamma^+}\right)^\alpha \left(1 - \frac{x}{\gamma^-}\right)^\alpha dx.$$

- It is a special case of Appell F_1:

$$F_1(\alpha, \beta, \beta'; \gamma; y, z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 x^{\alpha-1}(1-x)^{\gamma-\alpha-1}(1-yx)^{-\beta}(1-zx)^{-\beta'} dx.$$

Appell F_1

Appell F_1

- Power series expansion, differential equation and integral representation is known.
- Generalization of Gauss Hypergeometric function F:

$$F_1(\alpha, \beta, 0; \gamma; y, z) = F(\alpha, \beta; \gamma; y).$$

- Regular except for (depending on the values of parameters)

$$y = 0, 1, \infty, \quad z = 0, 1, \infty, \quad y = z.$$

$\Rightarrow$ important information for numerical stability.

Around 4-dim.

- For massive case (with identity of F_1)

$$I_2^{(\alpha)} = \frac{\gamma^-}{\alpha+1}\mathcal{D}(0)^\alpha F(1,-\alpha;\alpha+2;\frac{\gamma^-}{\gamma^+}) + \frac{1-\gamma^-}{\alpha+1}\mathcal{D}(1)^\alpha F(1,-\alpha;\alpha+2;\frac{1-\gamma^-}{1-\gamma^+}).$$

- Expansion around 4-dim. with

$$F(a\epsilon, b\epsilon; 1+b\epsilon; z) = 1 + ab\epsilon^2\,\mathrm{Li}_2(z) + O(\epsilon^3),$$

$$F(1,\epsilon,2-\epsilon;z) = \frac{1-\epsilon}{1-2\epsilon}\left[\frac{1+z}{2z} - \frac{(1-z)^{1-2\epsilon}}{2z} - \epsilon^2\frac{1-z}{z}\,\mathrm{Li}_2(z)\right] + \mathcal{O}(\epsilon^3).$$

  - usual expression in 4-dim with $\log$ and Li_2.
  - numerically calculable

- Other identities and expansion formulas are used for massless case.

3-point function

- 3-point function

$$I_3^{(\alpha)} = \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \, \mathcal{D}^\alpha.$$

where, $\mathcal{D}$ is a quadratic form of x_1 and x_2.

- Linearization of $\mathcal{D}$ for one variable (elimination of x_2^2 term):
Change variables $(x_1, x_2) \to (x_2, y)$: (projective transformation; 'tHooft-Veltman '79)

$$x_1 = y - rx_2, \quad y = x_1 + rx_2,$$

$$r : \text{ root of a quadratic equation.}$$

3-point function: 1st integration

- $\mathcal{D}$ is linear in x_2 $\Rightarrow$ trivial integration for x_2.

$$I_3^{(\alpha)} = \sum_{i:\text{edge of the triangle}} c_i \int_{\text{edge}_i} \frac{\mathcal{D}^{\alpha+1}}{a_i y + b_i}\, dy.$$

- $\mathcal{D}$: quadratic in terms of y $\Rightarrow$ product of linear factors of y.

$$I_3^{(\alpha)} = \sum_i c_i \int \frac{1}{a_i y + b_i} \left(1 - \frac{y}{\gamma_i^+}\right)^{\alpha+1} \left(1 - \frac{y}{\gamma_i^-}\right)^{\alpha+1} dy.$$

One more factor in the integrand than F_1.

$\Rightarrow$ Lauricella's F_D.

Lauricella F_D

- Integral representation

$$F_D(\alpha, \beta_1, \cdots, \beta_n; \gamma; x_1, \cdots, x_n) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1} \prod_{i=1}^{n} (1-x_i t)^{-\beta_i} dt$$

- A generalization of F (when $n=1$), F_1 (when $n=2$).
- **Analyticity**: Regular except for $x_j = 0, 1, \infty$, $x_j = x_k$ $(j \neq k)$.
- Identity: for constants $x_1, ..., x_n$

$$z^{p-1}(1-z)^{q-1} \prod_{i=1}^{n} (1-x_i z)^{-\beta_i} = \frac{d}{dz} \frac{z^p}{p} F_D\left(p, (\beta_i), 1-q; p+1; (x_i z), z\right).$$

$\Rightarrow$ **primitive** function of any product of linear factors with arbitrary powers.

