## New results for algebraic NLO

# Tensor reduction of Feynman integrals

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Based on work done in collaboration with Jochem Fleischer and Valery Yundin
ACAT, Sep 04 – Sep 09, 2011, Brunel Univ., London, England

# A simple example

## 1-loop self-energy:

$$I_2^\mu = \int \frac{d^d k}{i\pi^{d/2}} \frac{k^\mu}{[k^2 - M_1^2]\,[(k+p)^2 - M_2^2]} = p_\mu B_1$$

## Solve:

$$\begin{aligned}
p_\mu I_2^\mu &= p^2 B_1(p, M_1, M_2) \\
&= \int \frac{d^d k}{i\pi^{d/2}} \frac{pk}{[k^2 - M_1^2]\,[(k+p)^2 - M_2^2]} = \int \frac{d^d k}{i\pi^{d/2}} \frac{pk}{D_1\; D_2} \\
&= \int \frac{d^d k}{i\pi^{d/2}} \left[ \frac{D_2 - (p^2 - M_2^2 - M_1^2) - D_1}{D_1\; D_2} \right], \\
B_1(p, M_1, M_2) &= \frac{1}{2p^2} \left[ A_0(M_1) - A_0(M_2) - (p^2 - M_2^2 - M_1^2) B_0(p, M_1, M_2) \right]
\end{aligned}$$

A tensor Feynman integral is expressed in terms of scalar Feynman integrals.

Systematic approach to tensor reductions:
1- to 4-point functions: Passarino, Veltman 1978 [1]

Need in addition a library of scalar functions:
'tHooft, Veltman 1979 [2]

State of the art + open source programs:
K. Ellis and G. Zanderighi, QCDloop/FF [3, 4] 2007,1990
T. Hahn, LoopTools/FF [5, 4] 1998,1990

Open source programs for 5-point reductions:

- LoopTools/FF
- Golem
- others

$\rightarrow$ c++ code PJFry by V. Yundin, released this Summer 2011

## This talk: Efficient reduction formulae in the algebraic Fleischer-Davydychev-Tarasov approach

Recent developments in the Fleischer-Davydychev-Tarasov approach:

- Get tensor reduction with $\cdots$:
- $\cdots$ arbitrary masses
- $\cdots$ killed pentagon Gram determinants
- $\cdots$ treatment of full kinematics, also with small sub-diagram Gram determinants
  $\rightarrow$ presented by J. Fleischer at QCD@LHC@Trento2010
- $\rightarrow$ c++ code PJFry by V. Yundin [$\rightarrow$ GOLEM option]
  see talk at LHCphenonet Meeting 02/2011
- $\cdots$ multiple sums over tensor coefficients made efficient by contracting with external momenta arXiv/1104.4067, PLB 701(2011)646

# Outline

[6] 1991 Davydychev, . . . *Reducing Feynman diagrams to scalar integrals*
[7] 1996 Tarasov, *Connection [of] Feynman integrals [with] different . . . space-time dimensions*
[8] 1999 Fleischer et al., *Algebraic reduction of one-loop Feynman graph amplitudes*



References:
[9] 2010 Diakonidis et al., PLB 683, . . . *recursive reduction of tensor Feynman integrals*
[10] 2011 Fleischer, T.R., PRD 83, *Complete . . . reduction of . . . tensor Feynman integrals*
[11] 2011 Fleischer, T.R., PLB 701, . . . *contracted tensor Feynman integrals*
subm. Aug. 2011: V. Yundin, PhD thesis [with PJFry code]

# Notations: Gram and modified Cayley determinant, signed minors [Melrose:1965]

Gram determinant $G_n$:

$$G_n = |2q_i q_j|, i,j = 1,\dots n \tag{1}$$

Modified Cayley determinant $()_N$ of a diagram with $N$ internal lines and chords $q_j$; for a choice $q_n = 0$, both determinants are related:

$$()_N \equiv \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & Y_{11} & Y_{12} & \dots & Y_{1N} \\ 1 & Y_{12} & Y_{22} & \dots & Y_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & Y_{1N} & Y_{2N} & \dots & Y_{NN} \end{vmatrix} = -G_{N-1}, \tag{2}$$

where $D_i = (k - q_i)^2 - m_i^2$ [with $q_i$ = chord], and the matrix elements

$$Y_{ij} = -(q_i - q_j)^2 + m_i^2 + m_j^2, \quad (i,j = 1 \dots N) \tag{3}$$

⇒ The Gram determinant $()_N$ does not depend on the masses.

