New approaches for numerical techniques in higher order calculations

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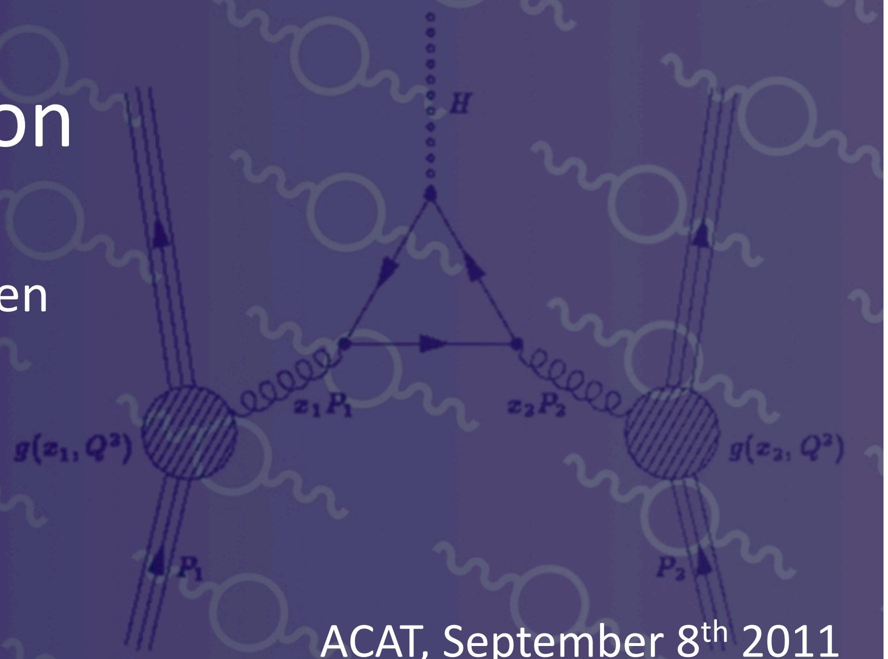


ACAT, September 8th 2011

Numerical techniques ?

- Discuss only problems relevant to next-to-next-to-leading order and higher
- For next-to-leading order techniques refer to talk by F. Tramontano
- What counts as a numerical approach ?

Plan

I. Virtual corrections
   1) Sector Decomposition + contour deformation
   2) Mellin-Barnes representations + contour deformation
   3) Differential equations

II. Real corrections
   1) SecToR Improved Phase sPacE for Real Radiation
   2) Non-linear variable transformations

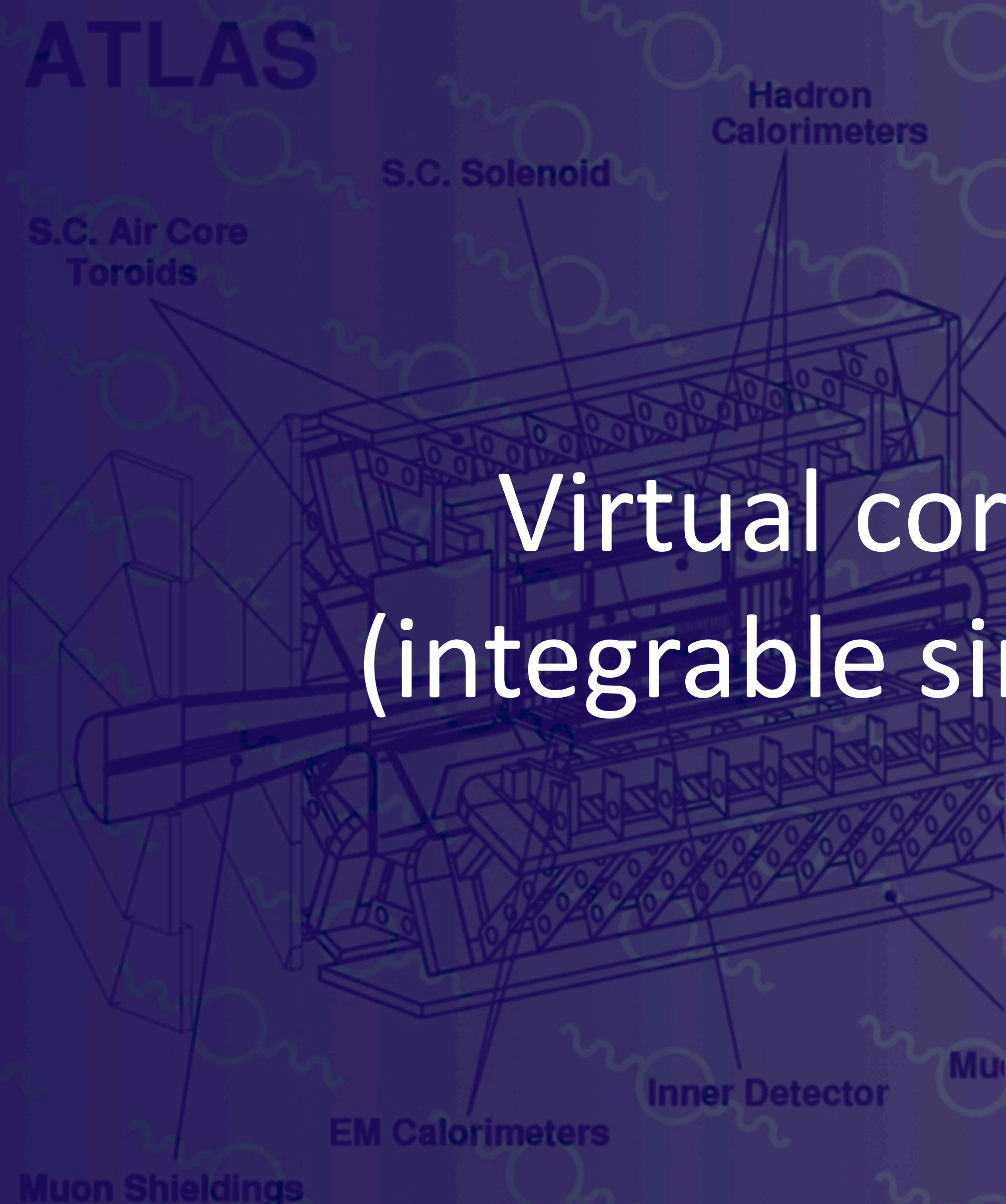
A view on higher order calculations

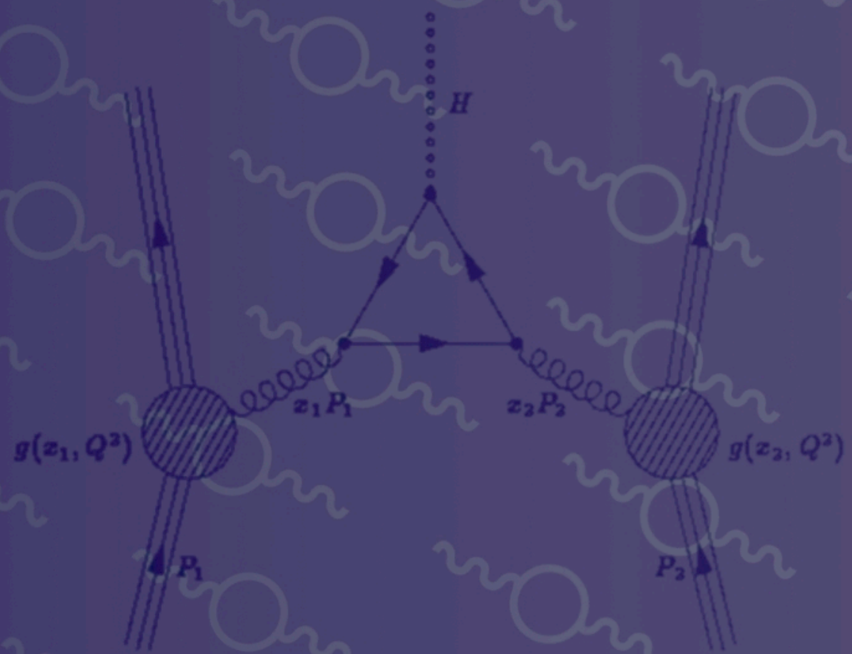
- Main problem: integration over singular integrands
- Two types of singularities:
    1. regularized, but actual – Laurent expansions in the dimensional regularization parameter
    2. non-regularized, but integrable – Minkowski region thresholds
- Two types of problems:
    1. virtual corrections
    2. real corrections

"In computer programming, the technique of choice is not necessarily the most efficient, or elegant, or fastest executing one. Instead, it may be the one that is quick to implement, general, and easy to check."

Numerical Recipes: The Art of Scientific Computing

Virtual corrections
(integrable singularities)





Sector Decomposition

Historical development of the method:

T. Binoth, G. Heinrich ’00 – sector decomposition for virtual graphs
Z. Nagy, D. Soper ’06 – contour deformation for one-loop graphs

Progress in numerical evaluation:

Ch. Anastasiou, S. Beerli, A. Daleo ’07

Not discussed: sector decomposition basics – see talk by J. Carter

Sector Decomposition

Generalities:

polynomials

• Feynman parameterization of integrals

$$I = C(\epsilon) \lim_{\delta\to 0} \int_0^1 dx_1 \cdots dx_n \frac{\mathcal{F}(\vec{x},\epsilon)}{[\mathcal{G}(\vec{x}, M_i^2, s_{kl}) - i\delta]^{\alpha+n_L\epsilon}}$$

• after sector decomposition

explicit singularity sources

$$I_s = C(\epsilon) \lim_{\delta\to 0} \int_0^1 \frac{dx_1 \cdots dx_n x_1^{-\alpha_1+\beta_1\epsilon} \cdots x_n^{-\alpha_n+\beta_n\epsilon} \mathcal{F}_s(\vec{x},\epsilon)}{[\mathcal{G}_s(\vec{x}, M_i^2, s_{kl}) - i\delta]^{\alpha+n_L\epsilon}}$$

• while singularities in ε almost explicit, integration problems due to singularities inside the integration region

• contour deformation inducing the same imaginary part as –iδ at small λ avoids problems

$$z_i = x_i - i\lambda x_i(1-x_i)\frac{\partial \mathcal{G}_s}{\partial x_i} \qquad \mathcal{G}_s(\vec{z}) = \mathcal{G}_s(\vec{x}) - i\lambda \sum_i x_i(1-x_i)\left(\frac{\partial \mathcal{G}_s}{\partial x_i}\right)^2 + \mathcal{O}(\lambda^2)$$

• starting point for the extraction of singularities

$$\int_0^1 \frac{\left(\prod_{j=1}^n dx_j x_j^{-\alpha_j+\beta_j\epsilon}\right) \mathcal{F}_s(\vec{x},\epsilon)}{[\mathcal{G}_s(\vec{x}, M_i^2, s_{kl}) - i\delta]^{\alpha+n_L\epsilon}} = \int_C \frac{\left(\prod_{j=1}^n dz_j z_j^{-\alpha_j+\beta_j\epsilon}\right) \mathcal{F}_s(\vec{z},\epsilon)}{[\mathcal{G}_s(\vec{z}, M_i^2, s_{kl})]^{\alpha+n_L\epsilon}}$$

Sector Decomposition

Generalities:

- change of variables

$$I_s = C(\epsilon)\int_0^1 \prod_{j=1}^{n} dx_j z_j^{-\alpha_j+\beta_j\epsilon}\,\mathcal{J}(\vec{x}\to\vec{z})\mathcal{L}(\vec{z}(\vec{x}),\epsilon)$$
$$= C(\epsilon)\int_0^1 \prod_{j=1}^{n} dx_j x_j^{-\alpha_j+\beta_j\epsilon}\left(\frac{z_j}{x_j}\right)^{-\alpha_j+\beta_j\epsilon}\mathcal{J}(\vec{x}\to\vec{z})\mathcal{L}(\vec{z}(\vec{x}),\epsilon)$$