3-point function with F_D

- The result is

$$I_3^{(\alpha)} = \frac{1}{(\alpha+1)\sqrt{D}} \sum_{k=0}^{2} \frac{\mathcal{D}_k(0)^{\alpha+1}}{x_{k0}} F_D(1,1,-\alpha-1,-\alpha-1;2;\frac{1}{x_{k0}},\frac{1}{\gamma_k^+},\frac{1}{\gamma_k^-})$$

  where,

  - D : discriminant appearing in the projective transformation.
  - k: indicates an edge of the triangular integration domain.
  - $\mathcal{D}_k$: value of $\mathcal{D}$ on the edge k.
  - x_{k0}: $-b/a$ on the edge k.
  - $\gamma_k^{\pm}$: roots of $\mathcal{D}_k$.

- Around 4 dim.: $F_D \Rightarrow F_1 \Rightarrow \log, \mathrm{Li}_2$.

4-point function

- 4-point function:

$$I_4^{(\alpha)} = \int_{\mathbb{R}^4_{\geq 0}} d^4x \, \delta\Big(1 - \sum x_j\Big) \, \mathcal{D}^\alpha.$$

$\mathcal{D}$ is a homogeneous quadratic form of x_j.

For scalar integration, $\alpha = -2 - \epsilon$ $(\epsilon < 0)$.

- 1st integration : $\Leftarrow$ Projective transformation : same as 3-point functions:

$$\int \mathcal{D}^\alpha dy = \frac{1}{\alpha+1} \frac{\mathcal{D}^{\alpha+1}}{\partial_y \mathcal{D}}.$$

$\mathcal{D}$ is quadratic in terms of other variables, while $\partial_y \mathcal{D}$ is a linear factor.

Second integration

- Projective transformation once more
  ⇒ Integrate with the formula (special case for F_D):

$$\beta \frac{z^{\beta-1}}{1-z} = \frac{d}{dz} z^{\beta} F(1,\beta;\beta+1;z),$$

  ⇒ The second integration is expressed with F (Gauss HGF).
- Integration domain : slightly complicated
  ⇒ differential form ⇒ Stokes' theorem.

The result of the second integration

- After integrations twice:

$$I_4^{(\alpha)} = \sum_{i=1}^{3} \sum_{m=1,m\neq i}^{4} J_{im}$$

$$J_{im} = -\frac{1}{(\alpha+1)(\alpha+2)} \int_V \delta(1-\sum x_j)\,\delta(x_i)\,\delta(x_m) \times \frac{e_{im}^{\alpha+1}}{d_{im}^{\alpha+2}} \left(\frac{d_{im}\mathcal{D}}{e_{im}}\right)^{\alpha+2} F(1,\alpha+2,\alpha+3,\frac{d_{im}\mathcal{D}}{e_{im}})\, d^4x,$$

where

- d_{im}: independent of x, brought from projective transformations.
- e_{im}: **quadratic form** of x.

Separation of the leading pole

- For $\alpha + 2 = -\epsilon \to +0$ (for the scalar integration),

$$F(1, \alpha+2, \alpha+3; z) = 1 + \frac{\alpha+2}{\alpha+3} z\, F(1, \alpha+3, \alpha+4; z).$$

- For J_{12} $(x_1 = x_2 = 0,\ x_3 = y,\ x_4 = 1 - y)$

$$J_{12} = -\frac{1}{(\alpha+1)(\alpha+2)} \int_0^1 \frac{\mathcal{D}^{\alpha+2}}{e_{12}}\, dy - \frac{1}{(\alpha+1)(\alpha+3)} \int_0^1 \frac{e_{12}^{\alpha+1}}{d_{12}^{\alpha+2}} \left(\frac{d_{12}\mathcal{D}}{e_{12}}\right)^{\alpha+3} F(1, \alpha+3, \alpha+4, \frac{d_{12}\mathcal{D}}{e_{12}})\, dy.$$

- The first term $\Rightarrow$ leading pole $\Rightarrow$ expressed by F_D.

  When $e_{12} \propto \mathcal{D} \Rightarrow 1/\epsilon^2$.

  The second term

  Argument of F: rational expression (quadratic)/(quadratic).