# Notations: signed minors [Melrose:1965]

signed minors of $()_N$ are constructed by deleting $m$ rows and $m$ columns from $()_N$, and multiplying with a sign factor:

$$\begin{pmatrix} j_1 & j_2 & \cdots & j_m \\ k_1 & k_2 & \cdots & k_m \end{pmatrix}_N \equiv$$

$$\equiv (-1)^{\sum_l (j_l+k_l)} \mathrm{sgn}_{\{j\}}\, \mathrm{sgn}_{\{k\}} \left| \begin{matrix} \text{rows } j_1 \cdots j_m \text{ deleted} \\ \text{columns } k_1 \cdots k_m \text{ deleted} \end{matrix} \right| \qquad (4)$$

where $\mathrm{sgn}_{\{j\}}$ and $\mathrm{sgn}_{\{k\}}$ are the signs of permutations that sort the deleted rows $j_1 \cdots j_m$ and columns $k_1 \cdots k_m$ into ascending order.

Example:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}_N \equiv \begin{vmatrix} Y_{11} & Y_{12} & \ldots & Y_{1N} \\ Y_{12} & Y_{22} & \ldots & Y_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1N} & Y_{2N} & \ldots & Y_{NN} \end{vmatrix}, \qquad (5)$$

# Example: Getting a 4-point function from a six-point function I

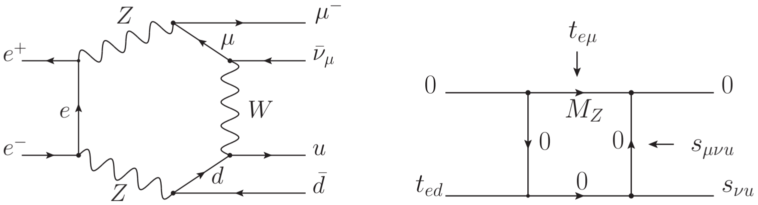



Figure: A six-point topology (a) leading to four-point functions (b) with realistically vanishing Gram determinants.

## Example: Getting a 4-point function from a six-point function II

The example is taken from [12].

The corresponding 4-point tensor integrals are, in LoopTools [5, 13] notation:

$$\mathrm{D0i}(\mathrm{id}, 0, 0, s_{\bar{\nu}u}, t_{ed}, t_{\bar{e}\mu}, s_{\mu\bar{\nu}u}, 0, M_Z^2, 0, 0). \qquad (6)$$

The Gram determinant is:

$$()_4 \quad = \quad -2t_{\bar{e}\mu}[s_{\mu\bar{\nu}u}^2 + s_{\bar{\nu}u}t_{ed} - s_{\mu\bar{\nu}u}(s_{\bar{\nu}u} + t_{ed} - t_{\bar{e}\mu})], \qquad (7)$$

It vanishes if:

$$t_{ed} \to t_{ed,\mathrm{crit}} = \frac{s_{\mu\bar{\nu}u}(s_{\mu\bar{\nu}u} - s_{\bar{\nu}u} + t_{\bar{e}\mu})}{s_{\mu\bar{\nu}u} - s_{\bar{\nu}u}}. \qquad (8)$$

In terms of a dimensionless scaling parameter $x$,

$$t_{ed} \quad = \quad (1+x)t_{ed,\mathrm{crit}}, \qquad (9)$$

# Example: Getting a 4-point function from a six-point function III

the Gram determinant becomes:

$$()_4 = 2\, x\, s_{\mu\bar{\nu}u} t_{\bar{e}\mu} (s_{\mu\bar{\nu}u} - s_{\bar{\nu}u} + t_{\bar{e}\mu}). \quad (10)$$