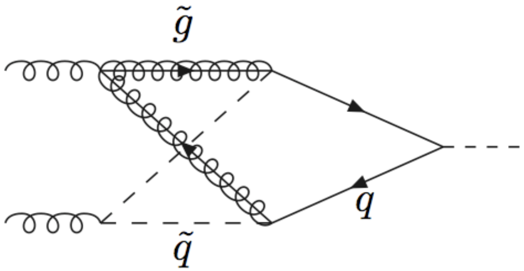
- extraction of singularities

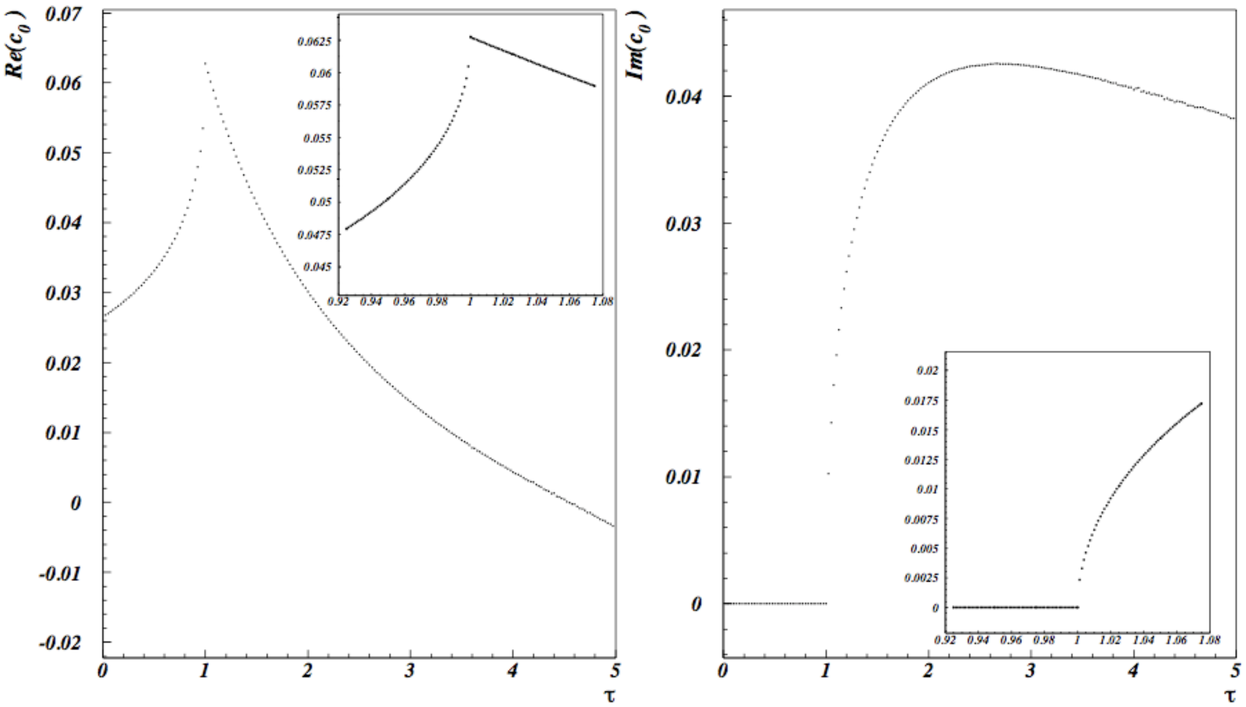
$$\int_0^1 dx x^{-n+\epsilon} f(x) = \int_0^1 dx x^{\epsilon}\frac{f(x)-\sum_{k=0}^{n-1} x^k \frac{f^{(k)}(0)}{k!}}{x^n} + \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!(k+1-n+\epsilon)}$$

- ready for numerical evaluation

Sector Decomposition

Example: multiscale vertex graph





Sector Decomposition

Available software:

FIESTA2 (Smirnov, Smirnov, Tentyukov ‘09)
sector_decomposition (Bogner, Weinzierl ’07)
CSectors (Gluza, Kajda, Riemann, Yundin ’10) – interface to sector_decomposition
SecDec (Carter, Heinrich ‘11) – SOON WITH CONTOUR DEFORMATION

Mellin-Barnes Representations

Historical development of the method:

M.C. Bergere, Y.-M.P. Lam ’74 – Mellin transforms and related techniques for Feynman integrals
N.I. Ussyukina ‘75, A.I. Davydychev ’89 – analytic evaluation of two-loop graphs
V.A. Smirnov ‘99, J.B. Tausk ’99 – resolution of singularities

Progress in numerical evaluation:

A. Freitas, Y.-C. Huang ‘10

Not discussed: resolution of singularities – implemented in automated packages

Mellin-Barnes Representations

Example: a general one-loop graph (in Feynman parameterization)

$$I_1 = \int \frac{d^D q}{i\pi^{D/2}} \left[[q^2 - m_0^2]^{\mu_0} [(q+p_1)^2 - m_1^2]^{\mu_1} \cdots [(q+p_1+p_2+\ldots+p_n)^2 - m_n^2]^{\mu_n} \right]^{-1}$$

$$I_1 = (-1)^M \frac{\Gamma(M-D/2)}{\Gamma(\nu_0)\cdots\Gamma(\nu_n)} \int_0^1 dx_0 \cdots dx_n \, \delta(1 - x_0 - \ldots - x_n)$$
$$\times \frac{x_0^{\mu_0 - 1} \cdots x_n^{\mu_n - 1}}{\left[-\sum_{i,j=1}^n (p_i - p_j)^2 x_i x_j - \sum_{i=1}^n (p_i^2 x_i^2 - m_i x_i) + m_0 x_0 - i\epsilon \right]^{M-D/2}}$$

- Split addition signs with

$$\frac{1}{(A_0 + \ldots + A_m)^Z} = \frac{1}{(2\pi i)^m} \int_{\mathcal{C}_1} dz_1 \cdots \int_{\mathcal{C}_m} dz_m \, A_1^{z_1} \cdots A_m^{z_m} A_0^{-Z-z_1-\ldots-z_m}$$
$$\times \frac{\Gamma(-z_1)\cdots\Gamma(-z_m)\Gamma(Z+z_1+\ldots+z_m)}{\Gamma(Z)}$$

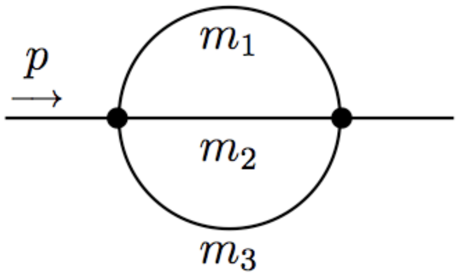
- Integrate

$$\int_0^1 dx_0 \cdots dx_n \, \delta(1 - x_0 - \ldots - x_n) \, x_0^{\alpha_0 - 1} \cdots x_n^{\alpha_n - 1} = \frac{\Gamma(\alpha_0)\cdots\Gamma(\alpha_n)}{\Gamma(\alpha_0 + \ldots + \alpha_n)}$$

- Generalize to arbitrary loops

Mellin-Barnes Representations

Example: the sunrise graph at two-loops



$$\frac{-1}{(2\pi i)^3}\int dz_1 dz_2 dz_3\, (m_1^2)^{-\varepsilon - z_1 - z_2} (m_2^2)^{z_2} (m_3^2)^{1-\varepsilon+z_1-z_3} (-p^2)^{z_3}\, \Gamma(-z_2)\Gamma(-z_3)$$
$$\times\, \Gamma(1+z_1+z_2)\Gamma(z_3-z_1)\frac{\Gamma(1-\varepsilon-z_2)\Gamma(\varepsilon+z_1+z_2)\Gamma(\varepsilon-1-z_1+z_3)}{\Gamma(2-\varepsilon+z_3)}.$$

- Analytic continuation needed in ε

- Numerical integration possible due to the behaviour of gamma functions

$$\Gamma(a+ib) \simeq \sqrt{2\pi}\; e^{i\frac{\pi}{4}(2a-1)}\; e^{ib\,(\log b - 1)}\; e^{-\frac{b\pi}{2}}\; b^{a-1/2},$$

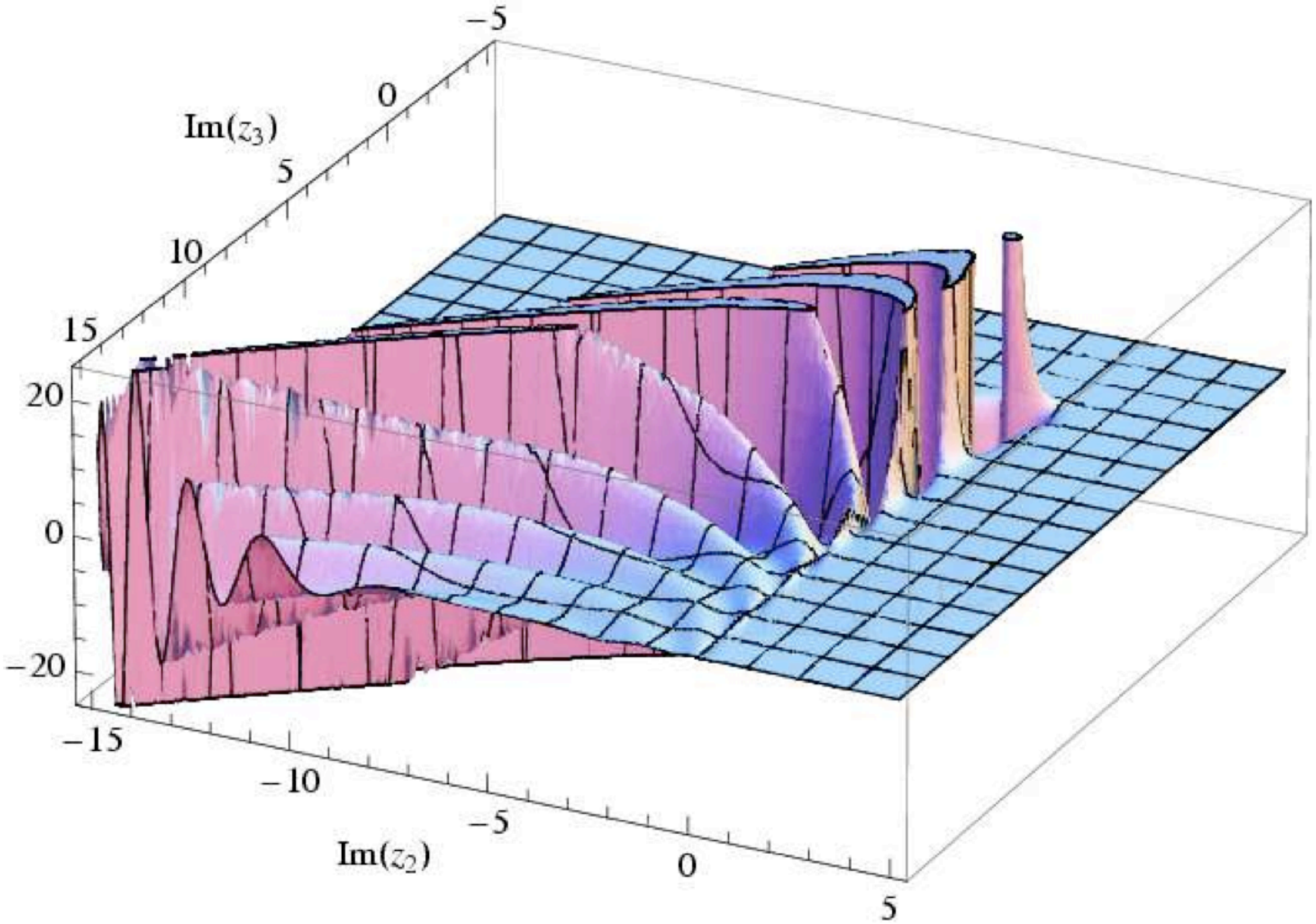
$$\Gamma(a-ib) \simeq \sqrt{2\pi}\; e^{-i\frac{\pi}{4}(2a-1)}\; e^{-ib\,(\log b - 1)}\; e^{-\frac{b\pi}{2}}\; b^{a-1/2}$$

