An analytical solution for a non-planar massive double box diagram

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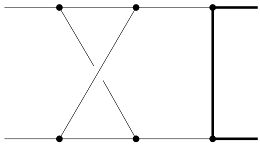
A non-planar massive double box

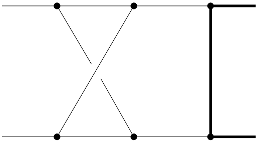


- needed for 2-loop corrections to **top quark pair production**:
  - light fermionic (gg channel)
  - subleading colour (qq/gg channel)
- two approaches to full mass dependence for top quark pair production at 2 loops
  - semi-analytical: Czakon, Bärnreuther (2008-2011) (see plenary talk by M. Czakon)
  - analytical: Bonciani, Ferroglia, Gehrmann, Maitre, A.v.M., Studerus (2008-2011)
- analytical solution for massless case: Tausk 1999

this talk:

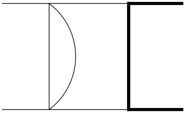
- analytical solution with full mass dependence A.v.M., Studerus (in preparation)
- program `Reduze 2` used for calculation, A.v.M., Studerus (in preparation) (see also talk by C. Studerus, tomorrow)

A three-scale problem

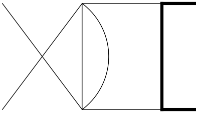


- IBPs $\Rightarrow$ 3 master integrals (+ subtopo masters)
- 3 + 1 scale problem: s, t, u and m appear naturally:

subtopo with s,t:



subtopo with s,u:



Organisation of the calculation

Recipe:

1. **reduce** integrals of sector via distributed Laporta's algorithm
2. calculate **Mellin-Barnes** representation, derive **leading poles** from it
3. solve **differential equations** up to constants in terms of GPLs
4. determine **integration constants** (regularity, symmetry, MB in limits)
5. outlook: **simplify** result using **symbols**

Reductions via IBP identities

integration by parts identities:

- lead to **linear relations** between different Feynman integrals
- exploit for **systematic reduction** to a few master integrals (MIs)
  Chetyrkin, Tkachov (1981)

reduction techniques:

- Laporta: order integrals, reduce lin. system
  public codes: Anastasiou: `AIR`, Smirnov: `FIRE`, Studerus: `Reduze`
- Smirnov, Smirnov: S-basis for differential operators
  public code: `FIRE`
- Gluza, Kajda, Kosower: Gram-determinants, Gröbner basis for n-tuples
- and lots of private codes ...

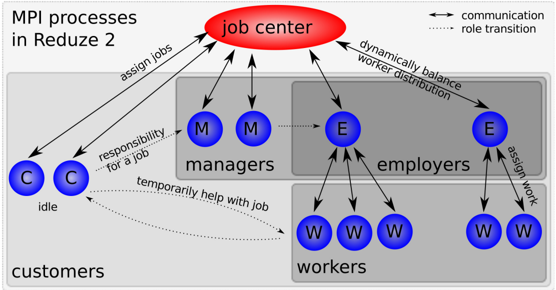
Reduze 2

reduction:

- fully distributed reductions
- generates differential equations

topological:

- finds shifts for diagrams
- finds shifts between sectors



implementation:

- to be released soon (A.v.M., Studerus)
- uses GiNaC by Bauer, Frink, Kreckel
- optional: Fermat CAS by Lewis
- more details: see talk by C. Studerus

Reduze 2 USAGE

on-line help for availabe job types:

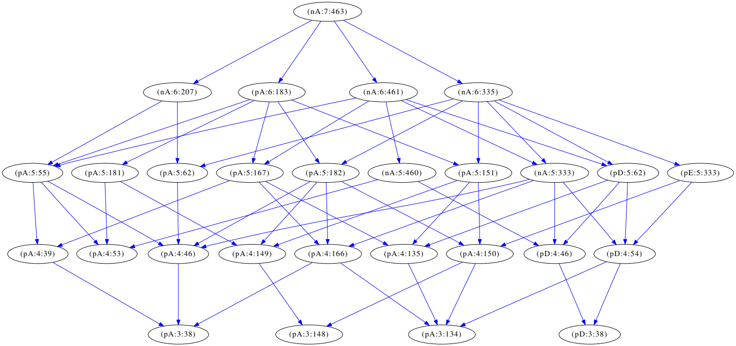
```
andreas : bash
Datei  Bearbeiten  Ansicht  Verlauf  Lesezeichen  Einstellungen  Hilfe
andreas@chili:~$ reduze -h jobs

List of available job types:

cat_files:                         Concatenates files.
collect_integrals:                 Collects all integrals appearing in the input file.
compute_diagram_interferences:     Computes interferences of diagrams.
compute_differential_equations:    Computes derivatives of integrals wrt invariants.
export:                            Exports to FORM or Mathematica format.
extract_database_contents:         Extracts intermediate results from aborted reduction.
find_diagram_shifts:               Matches diagrams to integral families by shifting loop momenta.
find_sector_shifts:                Finds equivalent sectors.
generate_identities:               Generates identities like IBPs for given seeds.
generate_seeds:                    Generates integrals from a sector.
insert_reductions:                 Inserts reductions in expressions.
normalize:                         Simplifies linear combinations and equations.
print_reduction_info_file:         Analyzes reductions in a file.
print_reduction_info_sectors:      Analyzes reductions available for sectors.
print_sector_info:                 Shows information about sectors.
reduce_files:                      Reduces identities in given files.
reduce_sectors:                    Reduces integrals from a selection of sectors.
run_reduction:                     Runs a reduction using a database.
select_reductions:                 Selects reductions for integrals.
setup_sector_mappings:             Finds zero sectors.
sum_terms:                         Sums terms.
verify_same_identities:            Verifies two files contain the same identities.

andreas@chili:~$
andreas : bash
```