We will also need the modified Cayley determinant:

$$\begin{aligned}
\binom{0}{0}_4 &= \begin{pmatrix} 2M_Z^2 & M_Z^2 & M_Z^2 - s_{\mu\bar{\nu}u} & M_Z^2 \\ M_Z^2 & 0 & -s_{\bar{\nu}u} & M_Z^2 \\ M_Z^2 - s_{\mu\bar{\nu}u} & -s_{\bar{\nu}u} & 0 & -t_{ed} \\ M_Z^2 & -t_{\bar{e}\mu} & -t_{ed} & 0 \end{pmatrix} \\
&= s_{\mu\bar{\nu}u}^2 t_{\bar{e}\mu}^2 + 2\, M_Z^2 t_{\bar{e}\mu} [-2 s_{\bar{\nu}u} t_{ed} + s_{\mu\bar{\nu}u}(s_{\bar{\nu}u} + t_{ed} - t_{\bar{e}\mu})] \\
&\quad + M_Z^4 (s_{\bar{\nu}u}^2 + (t_{ed} - t_{\bar{e}\mu})^2 - 2 s_{\bar{\nu}u}(t_{ed} + t_{\bar{e}\mu})).
\end{aligned}$$

# Dimensional shifts and recurrence relations for pentagons (I)

Following [Davydychev:1991 [6]]

Replace tensors by scalar integrals in higher dimensions:

Example $R = 3$:

$$
\begin{aligned}
I_5^{\mu\nu\lambda} &= \int \frac{d^{4-2\epsilon}k}{i\pi^{d/2}} \prod_{r=1}^{5} c_r^{-1} k^\mu k^\nu k^\lambda \qquad (11)\\
&= -\sum_{i,j,k=1}^{4} q_i^\mu q_j^\nu q_k^\lambda n_{ijk} I_{5,ijk}^{[d+]^3} + \frac{1}{2}\sum_{i=1}^{n-1} (g^{\mu\nu} q_i^\lambda + g^{\mu\lambda} q_i^\nu + g^{\nu\lambda} q_i^\mu) I_{5,i}^{[d+]^2},
\end{aligned}
$$

and $n_{ijk} = (1+\delta_{ij})(1+\delta_{ik}+\delta_{jk})$.

$[d+]^l = 4 - 2\epsilon + 2l$

$I_{5,i}^{[d+]^2}$ – scratch the line $i$ from $I_5^{[d+]^2}$.

## Dimensional shifts and recurrence relations for pentagons (II)

### 'Naive', direct approach – just perform dimensional recurrences

Following [Tarasov:1996,Fleischer:1999 [7, 8]]
apply recurrence relations, relating scalar integrals of different dimensions, in order to get rid of the dimensionalities $[d+]^l = 4 - 2\epsilon + 2l$:

$$\nu_j (\mathbf{j}^+ I_5^{[d+]}) = \frac{1}{()_5} \left[ -\binom{j}{0}_5 + \sum_{k=1}^{5} \binom{j}{k}_5 \mathbf{k}^- \right] I_5 \qquad (12)$$

$$(d - \sum_{i=1}^{5} \nu_i + 1) I_5^{[d+]} = \frac{1}{()_5} \left[ \binom{0}{0}_5 - \sum_{k=1}^{5} \binom{0}{k}_5 \mathbf{k}^- \right] I_5, \qquad (13)$$

where the operators $\mathbf{i}^\pm, \mathbf{j}^\pm, \mathbf{k}^\pm$ act by shifting the indices $\nu_i, \nu_j, \nu_k$ by $\pm 1$.

# The result of simplifying manipulations

...and collecting all contributions, our final result for e.g. the tensor of rank $R = 3$ can be written as follows:

$$I_5^{\mu\,\nu\,\lambda} = \sum_{i,j,k=1}^{4} q_i^{\mu}\, q_j^{\nu}\, q_k^{\lambda} E_{ijk} + \sum_{k=1}^{4} g^{[\mu\nu} q_k^{\lambda]} E_{00k}, \tag{14}$$

with:

$$E_{00j} = \sum_{s=1}^{5} \frac{1}{\binom{0}{0}_5} \left[ \frac{1}{2} \binom{0s}{0j}_5 I_4^{[d+],s} - \frac{d-1}{3} \binom{s}{j}_5 I_4^{[d+]^2,s} \right], \tag{15}$$

$$E_{ijk} = -\sum_{s=1}^{5} \frac{1}{\binom{0}{0}_5} \left\{ \left[ \binom{0j}{sk}_5 I_{4,i}^{[d+]^2,s} + (i \leftrightarrow j) \right] + \binom{0s}{0k}_5 \nu_{ij} I_{4,ij}^{[d+]^2,s} \right\}. \tag{16}$$

✓ no scalar 5-point integrals in higher dimensions
✓ no inverse Gram det. $()_5$

We have yet:

† scalar 4-point integrals in higher dimensions: $I_{4,ij}^{[d+]^2,s}$ etc.