- Problems may be caused by kinematic invariants

$$(-p^2)^{z_3} = (p^2)^{c_3+iy_3}(-1-i\epsilon)^{c_3+iy_3} = (p^2)^{c_3+iy_3} e^{-i\pi c_3} e^{\pi y_3}$$

Mellin-Barnes Representations

Example: the sunrise graph at two-loops



Mellin-Barnes Representations

Idea of Freitas & Huang:

$$c_i + iy_i \to c_i + (\theta + i)y_i.$$

cancel the exponent

$$(-p^2)^{z_3} = (p^2)^{c_3+iy_3} e^{-i\pi(c_3+\theta y_i)} e^{(\pi+\theta \log p^2)y_3}$$

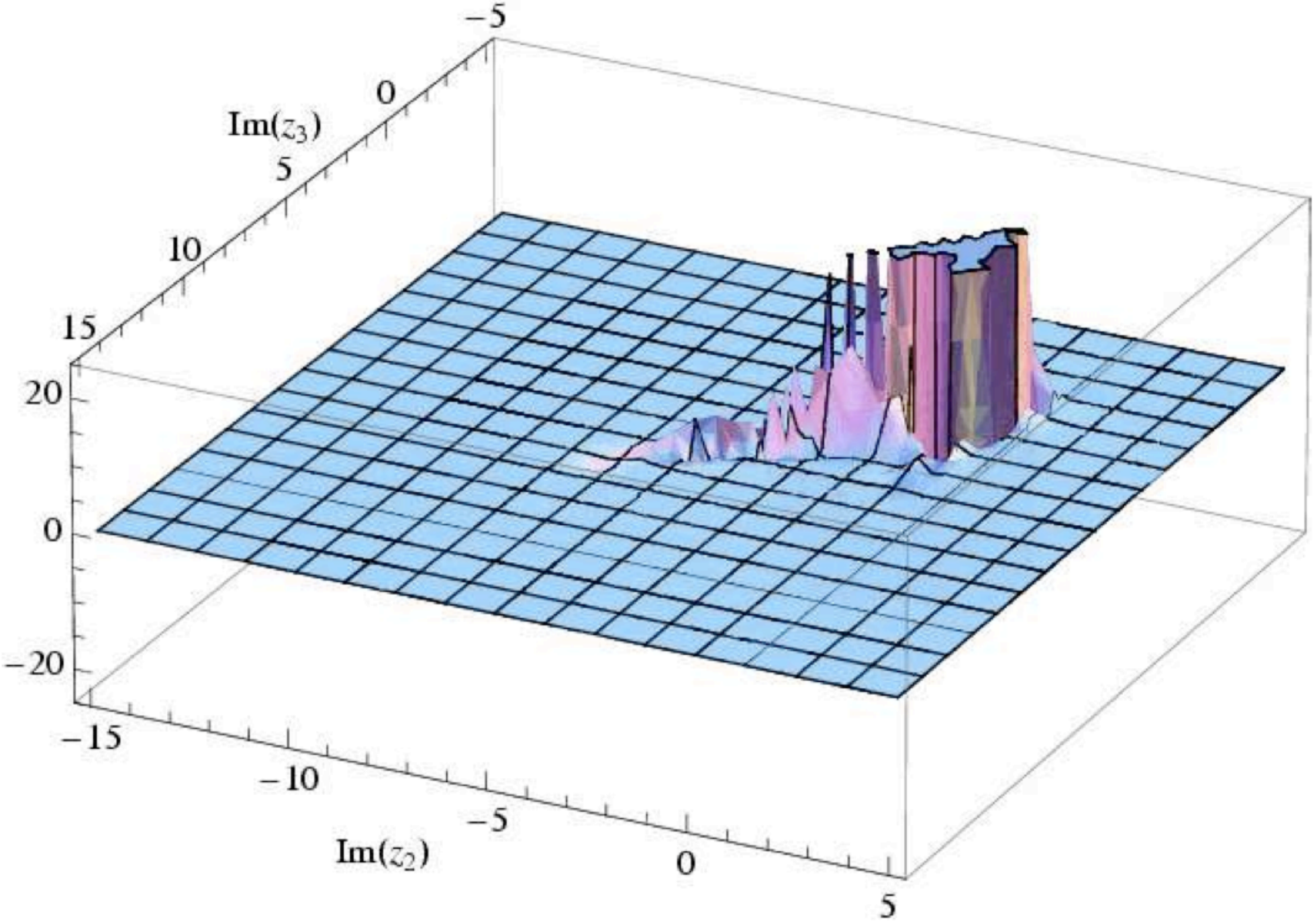
- Generalization for many variables:

$$\begin{aligned}
y_1 &= r\cos\phi_1, \\
y_2 &= r\sin\phi_1\cos\phi_2, \\
\vdots & \qquad\quad \vdots \\
y_{n-1} &= r\sin\phi_1\cdots\sin\phi_{n-2}\cos\phi_{n-1}, \\
y_n &= r\sin\phi_1\cdots\sin\phi_{n-2}\sin\phi_{n-1}.
\end{aligned}$$

- Transformation applied only to r

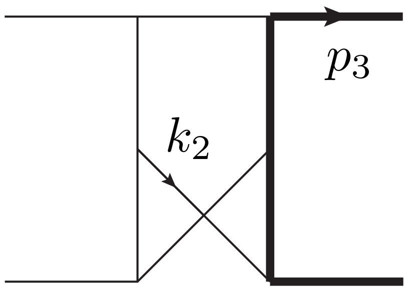
Mellin-Barnes Representations

Example: the sunrise graph at two-loops



Mellin-Barnes Representations

Example: two-loop non-planar box graph with massive lines



$\times (k_2 \cdot p_3)$

Kinematic point near threshold
(there is no singularity here, though)

$$m^2/s = 0.2458180106$$
$$-t/s = 0.4983674794$$

- Result of a numerical integration of a 7-fold Mellin-Barnes representation

$$\frac{0.83362}{\epsilon^4} - \frac{0.00905406 - 3.14166i}{\epsilon^3} - \frac{4.95519 + 0.174758i}{\epsilon^2} + \frac{36.0863 - 31.2154i}{\epsilon} + (283.622 - 63.0273i)$$

- Error estimates for real and imaginary parts of higher dimensional integrations

$$\frac{2.1 \times 10^{-8}}{\epsilon^2} + \frac{1.2 \times 10^{-5}}{\epsilon} + 0.29 \qquad \frac{2.1 \times 10^{-8}}{\epsilon^2} + \frac{3.9 \times 10^{-6}}{\epsilon} + 0.35$$

- For comparison: high precision result from differential equation integration (next method)

$$\frac{0.833620}{\epsilon^4} - \frac{0.00905 - 3.14166i}{\epsilon^3} - \frac{4.95519 + 0.17476i}{\epsilon^2} + \frac{36.0863 - 31.2154i}{\epsilon} + (283.674 - 62.967i)$$

Mellin-Barnes Representations

Available software:

http://projects.hepforge.org/mbtools/ - tools for Mellin-Barnes integrals

MB.m (MC '05) – numerical integration SOON WITH CONTOUR DEFORMATION
MBasymptotics.m (MC '06)
MBresolve (Smirnov, Smirnov '09)
AMBRE.m (Gluza, Kajda, Riemann '07)
barnesroutines.m (Kosower '07)

Differential Equations

Example of a special function: the hypergeometric ${}_2F_1$ function

$$
{}_2F_1(a,b,c;z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots
+ \frac{a(a+1)\ldots(a+j-1)b(b+1)\ldots(b+j-1)}{c(c+1)\ldots(c+j-1)}\frac{z^j}{j!} + \cdots
$$

- Differential equation defining the analytic continuation

$$
z(1-z)F'' = abF - [c-(a+b+1)z]F'
$$

- System of equations parameterized with a straight line

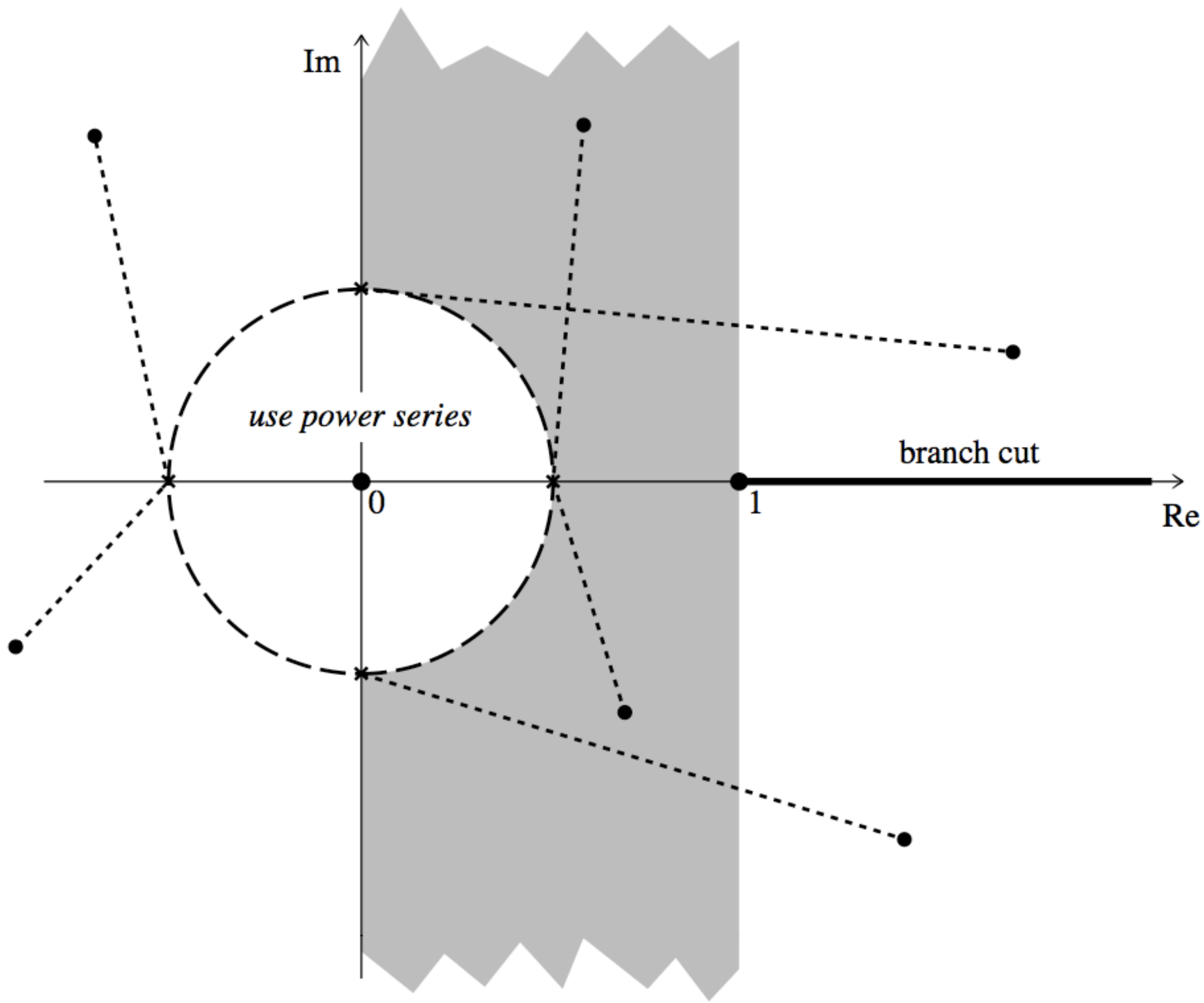
$$
z(s) = z_0 + s(z_1 - z_0)
$$

$$
\frac{dF}{ds} = (z_1 - z_0)F'
$$

$$
\frac{dF'}{ds} = (z_1 - z_0)\left(\frac{abF - [c-(a+b+1)z]F'}{z(1-z)}\right)
$$