Partial integration

- Primitive function of $e_{12}^{\alpha+1}$ $\Rightarrow$ partial integration.
- Product of linear terms with roots of $e_{12}(y) = 0$:

$$e_{12}(y) = \tilde{e}_{12}(y - y_+)(y - y_-)$$

- Primitive function.

$$e_{12}(y)^{\alpha+1} = \frac{dh(y)}{dy},$$
$$h(y) = -\frac{\tilde{e}_{12}(y_+ - y_+)^{\alpha+1}(y_- - y)^{\alpha+2}}{\alpha+2} \times F(-\alpha-1, \alpha+2; \alpha+3; \frac{y - y_-}{y_+ - y_-}).$$

Argument of F : linear for y.

Result of partial integration

- Result of partial integration

$$J_{12} = J_{12D} + J_{12S} + J_{12I}$$

$$J_{12D} := -\frac{1}{(\alpha+1)(\alpha+2)} \int_0^1 \frac{\mathcal{D}^{\alpha+2}}{e_{12}}\, dy$$

$$J_{12S} := -\frac{1}{(\alpha+1)(\alpha+3)d_{12}^{\alpha+2}} \left[h(y) \left(\frac{d_{12}\mathcal{D}}{e_{12}} \right)^{\alpha+3} F(1, \alpha+3, \alpha+4, \frac{d_{12}\mathcal{D}}{e_{12}}) \right]_0^1$$

$$J_{12I} := -\frac{1}{\alpha+1} \int_0^1 h(y) \left(\frac{\mathcal{D}}{e_{12}} \right)^{\alpha+2} P(y)\, dy,$$

$$P(y) := \frac{d}{dy} \log \left(\frac{e_{12} - d_{12}\mathcal{D}}{e_{12}} \right)$$ (sum of 1/(linear)).

- J_{12D} : leading pole term $\Rightarrow F_D$.
- J_{12S} : surface term $\Rightarrow F \times F$.
- J_{12I} : to be integrated.

Last integration

- Divide integration domain into $[0, y_-]$ and $[y_-, 1]$ and re-scale them to $[0, 1]$.
- Use integral representation of F.
- The result is (w, z:new variable, u_0:const.)

$$J_{12I} = \text{const.} \int_0^1 dw \int_0^{1-w} dz \left(\frac{\mathcal{D}(y_- w)}{(1-w)\{1-u_0(1-w)\}} \right)^{\alpha+2} \times P(y_- w) z^{\alpha+1} (1-u_0 z)^{\alpha+1} + \text{(one more similar term)}.$$

  - 2 dim. integration.
  - Integrand is a product of powers of linear factors.
  - Integration domain is a simplex.

⇒ Aomoto-Gelfand hypergeometric function.

Aomoto-Gelfand HGF

Integral representation:

$$F((\alpha_i),(\beta_j),\gamma;x)$$
$$= \frac{\Gamma(\gamma)}{\Gamma(\gamma-\sum_{i=1}^{n}\alpha_i)\prod_{i=1}^{n}\Gamma(\alpha_i)}\int_{\Delta^n}\prod_{i=1}^{n}u_i^{\alpha_i-1}\cdot\left(1-\sum_{i=1}^{n}u_i\right)^{\gamma-\sum_{i=1}^{n}\alpha_i-1}$$
$$\cdot\prod_{j=1}^{m-n-1}\left(1-\sum_{i=1}^{n}x_{ij}u_i\right)^{-\beta_j}d^n u.$$

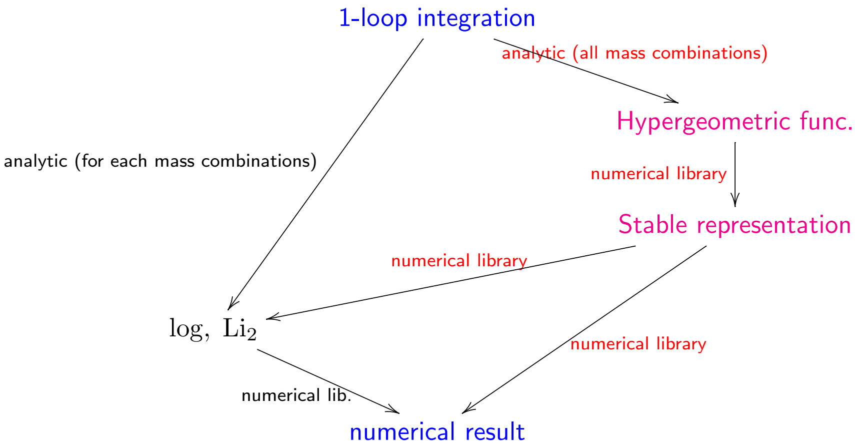
- Integrand : product of power of linear term.
- Integration domain : n-dimensional simplex Δ^n.
- Variable x : $m\times(m-n-1)$ matrix;
  Indices ν : $m\times(m-n-1)$ integer matrix;
  α : n-dim. vector; β : $(m-n-1)$-dim. vector.
- See: K. Aomoto and M. Kita, *"Theory of Hypergeometric Functions"*, Springer, 2011.