† inverse Gram det. $\binom{0}{0}_5 \equiv ()_4$

## Reduce $I_{4,ij\cdots}^{[d+]^l,s}$ to $I_4^{[d+]^L,s}$ plus simpler objects I

By nontrivial manipulations we get e.g.:

$$I_{4,i}^{[d+],s} = \frac{1}{\binom{0s}{0s}_5}\left[-\binom{0s}{is}_5 (d-3) I_4^{[d+],s} + \sum_{t=1}^{5}\binom{0st}{0si}_5 I_3^{st}\right] \tag{17}$$

$$\begin{aligned}\nu_{ij} I_{4,ij}^{[d+]^2} = \quad & \frac{\binom{0}{i}_4}{\binom{0}{0}_4}\frac{\binom{0}{j}_4}{\binom{0}{0}_4}(d-2)(d-1) I_4^{[d+]^2} + \frac{\binom{0i}{0j}_4}{\binom{0}{0}_4} I_4^{[d+]} \\ & -\frac{\binom{0}{j}_4}{\binom{0}{0}_4}\frac{d-2}{\binom{0}{0}_4}\sum_{t=1}^{4}\binom{0t}{0i}_4 I_3^{[d+],t} + \frac{1}{\binom{0}{0}_4}\sum_{t=1}^{4}\binom{0t}{0j}_4 I_{3,i}^{[d+],t}\end{aligned} \tag{18}$$

These equations are free of inverse Gram determinants $()_4$.
But they contain yet the generic 4-point and (partly indexed) 3-point functions in higher dimensions, $I_4^{[d+],s}$, $I_3^{[d+],t}$, etc.

## Last step: evaluate the $I_4^{[d+],s}$, $I_3^{[d+],t}$, etc. I

Several strategies are now possible:

- Just evaluate them analytically in $d+2l-2\epsilon$ dimensions – if you may do that
- Just evaluate them numerically in $d+2l-2\epsilon$ dimensions
- Reduce them further by recurrences – buy the towers of $1/()_4$ $\rightarrow$ apply (13)
- Make a small Gram determinant expansion $\rightarrow$ apply (13) another way round

Last two items are done here.

## Reduction of scalars $I_4^D$ to the generic dimension $\to I_4^d = D_0,\, I_3^d = C_0$ I

### Non-small 4-point Gram determinants:

Direct, iterative use of (13) yields e.g.:

$$I_4^{[d+]^l} = \left[ \frac{\binom{0}{0}_4}{()_4} I_4^{[d+]^{l-1}} - \sum_{t=1}^{4} \frac{\binom{t}{0}_4}{()_4} I_3^{[d+]^{l-1},t} \right] \frac{1}{d+2l-5} \qquad (19)$$

$$I_3^{[d+]^l,t} = \left[ \frac{\binom{0t}{0t}_4}{\binom{t}{t}_4} I_3^{[d+]^{l-1},t} - \sum_{u=1,u\neq t}^{4} \frac{\binom{ut}{0t}_4}{\binom{t}{t}_4} I_2^{[d+]^{l-1},tu} \right] \frac{1}{d+2l-4} \qquad (20)$$

And we are done.

This works fine if $()_4$ is not small [and also the $\binom{t}{t}_4$].

# Make a small Gram expansion I

Again use (13):

$$\left(\right)_4 (d - \sum_{i=1}^{4} \nu_i + 1) I_4^{[d+]} = \left[ \binom{0}{0}_4 I_4 - \sum_{k=1}^{4} \binom{0}{k}_4 I_3^k \right]$$

If $()_4 = 0$, then it follows ($n = 4$):

$$I_n^D = \sum_k^n \frac{\binom{0}{k}_n}{\binom{0}{0}_n} I_{n-1}^{D,k} \qquad (21)$$

If $()_4 << 1$, re-write (13), as follows:

$$I_n^D = \sum_k^n \frac{\binom{0}{k}_n}{\binom{0}{0}_n} I_{n-1}^{D,k} - \frac{()_n}{\binom{0}{0}_n} [(D+1) - \sum_i^n \nu_i] I_n^{D+2}. \qquad (22)$$

Effectively we may evaluate $I_n^D$ in terms of simpler functions $I_{n-1}^{D,k}$ with a small correction depending on $I_n^{D+2}$.