Differential Equations

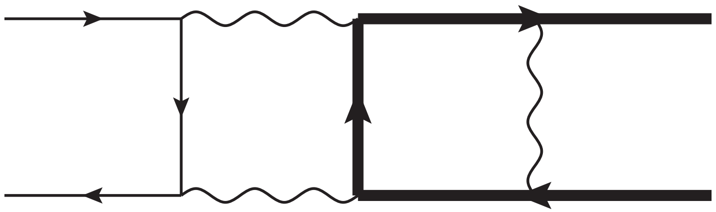
Example of a special function: the hypergeometric ${}_2F_1$ function



Differential Equations

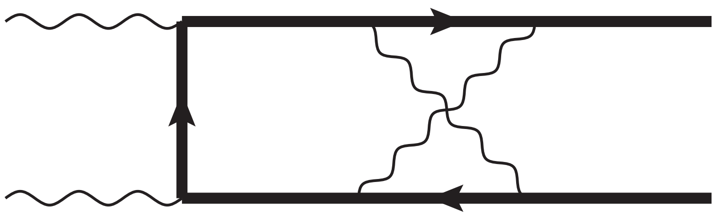
Example of a Feynman diagram calculation: virtual corrections in top quark pair production

• Computational complexity in the quark channel



190 diagrams
2812 integrals
145 master integrals

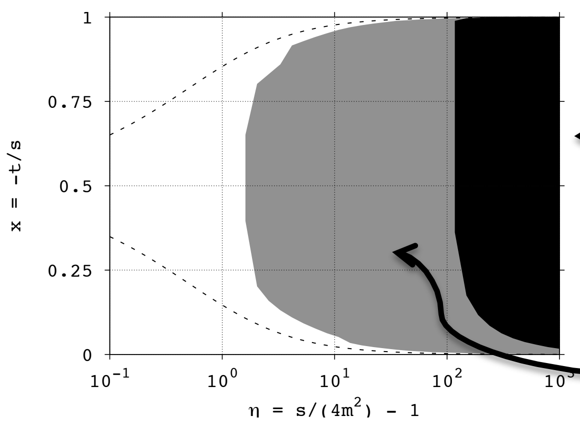
• Computational complexity in the gluon channel



726 diagrams
8676 integrals
422 master integrals

• Convergence of a series expansion around the small mass limit in the quark channel

MC '08

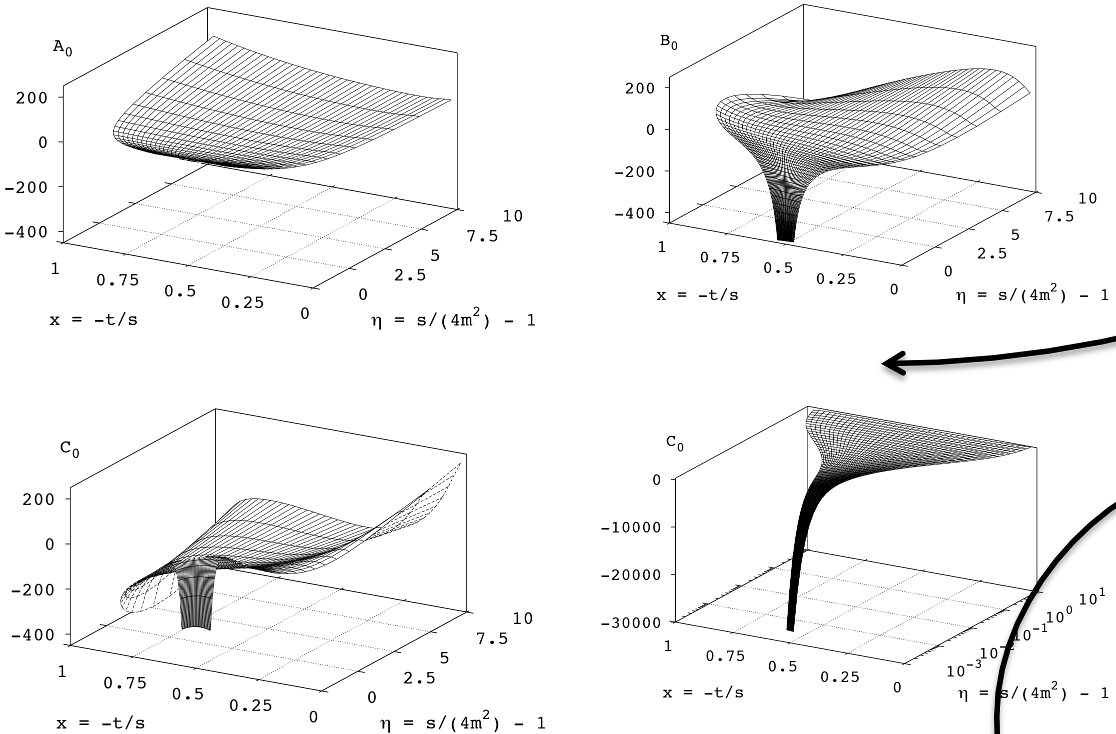


1% accuracy
MC, Mitov, Moch '07

0.1% from the expansion

Differential Equations

Example of a Feynman diagram calculation: virtual corrections in top quark pair production



quark channel results

MC '08

$$m^2 = 0.2s,\ t = -0.45s$$

	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
A	0.22625	1.391733154	-2.298174307	-4.145752449	17.37136599
B	-0.4525	-1.323646320	8.507455541	6.035611156	-35.12861106
C	0.22625	-0.06808683395	-18.00716652	6.302454931	3.524044913
D_l		-0.22625	0.2605057339	-0.7250180282	-1.935417247
D_h			0.5623350684	0.1045606449	-1.704747998
E_l		0.22625	-0.3323207300	7.904121951	2.848697837
E_h			-0.5623350684	4.528240788	12.73232424
F_l					-1.984228442
F_{lh}					-2.442562819
F_h					-0.07924540546

Differential Equations

Example of a Feynman diagram calculation: virtual corrections in top quark pair production

Bonciani, Ferroglia, Gehrmann, von Manteuffel, Studerus '10

MC, Bärnreuther, in preparation

Ferroglia, Neubert, Pecjak, Yang '09

gluon channel result

	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
A_{LC}	10.74942557	18.69389337	-156.8237244	262.1482588	12.72180680
A	10.74942557	18.69389337	-156.8237244	262.1482588	12.72180680
B	-21.28599123	-55.99039551	-235.0412564	1459.833288	-509.6019155
C		-6.199051597	-68.70297402	-268.1060373	804.0981895
D			94.08660818	-130.9619794	-283.3496755
E_l		-12.54099650	18.20646589	27.95708293	-112.6060988
E_h			0.012907497	11.79259573	-47.68412574
F_l		24.83365643	-26.60868620	-50.75380859	125.0537955
F_h			0.0	-23.32918072	132.5618962
G_l			3.099525798	67.04300456	-214.1081462
G_h				0.0	-179.3374874
H_l			2.388761238	-5.452031425	3.632861953
H_{lh}				-0.004302499	-3.945712447
H_h					0.00439856
I_l			-4.730220272	10.81032548	-7.182940516
I_{lh}				0.0	7.780900470
I_h					0.0
A^{Poles}	10.749	18.694	−156.82	262.15	
B^{Poles}	−21.286	−55.990	−235.04	1459.8	
C^{Poles}		−6.1991	−68.703	−268.11	
D^{Poles}			94.087	−130.96	
E_l^{Poles}		−12.541	18.207	27.957	
E_h^{Poles}			0.012908	11.793	
F_l^{Poles}		24.834	−26.609	−50.754	
F_h^{Poles}			0.0	−23.329	
G_l^{Poles}			3.0995	67.043	
G_h^{Poles}				0.0	
H_l^{Poles}			2.3888	−5.4520	
H_{lh}^{Poles}				−0.0043025	
H_h^{Poles}					
I_l^{Poles}			−4.7302	10.810	
I_{lh}^{Poles}				0.0	
I_h^{Poles}					

Differential Equations

Example of a Feynman diagram calculation: virtual corrections in top quark pair production

- The original idea to integrate numerically a system of differential equations for Feynman integrals stems from Czyz, Caffo, Remiddi '02

- Differential equations are obtained by reduction of the derivatives of the masters

$$m^2 \frac{d}{dm^2} M_i \left(\frac{m^2}{s}, -\frac{t}{s}, \epsilon \right) = \sum_j C_{ij} \left(\frac{m^2}{s}, -\frac{t}{s}, \epsilon \right) M_j \left(\frac{m^2}{s}, -\frac{t}{s}, \epsilon \right)$$

- They require expansion in ε for numerical evaluation

- The boundary conditions are obtained from the series expansion at small mass

- Values at arbitrary points are obtained by integrating the system of differential equations using contours in the complex plane in the mass and in the Mandelstam t variable

- In practice a professional Fortran system, ODEPACK (ZVODE), is used

- It is possible to include any fixed numerical precision by using available packages or native quadruple precision of Fortran compilers

- High numerical precision is needed due to cancellations

Differential Equations

Example of a Feynman diagram calculation: virtual corrections in top quark pair production

- Singularities of the differential equations in the quark channel

Jacobian singularity	branching	allowed	interpretation
$m_s = 0$	yes		collinear singularity
$m_s = 1/4$	yes		s-channel threshold
$m_s = -1/4$			
$x = 0$	yes		t-channel threshold
$x = 1$	yes		u-channel threshold
$x = 1/2$		yes	perpendicular scattering
$m_s = x\,(1-x)$			forward/backward scattering
$m_s = x$			
$m_s = 1-x$			
$m_s = -x$			
$m_s = x-1$			
$m_s = 1/2\,x\,(1-x)$		yes	
$m_s = 1/2\,x$		yes	
$m_s = 1/2\,(1-x)$		yes	
$m_s = 1/2\,(1-x^2)$			
$m_s = -1/2\,(1-x)^2$			

Differential Equations

Example of a Feynman diagram calculation: virtual corrections in top quark pair production

- Efficiency in the quark channel

	leading color			full color		
number of masters	36			145		
number of functions	155			595		
precision	quadruple		double	quadruple		double
evolution in m_s						
requested local error	10^{-20}	10^{-12}	10^{-12}	10^{-20}	10^{-12}	10^{-12}
contour deformation δm_s	0.1	0.1	0.1	0.1	0.1	0.1
number of steps taken	2319	618	534	2932	777	1302
Jacobian evaluation time [ms]	3.4	3.4	0.2	37	37	4.9
evolution in x						
requested local error	10^{-18}	10^{-10}	10^{-10}	10^{-18}	10^{-10}	10^{-10}
contour deformation δx	0.1	0.1	0.1	0.1	0.1	0.1
number of steps taken	545	139	139	739	174	432
Jacobian evaluation time [ms]	8.3	8.3	0.4	150	150	17
total evaluation time [s]	49	13	< 1	957	243	26

Differential Equations

When is the method useful ?