4-point function: summary

- General 4-point function is expressed by Aomoto-Gelfand hypergeometric functions and F_D for both massive and massless cases.
- Up to $\mathcal{O}(\epsilon^0)$ ($d = 4 - 2\epsilon$ for d-dimensional space-time), 4-point funct. is expressed by F_D.
- With further expansion in terms of ϵ, the expression is written with $\log$ and Li_2.
- Coefficients of $1/\epsilon^2$ and $1/\epsilon$ terms should cancel for massive case.

  $\Rightarrow$ usable for check.

Plan


1-loop integration
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log, Li2
numerical library
numerical lib.
numerical result

Numerical library

- It is hard to construct general numerical package to calculate F and F_D. (e.g. F includes Li_n for all n)
  $\Leftarrow$ We need values only for some special combination of parameters.
- Sample numerical calculation for 4 point functions:
  - All particles are massless.
  - At least one external particle is on-shell ($p_1^2 = 0$).
  - They have IR divergences (poles of ϵ).
  - Calculate up to $\mathcal{O}(\epsilon^0)$.
  - Tensor integrations up to rank $=$ 4 with analytic expressions.
- Library written in fortran90 (under development)
  1. Entry points of subroutines : F_D or F with parameters $a + b\epsilon$.
  2. Subroutines return : arrays of coefficients (a_n) in $F_D = a_{-2}/\epsilon^2 + a_{-1}/\epsilon + a_0 + \cdots$ up to necessary order.

Comparison

- 7560 points in physical and unphysical region, including tensor integrations (rank = 0, ..., 4).
- Compared with `golem95`.
- The maximum relative errors (measured by the distance on the complex plane) on these points in comparison with `golem95`

		maximum error
program-1(d)	program-2(d)	7.65×10^{-7}
program-1(d)	golem95(d)	9.13×10^{-10}
program-1(d)	program-1(q)	3.98×10^{-10}
golem95(d)	golem95(q)	5.17×10^{-10}
program-1(q)	golem95(q)	1.38×10^{-18}

(d) : double precision, (q) : quadruple precision.

program-1 : sample program

program-2 : last integration $\Rightarrow$ numerical

- Accuracy of the library is similar to golem95 package.

Summary

- 2-, 3-point functions are expressed with F_D, exactly for any combination of physical parameters and any space-time dimensions.
- 4-point functions are expressed with Aomoto-Gelfand hypergeometric functions for any combination of physical parameters.

  Up to $\mathcal{O}(\epsilon^0)$, they are expressed with F_D.
- A program library of F and F_D is under developing.
- Sample numerical calculations of box integration for massless QCD with IR divergences agree with `golem95` package.

Discussion

- GKZ-hypergeometric functions.
  - I.M.Gel'fand, A.V.Zelevinsky, M.M.Kapranov, *Funk. Anal. Appl.* **23** (1989), 94–106.
  - I.M.Gel'fand, A.V.Zelevinsky, M.M.Kapranov, *Adv. in Math.* **84** (1990), 255-271
- Power series (Γ-series) expansions, differential equations and integral representations are known.
- Integral representations:

$$F_\sigma(\alpha, \beta; P) = \int_\sigma \prod_j P_j(x_1, ..., x_n)^{\alpha_i}\ x_1^{\beta_1} \cdots x_n^{\beta_n}\ dx_1 \cdots dx_n,$$

where P_j are (Laurent) polynomials.

They say "practically all integrals which arise in quantum field theory."

Thank you!