We may go a step further, and insert into (22) for $I_n^{D+2}$ the rhs. of (21), taken now at $D' = D + 2$:

$$
\begin{aligned}
I_n^D &= \sum_k^n \frac{\binom{0}{k}_n}{\binom{0}{0}_n} I_{n-1}^{D,k} \\
&\quad - \frac{()_n}{\binom{0}{0}_n}[(D+1) - \sum_i^n \nu_i] \\
&\quad \times \left[ \sum_k^n \frac{\binom{0}{k}_n}{\binom{0}{0}_n} I_{n-1}^{D+2,k} - \frac{()_n}{\binom{0}{0}_n}[(D+3) - \sum_i^n \nu_i] I_n^{D+4} \right].
\end{aligned}
$$

The terms proportional to $[()_n/\binom{0}{0}_n]^a, a = 0, 1$ may be evaluated at the correct kinematics. They depend on three-point functions, and their reduction by normal recurrences will not introduce the unwanted powers of $1/()_4$. The last term, suppressed by the factor $[()_n/\binom{0}{0}_n]^2$, depends on $I_n^{D+4}$. It may either be taken approximately at $()_n = 0$, where it can also be represented by 3-point functions (and their reductions), or it may be evaluated more correctly by another iteration based on (21).

And so on and so on ...

In the numerical example – next section – we worked out up to 10 stable iterations.

A quite similar attempt to perform such a series of approximations was undertaken in [14] (see equation (5) there), where a specific example, forward light-by-light scattering through a massless fermion loop, was studied. The approach was then not further followed.

W. Giele, E. W. N. Glover, and G. Zanderighi,
in: Proceedings of Loops ans Legs 2004:
*Numerical evaluation of one-loop diagrams near exceptional momentum configurations*,

Following Davydychev, [6], one gets

$$I_4^{\mu\nu\lambda} = \int^d \frac{k^\mu k^\nu k^\lambda}{\prod_{r=1}^n c_r} = -\sum_{i,j,k=1}^n q_i^\mu q_j^\nu q_k^\lambda \nu_{ijk} I_{n,ijk}^{[d+]^3} + \frac{1}{2}\sum_{i=1}^n g^{[\mu\nu} q_i^{\lambda]} I_{n,i}^{[d+]^2} \quad (23)$$

We identify the tensor coefficients $D_{11\ldots}$ a la LoopTools, e.g.:

$$D_{111} = I_{4,222}^{[d+]^3} \quad (24)$$

Similarly:

$$D_{1111} = I_{4,2222}^{[d+]^4} \quad (25)$$

# Rank $R = 4$ tensor $D_{1111}$ – Numerics with dimensional recurrences

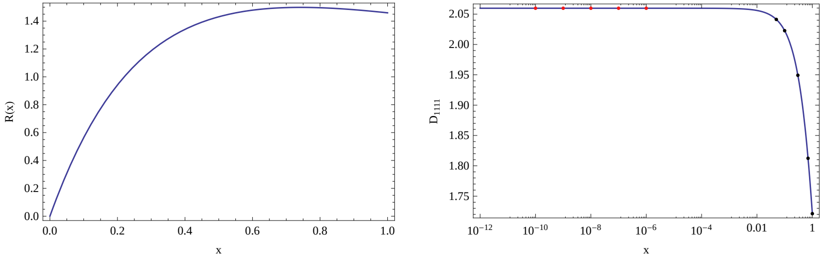
From (22) we see that a “small Gram determinant” expansion will be useful when the following dimensionless parameter becomes small:

$$R = \frac{()_4}{\binom{0}{0}_4} \times s, \qquad (26)$$

where $s$ is a typical scale of the process, e.g. we will choose $s = s_{\mu\bar{\nu}u}$.
Following [12], we further choose:

$$s_{\mu\bar{\nu}u} = 2 \times 10^4 \mathrm{GeV}^2,$$
$$s_{\bar{\nu}u} = 1 \times 10^4 \mathrm{GeV}^2,$$
$$t_{\bar{e}\mu} = -4 \times 10^4 \mathrm{GeV}^2,$$

and get $t_{ed,\mathrm{crit}} = -6 \times 10^4 \mathrm{GeV}^2$. For $x$=1, the Gram determinant becomes $()_4 = 4.8 \times 10^{13}$ GeV$^3$.
The small expansion parameter $R(x)$ and $D_{1111}$ are shown in figure 2.