• When differential equations can be derived, which may be a problem with many-scales computations, thus the primary use is for one- and two-scale applications

• When boundary conditions can be obtained, which is often possible with the help of diagrammatic expansions, but can sometimes require other methods, such as expansion of Mellin-Barnes representations in the top quark pair production case

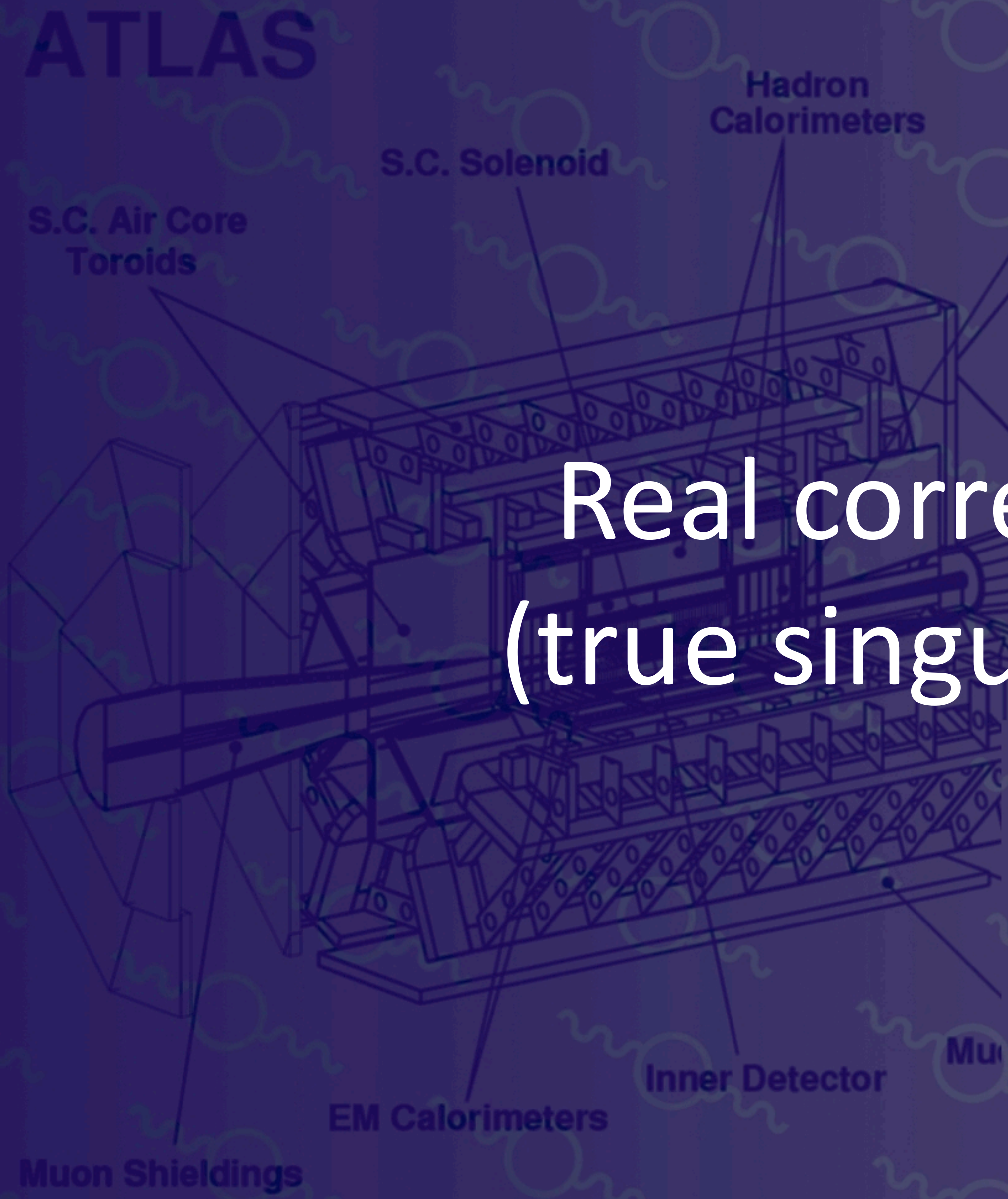
Main advantage:

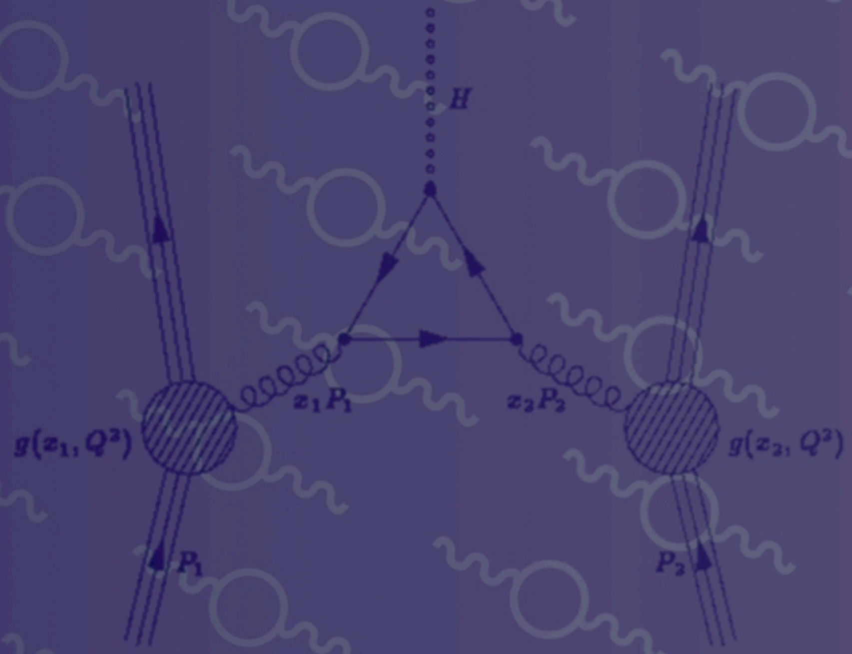
• high precision reachable in contrast to high dimensional Monte Carlo integration

Available software:

http://www.netlib.org/odepack/
http://www.netlib.org/ode/zvode.f

Real corrections
(true singularities)





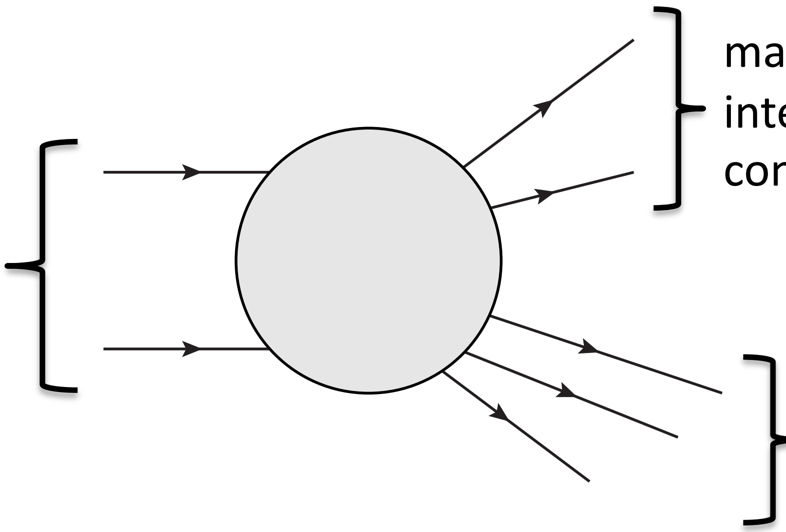
Subtraction Schemes

$$\sigma^{NLO} = \int_{m+1} \left[\left(d\sigma^R \right)_{\epsilon=0} - \left(d\sigma^A \right)_{\epsilon=0} \right] + \int_m \left[d\sigma^V + \int_1 d\sigma^A \right]_{\epsilon=0}$$

analytic integration ?

In this talk: progress at NNLO

massless initial state



massless final state to be integrated over unresolved configurations

arbitrary final state with resolved configurations

Subtraction Schemes

- NLO
  1) Catani-Seymour (smooth interpolation between limits, remapping of phase space allows for arbitrary phase space generators)
  2) FKS (Frixione-Kunszt-Signer) (decomposition of phase space according to collinear singularities, energy-angle parameterization, residue subtraction)
- NNLO general and successful
  1) Sector Decomposition (Binoth,Heinrich '04, Anastasiou,Melnikov,Petriello '05)
  2) Antenna Subtraction (Gehrmann De-Ridder, Gehrmann, Glover '05)
- NNLO special for colourless particles – Catani, Grazzini '07
- NNLO in the making (main problem - integration of subtraction terms)
  1) "generalized" Catani-Seymour – Weinzierl '03 (many others unfinished)
  2) "generalized" FKS – Somogyi, Trocsanyi, Del Duca '06
  3) "new stuff with variable change" - Anastasiou, Herzog, Lazopoulos '10

STRIPPER (SecToR ImProved Phase space for real Radiation) MC '10

STRIPPER Ideas

About the phase space:

1. parameterization of the massless system with energies and angles modified to allow for a description of all collinear singular configurations with only two variables

2. level 1 decomposition into sectors allowing for only one type of collinear singularities

3. level 2 decomposition into sectors defining the order of singular limits

About the subtraction terms:

1. Subtraction at the endpoint derived from known soft and collinear limits of QCD amplitudes

2. No analytic integration of the subtraction terms

Cross Sections for Top Production

$$\sigma^{RR}_{ab\to t\bar{t}cd}(s, m^2, \mu^2 = m^2, \alpha_s, \epsilon) = \frac{\alpha_s^4}{m^2} f_{ab\to t\bar{t}cd}(\beta, \epsilon)$$