Sectors for non-planar double box

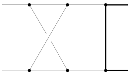


[diagram generated with GraphViz from dot-file produced by Reduze 2]

reduction result shows:

- non-planar double box has 3 master integrals
- masters for most subtopologies known, we solved 1 new topology with 2 MIs:

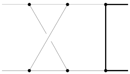
Mellin-Barnes representation for non-planar double box



- for planar integrals: MB rep. with `AMBRE.m` by Gluza, Kajda, Riemann (2007)
- here: use Feynman parameters
- **Mellin-Barnes rep. for massive propagator power** a_9, $t_1 = t + m^2$, $u_1 = u + m^2$:

$$I(a_9) = (-1)^{-1+a_9}\pi^d \frac{\Gamma(-2+d/2)^2}{\Gamma(a_9)\Gamma(-4+d)\Gamma(-6-a_9+3d/2)}$$
$$\int_{-i\infty}^{i\infty}\frac{\mathrm{d}z_1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\mathrm{d}z_2}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\mathrm{d}z_3}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\mathrm{d}z_4}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\mathrm{d}z_5}{2\pi i}$$
$$(-s)^{-6-a_9+d-z_1-z_2-z_5}(-t_1)^{z_1}(-u_1)^{z_2}(m^2)^{z_5}$$
$$\Gamma(-z_1)\Gamma(-z_2)\Gamma(-z_3)\Gamma(-z_4)\Gamma(-z_5)$$
$$\Gamma(1+z_1+z_3)\Gamma(1+z_2+z_3)\Gamma(1+z_1+z_4)\Gamma(1+z_2+z_4)$$
$$\Gamma(4-d/2+z_1+z_2+z_3+z_4)\Gamma(-5-a_9+d-z_1-z_2-z_3-z_5)$$
$$\Gamma(-5-a_9+d-z_1-z_2-z_4-z_5)\Gamma(6+a_9-d+z_1+z_2+z_3+z_4+z_5)$$
$$\Gamma(a_9+z_1+z_2+2z_5))/\Gamma(2+z_1+z_2+z_3+z_4)^2$$

- $I(1)$, $I(2)$, $I(3)$ form a basis
- analytical continuation: `MB.m` by Czakon (2005)
- limits (later): `MBasymptotics.m` by Czakon



- **leading poles** easily obtained from M.B. for general kinematics
- e.g. for **scalar integral** very simple:

$$
\begin{aligned}
\frac{1}{N(\epsilon)} I(1) = & \frac{1}{\epsilon^4} \cdot \frac{y_1 + z_1}{32 x_s^2 y_1 z_1} \\
& + \frac{1}{\epsilon^3} \cdot \frac{7(y_1 + z_1) - 6 * (y_1 + z_1) * \log x_s + 3(y_1 - z_1)(\log y_1 - \log z_1)}{96 x_s^2 y_1 z_1} \\
& + \frac{1}{\epsilon^2} \cdot \big(c_1 \log x_s + c_2 \log^2 x_s + c_3 \log y_1 + c_4 \log^2 y_1 + c_5 \log z_1 + c_6 \log^2 z_1 \\
& \qquad + c_7 \log x_s \log y_1 + c_8 \log x_s \log z_1 + c_9 \log y_1 \log z_1 \big) \\
& + \mathcal{O}(1/\epsilon)
\end{aligned}
$$

numerical agreement with SecDec by Carter, Heinrich (2010)
(possible only for unphysical kinematics: $s + t + u \neq 2m^2$)

recipe:

1. take **derivatives** of MIs w.r.t. kinematical invariants
$$\frac{\partial}{\partial s} I(s,t) = \sum c_j I_j$$
$$\frac{\partial}{\partial t} I(s,t) = \sum d_j I_j$$
2. **reduce** r.h.s. with IBPs
3. **expand** in $\epsilon = (4-d)/2$ to decouple DEQs
4. insert solutions for **subtopologies**
5. **solve** DEQs iteratively for increasing order o in ϵ:
    1. $$\frac{\partial}{\partial s} I \quad \Rightarrow \quad I^{(o)} = f_1(s,t) + g_1(s,t)\mathbf{h_1(t)}, \qquad \text{with } h_1(t) \text{ unknown}$$
    $$\frac{\partial}{\partial t} I \quad \Rightarrow \quad I^{(o)} = f_2(s,t) + g_2(s,t)\mathbf{h_2(s)}, \qquad \text{with } h_2(s) \text{ unknown}$$
    2. set solutions equal:
    $$I^{(o)}(s,t) = f_3(s,t) + g_3(s,t) \cdot \mathit{const}$$
6. determine *const* by additional input (see later)

Generate differential equations

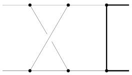
- find choice of:
  - masters integrals
  - variables

  in which DEQs:
  - decouple for different masters
  - rational hom. solution, inhom. solution in terms of GPLs
- job file for `Reduze 2`:

```
jobs:
  - compute_differential_equations:
      integrals_file: "curr.masters.m"
      differentiation_invariants:
        - "(sqrt(-s+4*mt^2) - sqrt(-s))/(sqrt(-s+4*mt^2) + sqrt(-s))"  # x
        - "-t/(mt^2)"  # y
        - "mt^2"
      auxiliary_invariants:
        - "SP(p1,p1)"
        - "SP(p2,p2)"
        - "SP(p4,p4)-SP(p3,p3)"
      output_file: "tmp/deq.unred"
  - collect_integrals:
      input_file: "tmp/deq.unred"
      output_file: "tmp/ints.deq"
```

  computes DEQs, inserts reductions, performs change of basis

Choice of basis and variables for DEQs



- choice which satisfies above criteria:
  - **master integrals**:

$$M_1 = \text{scalar}$$
$$M_2 = \text{scalar} * k_1 k_2$$
$$M_3 = \text{scalar} * (k_1 k_2)^2$$

  - **variables**:

$$x := \frac{\sqrt{-s+4m^2} - \sqrt{-s}}{\sqrt{-s+4m^2} + \sqrt{-s}}, \qquad y := -\frac{t}{m^2}$$

  - **decoupling**: from MB we know all M_i start at $1/\epsilon^4$

coeff	M_1	M_2	M_3	inhom
$\partial M_1/\partial x$	$*$	$*$	0	$1/\epsilon^4$
$\partial M_2/\partial x$	0	$*$	0	$1/\epsilon^3$
$\partial M_3/\partial x$	0	$*$	$*$	$1/\epsilon^4$

coeff	M_1	M_2	M_3	inhom
$\partial M_1/\partial y$	$*$	$*$	0	$1/\epsilon^4$
$\partial M_2/\partial y$	0	$*$	0	$1/\epsilon^3$
$\partial M_3/\partial y$	0	$*$	0	$1/\epsilon^4$

- canonically leads to result in terms of **GPLs**

Generalised polylogarithms

Remiddi, Gehrmann; Goncharov:

Definition of generalised polylogarithms (GPLs)

$$G(\vec{0}_n; x) = \frac{1}{n!} \log^n(x),$$
$$G(a; x) = \int_0^x dt \ f_a(t), \qquad \text{for } a \neq 0$$
$$G(a, \vec{b}; x) = \int_0^x dt \ f_a(t) G(\vec{b}; t), \qquad \text{for } a \neq 0$$

with weight functions for (complex) weight a:

$$f_a(x) = \frac{1}{x - a}$$

- $G(a \neq 0; x) = \log\left(\frac{a-x}{a}\right)$, $G(\vec{0}_{n-1}, 1; x) = Li_n(x)$, $G(\vec{0}_n, \vec{1}_p; x) = S_{n,p}(x)$
- for weights $0, 1, -1$, GPLs specialize to harmonic polylogarithms (HPLs), Remiddi, Vermaseren (1999)
- shuffle algebra, **symbols** (see $N = 4$ remainder func), ...
- here: need 2-dim. GPLs up to weight 4, e.g.: $G\left(-1, \frac{1}{z}, 0, -1; x\right)$
- **numerical evaluation**: with `GiNaC` implementation by Vollinga, Weinzierl (2005)