# New for QCD at LHC: small Gram expansion and Pade approximation I

[Fleischer,TR: PRD 2011 [10] ]

Tables have been taken from there.

They were shown already at QCD@LHC@Trento2010

The use of appropriate Pade approximations is explained there. Convergence in the small Gram determinant region is considerably improved.

| $x$ | | $\Re e\ D_{1111}$ | $\Im m\ D_{1111}$ |
|---|---|---|---|
| 0. | [exp 0,0] | 2.05969289730 E-10 | 1.55594910118 E-10 |
| $10^{-8}$ | [exp x,2] | 2.05969289342 E-10 | 1.55594909187 E-10 |
| | [exp 0,2] | 2.05969289349 E-10 | 1.55594909187 E-10 |
| $10^{-4}$ | [exp x,5] | 2.05965609497 E-10 | 1.55585605343 E-10 |
| | [exp 0,5] | 2.05965609495 E-10 | 1.55585605343 E-10 |
| 0.001 | [exp 0,6] | 2.05932484380 E-10 | 1.55501912433 E-10 |
| | [exp x,6] | 2.05932484381 E-10 | 1.55501912433 E-10 |
| | $I_{4,2222}^{[d+]^4}$ | 2.02292295240 E-10 | 1.54974785467 E-10 |
| | $D_{1111}$ | 2.01707671668 E-10 | 1.62587142251 E-10 |
| 0.005 | [exp 0,6] | 2.05786054801 E-10 | 1.55131031024 E-10 |
| | [pade 0,3] | 2.05785198947 E-10 | 1.55131031003 E-10 |
| | [exp x,6] | 2.05786364440 E-10 | 1.55131031024 E-10 |
| | [pade x,3] | 2.05785199805 E-10 | 1.55131030706 E-10 |
| | $I_{4,2222}^{[d+]^4}$ | 2.05778894114 E-10 | 1.55135794453 E-10 |
| | $D_{1111}$ | 2.05779811490 E-10 | 1.55136343923 E-10 |
| 0.01 | [exp 0,6] | 2.05703298143 E-10 | 1.54669910676 E-10 |
| | [pade 0,3] | 2.05600940065 E-10 | 1.54669907784 E-10 |
| | [exp 0,10] | 2.05600964693 E-10 | 1.54669910676 E-10 |
| | [pade 0,5] | 2.05600955381 E-10 | 1. 54669910676E-10 |
| | [exp x,10] | 2.05600963675 E-10 | 1.54669910676 E-10 |
| | [pade x,5] | 2.05600955381 E-10 | 1.54669910676 E-10 |
| | $I_{4,2222}^{[d+]^4}$ | 2.05600013702 E-10 | 1.54670651917 E-10 |
| | $D_{1111}$ | 2.05600239280 E-10 | 1.54670771210 E-10 |

Table: Numerical values for the tensor coefficient $D_{1111}$. Values marked by $D_{1111}$ are evaluated with LoopTools, the $I_{4,2222}^{[d+]^4}$ corresponds to (28) The labels [exp 0,2n] and [pade 0,n] denote iteration $2n$ and Pade approximant $[n, n]$ when the small Gram determinant expansion starts at $x = 0$, and [exp x,2n] and [pade x,n] are the corresponding numbers for an expansion starting at $x$.