$$\beta = \sqrt{1 - \frac{4m^2}{s}}$$

$gg \to t\bar{t}gg$

$gg \to t\bar{t}q\bar{q}$

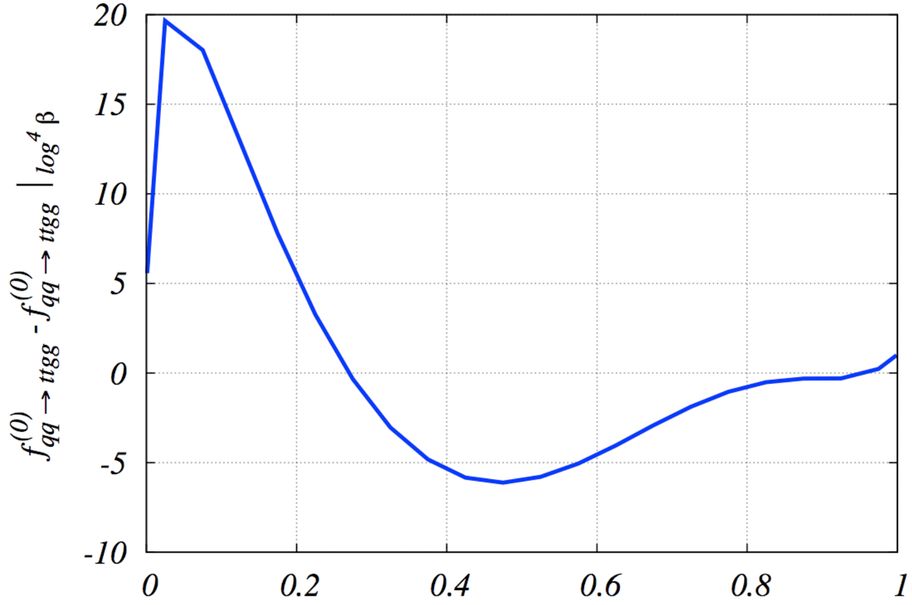
$q\bar{q} \to t\bar{t}gg$

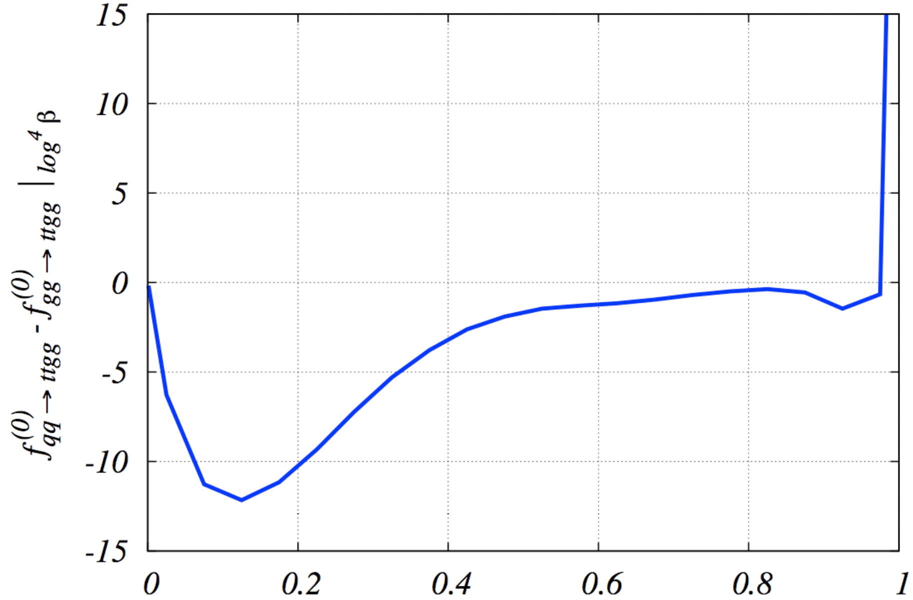
$q\bar{q} \to t\bar{t}q'\bar{q}', \; q' \neq q$

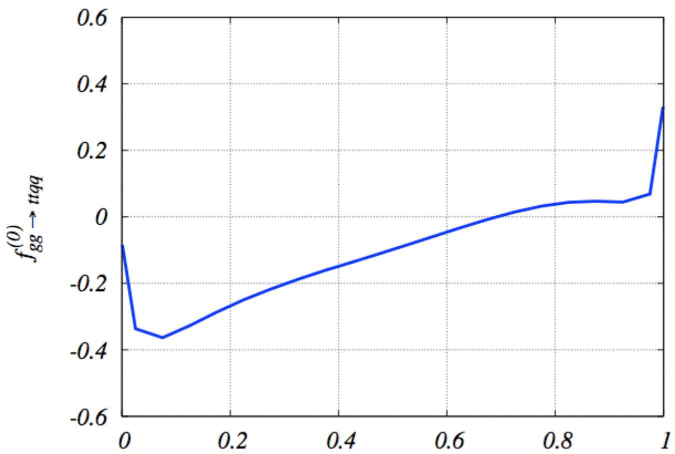
β	ϵ^{-1}			ϵ^{0}		
0.001	$+2.947 \times 10^{0}$	$\pm$	1.4×10^{-3}	$+5.218 \times 10^{1}$	$\pm$	2.9×10^{-2}
0.025	$+1.196 \times 10^{1}$	$\pm$	4.4×10^{-3}	$+1.143 \times 10^{2}$	$\pm$	4.6×10^{-2}
0.075	$+1.293 \times 10^{1}$	$\pm$	4.8×10^{-3}	$+8.660 \times 10^{1}$	$\pm$	3.6×10^{-2}
0.125	$+1.127 \times 10^{1}$	$\pm$	4.5×10^{-3}	$+5.975 \times 10^{1}$	$\pm$	3.0×10^{-2}
0.175	$+9.161 \times 10^{0}$	$\pm$	4.1×10^{-3}	$+3.949 \times 10^{1}$	$\pm$	2.9×10^{-2}
0.225	$+7.062 \times 10^{0}$	$\pm$	4.3×10^{-3}	$+2.454 \times 10^{1}$	$\pm$	2.6×10^{-2}
0.275	$+5.149 \times 10^{0}$	$\pm$	3.1×10^{-3}	$+1.375 \times 10^{1}$	$\pm$	1.9×10^{-2}
0.325	$+3.469 \times 10^{0}$	$\pm$	2.7×10^{-3}	$+6.118 \times 10^{0}$	$\pm$	1.6×10^{-2}
0.375	$+2.063 \times 10^{0}$	$\pm$	2.4×10^{-3}	$+9.998 \times 10^{-1}$	$\pm$	1.5×10^{-2}
0.425	$+9.289 \times 10^{-1}$	$\pm$	2.1×10^{-3}	-2.245×10^{0}	$\pm$	1.3×10^{-2}
0.475	$+7.292 \times 10^{-2}$	$\pm$	2.2×10^{-3}	-3.973×10^{0}	$\pm$	1.3×10^{-2}
0.525	-5.197×10^{-1}	$\pm$	2.7×10^{-3}	-4.575×10^{0}	$\pm$	1.1×10^{-2}
0.525	-5.197×10^{-1}	$\pm$	2.7×10^{-3}	-4.575×10^{0}	$\pm$	1.1×10^{-2}
0.575	-8.661×10^{-1}	$\pm$	1.7×10^{-3}	-4.393×10^{0}	$\pm$	1.1×10^{-2}
0.625	-9.995×10^{-1}	$\pm$	1.6×10^{-3}	-3.711×10^{0}	$\pm$	8.9×10^{-3}
0.675	-9.525×10^{-1}	$\pm$	1.5×10^{-3}	-2.772×10^{0}	$\pm$	8.9×10^{-3}
0.725	-7.695×10^{-1}	$\pm$	1.9×10^{-3}	-1.823×10^{0}	$\pm$	8.5×10^{-3}
0.775	-5.075×10^{-1}	$\pm$	1.1×10^{-3}	-1.023×10^{0}	$\pm$	5.6×10^{-3}
0.825	-2.310×10^{-1}	$\pm$	8.5×10^{-4}	-5.028×10^{-1}	$\pm$	4.5×10^{-3}
0.875	-2.577×10^{-2}	$\pm$	6.8×10^{-4}	-3.085×10^{-1}	$\pm$	3.5×10^{-3}
0.925	$+1.082 \times 10^{-3}$	$\pm$	5.2×10^{-4}	-2.996×10^{-1}	$\pm$	2.6×10^{-3}
0.975	-2.722×10^{-1}	$\pm$	3.0×10^{-4}	$+2.370 \times 10^{-1}$	$\pm$	1.5×10^{-3}
0.999	-2.158×10^{-1}	$\pm$	1.9×10^{-4}	$+9.970 \times 10^{-1}$	$\pm$	1.5×10^{-3}

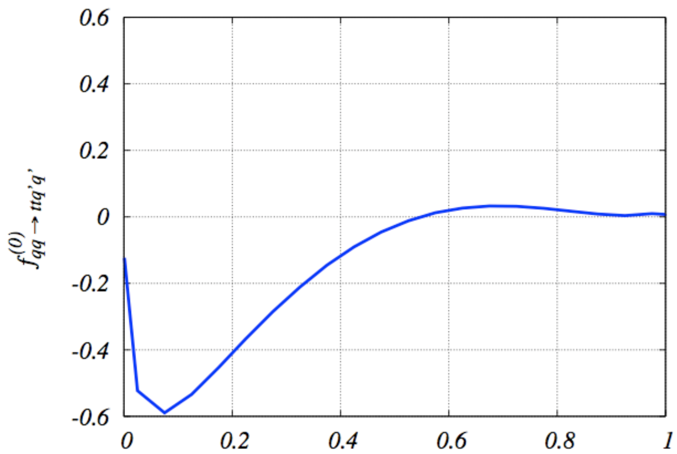
β	$f^{(0)}\vert_{10^{-7}} - f^{(0)}\vert_{10^{-6}}$	$\Delta f^{(0)}\vert_{10^{-7}} + \Delta f^{(0)}\vert_{10^{-6}}$	$f^{(\epsilon,0)}$
0.001	$+1.9 \times 10^{-2}$	4.8×10^{-2}	-3.7×10^{-8}
0.025	$+6.5 \times 10^{-2}$	8.8×10^{-2}	-3.0×10^{-5}
0.075	$+8.5 \times 10^{-2}$	7.1×10^{-2}	-3.9×10^{-4}
0.125	$+8.8 \times 10^{-2}$	6.6×10^{-2}	-1.3×10^{-3}
0.175	$+7.9 \times 10^{-2}$	5.5×10^{-2}	-2.7×10^{-3}
0.225	$+6.3 \times 10^{-2}$	4.8×10^{-2}	-4.8×10^{-3}
0.275	$+5.9 \times 10^{-2}$	3.8×10^{-2}	-6.6×10^{-3}

Cross Sections for Top Production









Phase Space

$$d\Phi_4 = \frac{d^{d-1}k_1}{(2\pi)^{d-1}2k_1^0}\frac{d^{d-1}k_2}{(2\pi)^{d-1}2k_2^0}\frac{d^{d-1}q_1}{(2\pi)^{d-1}2q_1^0}\frac{d^{d-1}q_2}{(2\pi)^{d-1}2q_2^0}(2\pi)^d\delta^{(d)}(k_1+k_2+q_1+q_2-p_1-p_2)$$

$$\begin{aligned}
p_1^\mu &= \frac{\sqrt{s}}{2}(1,0,0,1)\ ,\\
p_2^\mu &= \frac{\sqrt{s}}{2}(1,0,0,-1)\ ,\\
n_1^\mu &= \frac{\sqrt{s}}{2}\beta^2(1,0,\sin\theta_1,\cos\theta_1)\ ,\\
n_2^\mu &= \frac{\sqrt{s}}{2}\beta^2(1,\sin\phi\sin\theta_2,\cos\phi\sin\theta_2,\cos\theta_2)\ ,\\
k_1^\mu &= \hat{\xi}_1\, n_1^\mu\ ,\\
k_2^\mu &= \hat{\xi}_2\, n_2^\mu\ ,
\end{aligned}$$

$$\begin{aligned}
\hat{\eta}_{1,2} &= \frac{1}{2}(1-\cos\theta_{1,2})\ ,\\
\eta_3 &= \frac{1}{2}(1-\cos\theta_3)\\
&= \frac{1}{2}(1-\cos\phi\sin\theta_1\sin\theta_2-\cos\theta_1\cos\theta_2)\\
&= \frac{1}{2}(1-\cos(\theta_1-\theta_2)+(1-\cos\phi)\sin\theta_1\sin\theta_2)\ ,
\end{aligned}$$

$$\eta_3 = \frac{(\hat{\eta}_1-\hat{\eta}_2)^2}{\hat{\eta}_1+\hat{\eta}_2-2\hat{\eta}_1\hat{\eta}_2-2(1-2\zeta)\sqrt{\hat{\eta}_1(1-\hat{\eta}_1)\hat{\eta}_2(1-\hat{\eta}_2)}}$$

$$\zeta = \frac{1}{2}\frac{(1-\cos(\theta_1-\theta_2))(1+\cos\phi)}{1-\cos(\theta_1-\theta_2)+(1-\cos\phi)\sin\theta_1\sin\theta_2}$$

all collinear limits with only two variables

$$d\Phi_4 = d\Phi_3(p_1+p_2;k_1,k_2)\ d\Phi_2(Q;q_1,q_2)$$

$$\begin{aligned}
d\Phi_3(p_1+p_2;k_1,k_2) &= \frac{\pi^{2\epsilon}}{8(2\pi)^5\Gamma(1-2\epsilon)}s^{2-2\epsilon}\beta^{8-8\epsilon}\,(\zeta(1-\zeta))^{-\frac{1}{2}-\epsilon}\\
&\times\ (\hat{\eta}_1(1-\hat{\eta}_1))^{-\epsilon}(\hat{\eta}_2(1-\hat{\eta}_2))^{-\epsilon}\frac{\eta_3^{1-2\epsilon}}{|\hat{\eta}_1-\hat{\eta}_2|^{1-2\epsilon}}\ \hat{\xi}_1^{1-2\epsilon}\hat{\xi}_2^{1-2\epsilon}\\
&\times\ d\zeta\ d\hat{\eta}_1\ d\hat{\eta}_2\ d\hat{\xi}_1\ d\hat{\xi}_2\ .
\end{aligned}$$