Integrating the DEQs

- obtaining GPLs from integrations requires **rewriting GPLs**, e.g.:

$$G(a(x), b(x), \dots, y) = \sum_i c_i G(a_i(y), b_i(y), \dots, x)$$

- generic **recursive recipe**:

$$G(a(x), b(x), \dots, y) = \int_{x_0}^{x} dx' \, \frac{\partial}{\partial x'} G(a(x'), b(x'), \dots, y) + G(a(x_0), b(x_0), \dots, y)$$

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- **example** for algorithmic solution:

$$\begin{aligned} G(1/x, y) =& \int_{x_0}^{x} dx' \frac{\partial}{\partial x'} \int_0^y \frac{dy'}{y' - 1/x} + G(1/x_0, y) \\ =& G(1/y, x) - G(1/y, x_0) + G(1/x_0, y) \end{aligned}$$

take **limit** $x_0 \to 0$:

$$G(1/x, y) = G(1/y, x)$$

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- for **non-planar**: GPL conversions **involved** and **require analyt. cont.**:
"crossed subtopos" naturally carry $z = -u/m^2$ instead of $y = -t/m^2$

$$\begin{aligned} G(-1, 0, -1; z) &= G\left(-1, 0, -1; -\frac{1}{x} - x - y\right) \\ &= G\left(1 - \frac{1}{x} - x, -\frac{1}{x} - x, 1 - \frac{1}{x} - x; y\right) + 42 \text{ more GPLs} \\ &= G\left(\frac{1 - y + \sqrt{-3 - 2y + y^2}}{2}, \frac{-y + \sqrt{-4 + y^2}}{2}, \frac{1 - y + \sqrt{-3 - 2y + y^2}}{2}; x\right) + 39 \text{ more GPLs} \end{aligned}$$

Result

solved:

- all masters up to and including finite parts
- determined integration constants from
  - regularity conditions
  - symmetry conditions
  - Mellin-Barnes in kinematical limits
    (finite constants up to 3-fold scaleless Mellin-Barnes integrals)

found **result** in terms of (x, y):

- linear combinations of GPLs with rational prefactors
- transcendality up to 4
- 805 GPLs total, 166 two-dim.
- GPLs with argument y: weights $\{0, -1, -x, -1/x, -1/x - x, -1/x - x + 1\}$
- GPLs with argument x: weights $\{0, \pm 1, \pm i, (1 \pm i\sqrt{3})/2\}$

Outlook: simplification via symbols

symbols for GPLs introduced by Goncharov (2009):
drastic simplification of $N = 4$ rem. func.: Goncharov, Spradlin, Vergu, Volovich (2010)

Definition: Symbol

for a rational function $R(x)$

$$\text{symbol}(\log R(x)) = R(x)$$

for function f(x) with $df = \sum_i g_i(x) d\log(R_i(x))$

$$\text{symbol}(f(x)) = \sum_i (\text{symbol}\, g_i(x)) \otimes (R_i(x))$$

e.g.: symbol$(G(1, -1, 0, x)) = x \otimes (x + 1) \otimes (x - 1)$

Logarithm law for symbols

$$R_1 \cdot \otimes (R_a R_b) \otimes \cdots R_k = R_1 \cdot \otimes R_a \otimes \cdots R_k + R_1 \cdot \otimes R_b \otimes \cdots R_k$$

$$R_1 \cdot \otimes (c R_a) \otimes \cdots R_k = R_1 \cdot \otimes R_a \otimes \cdots R_k \qquad \text{for constant } c$$

Simplification via symbols: first results

result for $1/\epsilon$ of scalar master:

- using GPLs with variables (x, y): **62 different GPLs** (23 two-dim.)
- choosing different arguments via **symbols**, this **simplifies to**:

$$
\begin{aligned}
M_1|_{1/\epsilon} = & \frac{1}{16 m^2 y_1 z_1 (y_1 + z_1)} \times \\
& \times \left(\mathrm{Li}_3\left(\frac{y_1 z_1}{y_1 + z_1}\right) \right. \\
& \quad + \mathrm{Li}_2\left(\frac{y_1 z_1}{y_1 + z_1}\right) (\log(-y_1 - z_1) - \log y_1 - \log z_1) \\
& \quad \left. + \text{polynomial in } \log(y_1),\ \log(z_1),\ \log(-y_1 - z_1),\ \log\left(1 - \frac{y_1 z_1}{y_1 + z_1}\right)\right)
\end{aligned}
$$

with $y_1 := -t/m^2 + 1$,
$z_1 := -u/m^2 + 1$

Conclusions

- **analytical solution** obtained for a non-planar massive double box
- **differential equations** work
- `Reduze 2`: useful tool in calculation
- **choice of polylogs** via symbols simplify results for massive QCD integrals