| $x$ | | $\Re e\, D_{1111}$ | $\Im m\, D_{1111}$ |
|---|---|---|---|
| 0.01 | [exp 0,6] | 2.05703298143 E-10 | 1.54669910676 E-10 |
| | [pade 0,3] | 2.05600940065 E-10 | 1.54669907784 E-10 |
| | [exp 0,10] | 2.05600964693 E-10 | 1.54669910676 E-10 |
| | [pade 0,5] | 2.05600955381 E-10 | 1. 54669910676E-10 |
| | [exp x,10] | 2.05600963675 E-10 | 1.54669910676 E-10 |
| | [pade x,5] | 2.05600955381 E-10 | 1.54669910676 E-10 |
| | $I_{4,2222}^{[d+]^4}$ | 2.05600013702 E-10 | 1.54670651917 E-10 |
| | $D_{1111}$ | 2.05600239280 E-10 | 1.54670771210 E-10 |
| 0.05 | [exp 0,6] | 4.83822963052 E-09 | 1.51077429118 E-10 |
| | [pade 0,3] | 2.01518061131 E-10 | 1.50591643209 E-10 |
| | [exp 0,20] | 2.04218962072 E-10 | 1.51077424143 E-10 |
| | [pade 0,10] | 2.04122727654 E-10 | 1.51077424149 E-10 |
| | [exp x,20] | 2.04190274030 E-10 | 1.51077424143 E-10 |
| | [pade x,10] | 2.04122727971 E-10 | 1.51077423985 E-10 |
| | $I_{4,2222}^{[d+]^4}$ | 2.04122726387 E-10 | 1.51077422901 E-10 |
| | $D_{1111}$ | 2.04122726601 E-10 | 1.51077423320 E-10 |
| 0.1 | [exp 0,26] | 2.20215264409 E-08 | 1.46815247004 E-10 |
| | [pade 0,13] | 2.01749674352 E-10 | 1.46681287362 E-10 |
| | [exp x,26] | 2.08190721550 E-08 | 1.46815247004 E-10 |
| | [pade x,13] | 2.03995221326 E-10 | 1.46785977364 E-10 |
| | $I_{4,2222}^{[d+]^4}$ | 2.02269485177 E-10 | 1.46815247061 E-10 |
| | $D_{1111}$ | 2.02269485217 E-10 | 1.46815247051 E-10 |
| 1. | $I_{4,2222}^{[d+]^4}$ | 1.72115440143 E-10 | 9.74550747662 E-11 |
| | $D_{1111}$ | 1.72115440148 E-10 | 9.74550747662 E-11 |

Table: Numerical values for the tensor coefficient $D_{1111}$. Values marked by $D_{1111}$ are evaluated with LoopTools, the $I_{4,2222}^{[d+]^4}$ corresponds to (28) The labels [exp 0,2n] and [pade 0,n] denote iteration $2n$ and Pade approximant $[n, n]$ when the small Gram determinant expansion starts at $x = 0$, and [exp x,2n] and [pade x,n] are the corresponding numbers for an expansion starting at $x$.

## PJFry - an open source c++ program by V. Yundin I

### PJFry 1.0.0 - one loop tensor integral library

- 
- More information and the latest source code:
  project page: https://github.com/Vayu/PJFry/
- $\rightarrow$ how to install
- $\rightarrow$ how to use
- $\rightarrow$ samples
- See also:
  V. Yundin's talk at LHCphenoNet meeting, Valencia, Feb 2011:
  "One loop tensor reduction program PJFRY"

## PJFry - an open source c++ program by V. Yundin II

- Yundin's PhD thesis, submitted Aug 2011 at Humboldt University

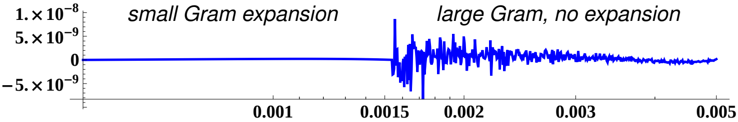
## PJFry — numerical package [from V.Y. Valencia 2011] I

Numerical implementation of described algorithms:
C++ package **PJFry** by V. Yundin [see project webpage]

- Reduction of **5-point** 1-loop tensor integrals up to **rank 5**
- No limitations on internal/external masses combinations
- Small Gram determinants treatment by expansion
- Interfaces for C, C++, FORTRAN and MATHEMATICA

**Example:**

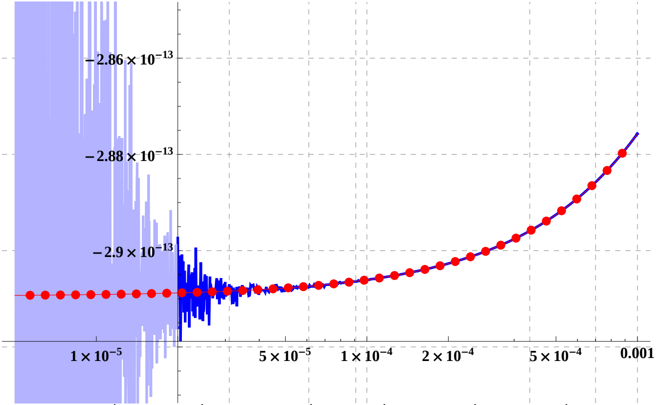
Relative accuracy of $E_{3333}$ coef. around small Gram4 region