$d\mu_{\eta\xi}$

Level 1 Decomposition

$$
\begin{aligned}
1 = \;& \left.\begin{matrix} +\theta_1(k_1)\theta_1(k_2) \\ +\theta_2(k_1)\theta_2(k_2) \end{matrix}\right\} \text{ triple-collinear sector} \\
& \left.\begin{matrix} +\theta_1(k_1)\theta_2(k_2)(1-\theta_3(k_1,k_2)) \\ +\theta_2(k_1)\theta_1(k_2)(1-\theta_3(k_1,k_2)) \end{matrix}\right\} \text{ double-collinear sector} \\
& + (\theta_1(k_1)\theta_2(k_2)+\theta_2(k_1)\theta_1(k_2))\theta_3(k_1,k_2)\} \text{ single-collinear sector}
\end{aligned}
$$

most difficult (→ triple-collinear sector)

non-trivial only because of soft-collinear divergences (→ double-collinear sector)

trivial, because NLO type attach to first sector (contains same divergences) (→ single-collinear sector)

top quark pair production

general case

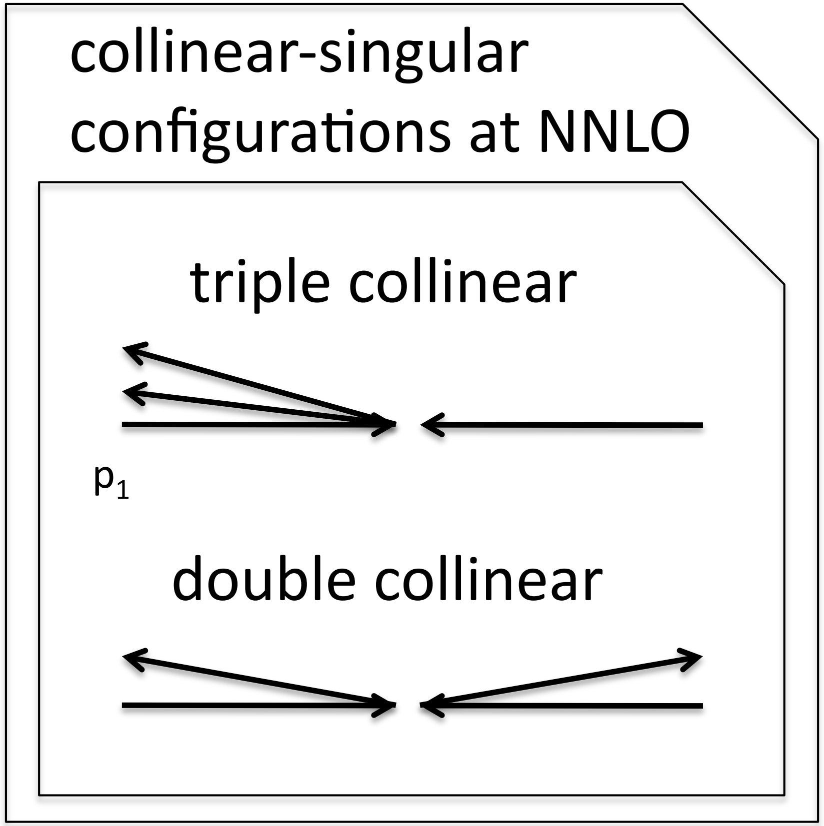
$$1 = \sum_{\substack{\text{pairs} \\ i,j\in\mathcal{F}}} \sum_{\substack{k\in\mathcal{I}\cup\mathcal{F} \\ k\notin\{i,j\}}} \left[\theta_{ij,k} + \sum_{\substack{l\in\mathcal{I}\cup\mathcal{F} \\ l\notin\{i,j,k\}}} \theta_{ij,kl}\right]$$

$$d_{ij} = \left[\left(\frac{2E_i}{\sqrt{s}}\right)\left(\frac{2E_j}{\sqrt{s}}\right)\right]^\alpha (1-\cos\theta_{ij})^\beta,$$

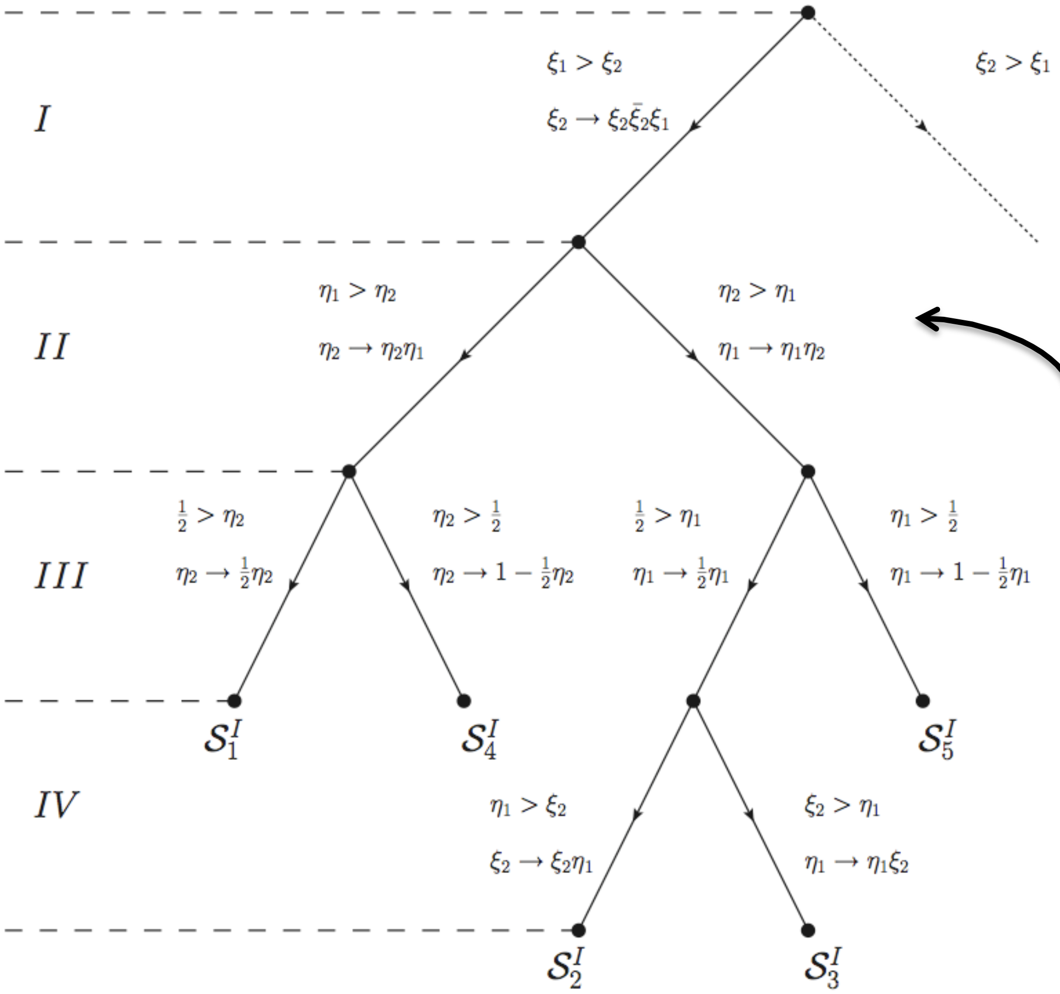
$$d_{ijk} = \left[\left(\frac{2E_i}{\sqrt{s}}\right)\left(\frac{2E_j}{\sqrt{s}}\right)\left(\frac{2E_k}{\sqrt{s}}\right)\right]^\alpha \left[(1-\cos\theta_{ij})(1-\cos\theta_{ik})(1-\cos\theta_{jk})\right]^\beta$$

$$\theta_{ij,k} = \frac{1}{\mathcal{D}}\frac{h_{ij,k}}{d_{ijk}}, \qquad \theta_{ij,kl} = \frac{1}{\mathcal{D}}\frac{h_{i,k}}{d_{ik}}\frac{h_{j,l}}{d_{jl}}$$

$$\mathcal{D} = \sum_{\substack{\text{pairs} \\ i,j\in\mathcal{F}}} \sum_{\substack{k\in\mathcal{I}\cup\mathcal{F} \\ k\notin\{i,j\}}} \left[\frac{h_{ij,k}}{d_{ijk}} + \sum_{\substack{l\in\mathcal{I}\cup\mathcal{F} \\ l\notin\{i,j,k\}}} \frac{h_{i,k}}{d_{ik}}\frac{h_{j,l}}{d_{jl}}\right]$$



Level 2 Decomposition



I) factorization of the soft singularities;

II, III) factorization of the collinear singularities;

IV) factorization of the soft-collinear singularities.

triple-collinear sector

Example from S_1^I

$$s_{156} = -\beta^2(\hat{\eta}_1\hat{\xi}_1 + \hat{\eta}_2\hat{\xi}_2 - \beta^2\hat{\xi}_1\hat{\xi}_2\eta_3)$$

$$-\frac{1}{2}\beta^2\eta_1\xi_1\left(2 + \eta_2\xi_2\bar{\xi}_2 - 2\beta^2\xi_1\xi_2\eta_{31}\bar{\xi}_2\right)$$

remove II & III in the double-collinear sector

Subtraction Terms

$$d\mu_{\eta\xi} = \eta_1^{a_1+b_1\epsilon}\eta_2^{a_2+b_2\epsilon}\xi_1^{a_3+b_3\epsilon}\xi_2^{a_4+b_4\epsilon}\ \mu_S^{\rm reg}\ d\eta_1 d\eta_2 d\xi_1 d\xi_2$$

$$\sigma_{\mathcal{O}} = \sum_{\mathcal{S}} \sigma_{\mathcal{O}}^{(\mathcal{S})} \qquad \sigma_{\mathcal{O}}^{(\mathcal{S})} = \int d\zeta\ d\eta_1\ d\eta_2\ d\xi_1\ d\xi_2\ d\cos\theta_Q\ d\phi_Q\ d\cos\rho_Q\ \Sigma_{\mathcal{O}}^{(\mathcal{S})}$$

$$\Sigma_{\mathcal{O}}^{(\mathcal{S})} = \frac{1}{2s}\left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{3\epsilon} \mu_\zeta\ \mu_S^{\rm reg}\ \mu_2\ \theta_S\ F_J \frac{1}{\eta_1^{1-b_1\epsilon}}\frac{1}{\eta_2^{1-b_2\epsilon}}\frac{1}{\xi_1^{1-b_3\epsilon}}\frac{1}{\xi_2^{1-b_4\epsilon}}\,\mathfrak{M}_S$$

$$\mathfrak{M}_S = \eta_1^{1+a_1}\eta_2^{1+a_2}\xi_1^{1+a_3}\xi_2^{1+a_4}\ \overline{|\mathcal{M}_4|^2}$$

$$\int_0^1 \frac{d\lambda}{\lambda^{1-b\epsilon}} f(\lambda) \longrightarrow \int_0^1 d\lambda \left[\frac{f(0)}{b\epsilon} + \frac{f(\lambda)-f(0)}{\lambda^{1-b\epsilon}}\right]$$

apply four times

$$\Sigma_{\mathcal{O}}^{(\mathcal{S})} \longrightarrow \left[\Sigma_{\mathcal{O}}^{(\mathcal{S})}\right]$$

Subtraction Terms

$$\mathbf{X} \subseteq \{\eta_1, \eta_2, \xi_1, \xi_2\}$$

$$\lim_{\mathbf{X}\to\mathbf{0}} \mathfrak{M}_{\mathcal{S}} = g^2 \langle \mathcal{M}_3|\mathbf{V}|\mathcal{M}_3\rangle \quad \text{or} \quad \lim_{\mathbf{X}\to\mathbf{0}} \mathfrak{M}_{\mathcal{S}} = g^4 \langle \mathcal{M}_2|\mathbf{V}|\mathcal{M}_2\rangle$$

Example:

$$\mathfrak{R}_{\mathcal{S}} = \eta_1^{1+a_1}\eta_2^{1+a_2}\xi_1^{1+a_3}\xi_2^{1+a_4}$$

$$\boxed{\hat{\eta}_1 = \hat{\eta}_2 = 0}$$

$$\mathbf{V}^{ss'}_{a_1a_5a_6} = \lim_{\mathbf{X}\to\mathbf{0}} \mathfrak{R}_{\mathcal{S}} \frac{4\hat{P}^{ss'}_{a_1a_5a_6}}{s^2_{156}}$$

$$x_1 = -1\,, \quad x_5 = \beta^2\hat{\xi}_1\,, \quad x_6 = \beta^2\hat{\xi}_2\,,$$

$$k^\mu_{\perp 1} = 0\,, \quad k^\mu_{\perp 5} = \beta^2\hat{\xi}_1\sqrt{\hat{\eta}_1}\,\bar{k}^\mu_{\perp 5}\,, \quad k^\mu_{\perp 6} = \beta^2\hat{\xi}_2\sqrt{\hat{\eta}_2}\,\bar{k}^\mu_{\perp 6}(\hat{\eta}_1,\hat{\eta}_2)\,,$$

$$\begin{aligned}
\bar{k}^\mu_{\perp 5} &= (0,0,1,0)\,, \\
\bar{k}^\mu_{\perp 6}(\hat{\eta}_1,\hat{\eta}_2) &= \frac{1}{\hat{\eta}_1+\hat{\eta}_2-2(1-2\zeta)\sqrt{\hat{\eta}_1\hat{\eta}_2}} \\
&\times \Big(0,\ 2|\hat{\eta}_1-\hat{\eta}_2|\sqrt{\zeta(1-\zeta)},\ 2\sqrt{\hat{\eta}_1\hat{\eta}_2}-(\hat{\eta}_1+\hat{\eta}_2)(1-2\zeta),\ 0\Big)
\end{aligned}$$

Summary for STRIPPER

Application domain: any NNLO corrections ?

Non-linear variable transformations

Ch. Anastasiou, F. Herzog, A. Lazopoulos '11

Idea: factorize singularities with non-linear remappings

Advantage: reduces the number of integrals in comparison to sector decomposition

Examples:

$$I_2 = \int_0^1 dx\, dy\, dz \frac{1}{(x+yz)^{2+\epsilon}} \quad \Rightarrow \quad x \to \frac{xyz}{1-x+yz} \quad \Rightarrow \quad I_2 = \int_0^1 \frac{dx\, dy\, dz}{(yz)^{1+\epsilon}} \frac{(1-x+yz)^{\epsilon}}{(1+yz)^{1+\epsilon}}$$

$$I_8 = \int_0^1 \frac{dx_1\, dx_2\, dx_3\, dx_4\, dx_5}{(x_1+x_2x_3+x_2x_4+x_4x_5)^{3+\epsilon}} \quad \Rightarrow \quad x_1 \to \frac{x_1(x_2x_3+x_2x_4+x_4x_5)}{1-x_1+(x_2x_3+x_2x_4+x_4x_5)}$$

$$I_8 = \int_0^1 dx_1\, dx_2\, dx_3\, dx_4\, dx_5 \frac{(1-x_1+x_2x_3+x_2x_4+x_4x_5)^{1+\epsilon}}{(x_2(x_3+x_4)+x_4x_5)^{2+\epsilon}(1+x_2x_3+x_2x_4+x_4x_5)^{2+\epsilon}}$$

$$I_8 = \int_0^1 dx_1\, dx_2\, dx_3\, dx_4\, dx_5 \frac{F(x_i)}{x_4^{1+\epsilon}x_5^{1+\epsilon}(x_3+x_4+x_4x_5)^{1+\epsilon}}$$

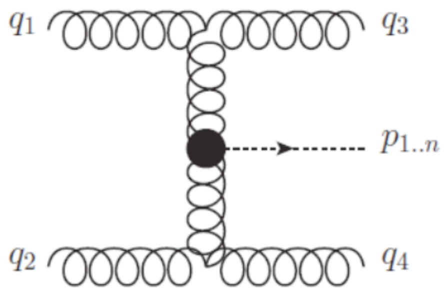
Non-linear variable transformations

Successfully applied to top quark pair production topologies

Example:

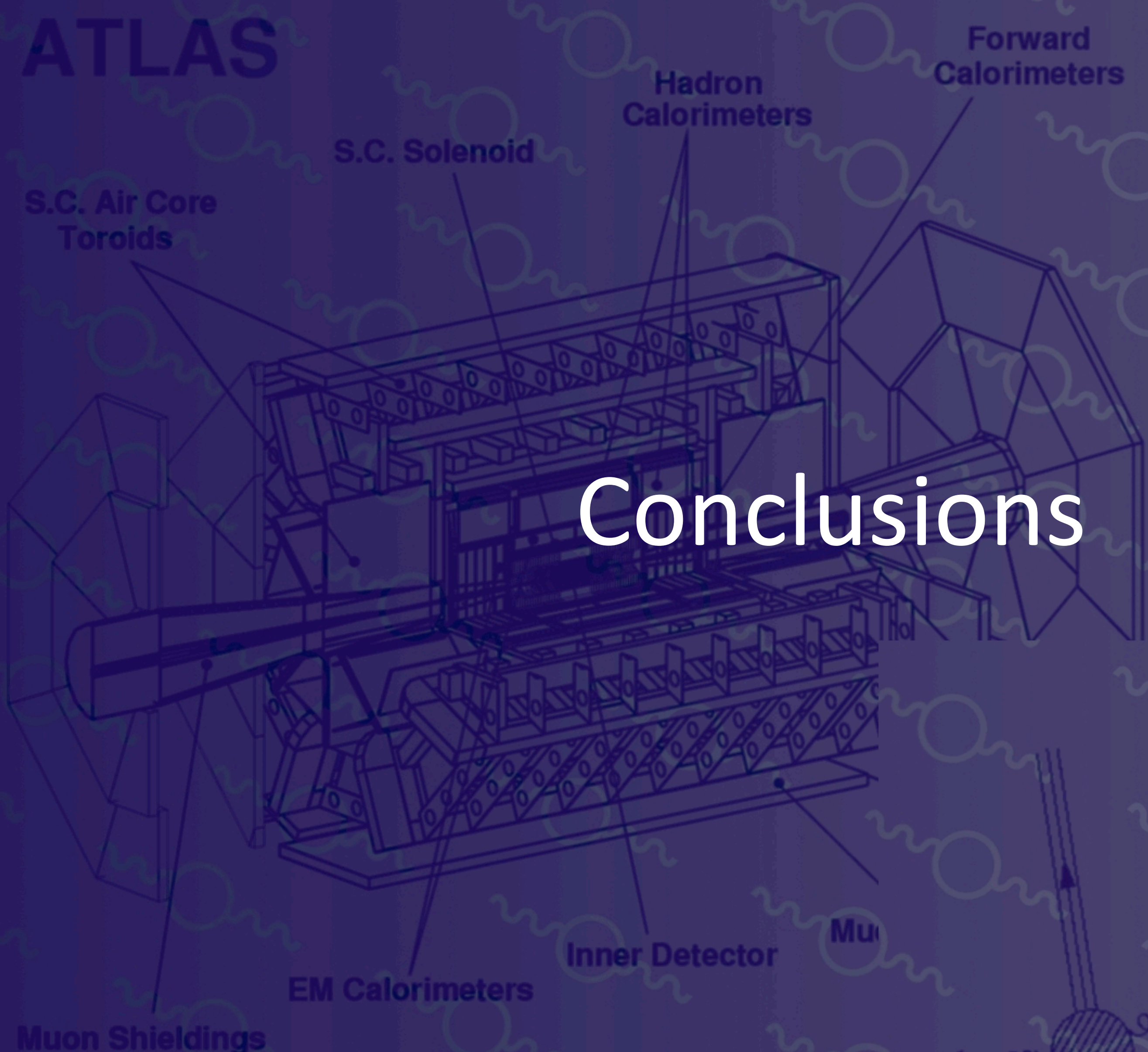
$$I_{11a} = \int \frac{d\Phi_3}{s_{13}s_{23}s_{14}s_{24}}$$

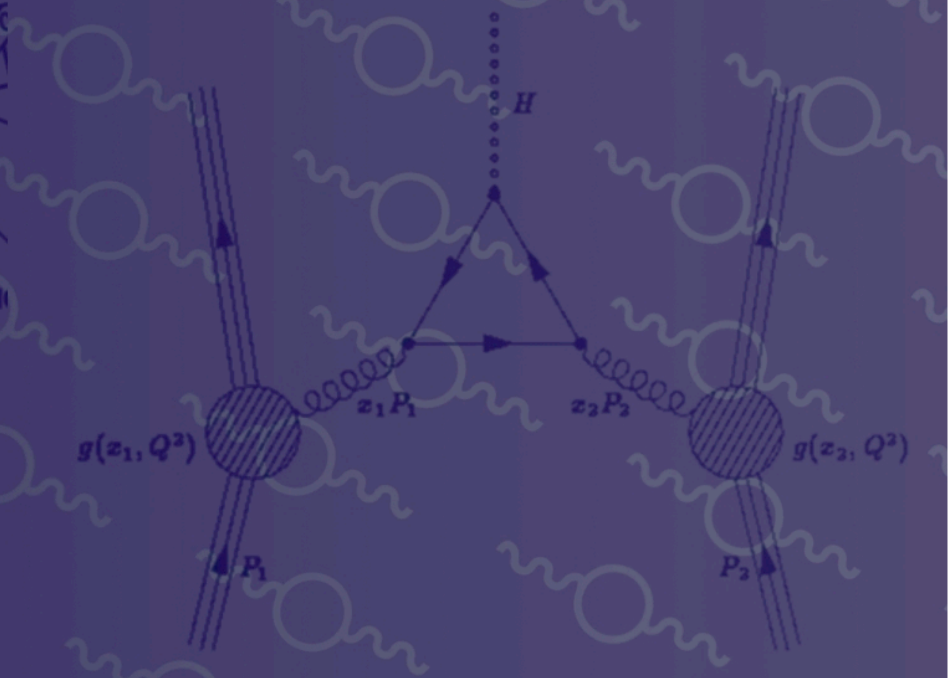
$$I_{11a} = 0.09400(2) + \frac{0.010951(4)}{\epsilon} - \frac{0.0035586(5)}{\epsilon^2} - \frac{0.001119844946(1)}{\epsilon^3}$$



$\sim 1/(s_{13}s_{24})$

Conclusions





Conclusions

• We have reached a point, where numerical techniques exist that provide integrands, which can be integrated with standard integration software

• Numerical approaches will certainly remain the best testing tool, because they are usually more foolproof than complicated analytical calculations

• Will we turn to numerics for problems without special enhancements (once large logs have been eliminated) ?

↑

sociology/psychology at play