## PJFry — small Gram region example [from V.Y. Valencia 2011]

**Example:** $E_{3333}$ coefficient in small Gram region ($x = 0$)

Comparison of Regular and Expansion formulae:




$x{=}0$: $E_{3333}(0, 0, -6\times10^4, 0, 0, 10^4, -3.5\times10^4, 2\times10^4, -4\times10^4, 1.5\times10^4, 0, 6550, 0, 0, 8315)$

## Contractions with external momenta [or with CHORDS] I

[Fleischer,TR: PLB 2011 [11] ]

After having tensor reductions with basis functions $I_n^D$,
which are independent of the indices $i, j, k, ...$,
one may use contractions with external momenta in order to perform all the sums over $i, j, k, ....$
This leads to a significant simplification and shortening of calculations.

Reminder:

One option was to avoid the appearance of inverse Gram determinants $1/()_5$. For rank $R = 5$, e.g.,

$$I_5^{\mu\nu\lambda\rho\sigma} = \sum_{s=1}^{5} \left[ \sum_{i,j,k,l,m=1}^{5} q_i^{\mu} q_j^{\nu} q_k^{\lambda} q_l^{\rho} q_m^{\sigma} E_{ijklm}^{s} + \sum_{i,j,k=1}^{5} g^{[\mu\nu} q_i^{\lambda} q_j^{\rho} q_k^{\sigma]} E_{00ijk}^{s} + \sum_{i=1}^{5} g^{[\mu\nu} g^{\lambda\rho} q_i^{\sigma]} E_{0000i}^{s} \right] \quad (27)$$

# Contractions with external momenta [or with CHORDS] I

The tensor coefficients are expressed in terms of integrals $I_{4,i\cdots}^{[d+]^l,s}$, e.g.:

$$E_{ijklm}^{s} = -\frac{1}{\binom{0}{0}_5}\left\{\left[\binom{0l}{sm}_5 n_{ijk} I_{4,ijk}^{[d+]^4,s} + (i\leftrightarrow l) + (j\leftrightarrow l) + (k\leftrightarrow l)\right] + \binom{0s}{0m}_5 n_{ijkl} I_{4,ijkl}^{[d+]^4,s}\right\}.$$

Now, in a next step, one may avoid the appearance of inverse sub-Gram determinants $()_4$.
The complete dependence on the indices $i$ of the tensor coefficients is contained now in the pre-factors with signed minors. One can say that the indices *decouple* from the integrals.
As an example, we reproduce the 4-point part of

$$n_{ijkl} I_{4,ijkl}^{[d+]^4} = \frac{\binom{0}{i}}{\binom{0}{0}}\frac{\binom{0}{j}}{\binom{0}{0}}\frac{\binom{0}{k}}{\binom{0}{0}}\frac{\binom{0}{l}}{\binom{0}{0}} d(d+1)(d+2)(d+3) I_4^{[d+]^4}$$
$$+ \frac{\binom{0i}{0j}\binom{0}{k}\binom{0}{l} + \binom{0i}{0k}\binom{0}{j}\binom{0}{l} + \binom{0j}{0k}\binom{0}{i}\binom{0}{l} + \binom{0i}{0l}\binom{0}{j}\binom{0}{k} + \binom{0j}{0l}\binom{0}{i}\binom{0}{k} + \binom{0k}{0l}\binom{0}{i}\binom{0}{j}}{\binom{0}{0}^3} d(d+1) I_4^{[d+]^3}$$
$$+ \frac{\binom{0i}{0l}\binom{0j}{0k} + \binom{0j}{0l}\binom{0i}{0k} + \binom{0k}{0l}\binom{0i}{0j}}{\binom{0}{0}^2} I_4^{[d+]^2} + \cdots \qquad (28)$$

] II

## Contractions with external momenta [or with CHORDS]

In (28), one has to understand the 4-point integrals to carry the corresponding index $s$ and the signed minors are $\binom{0}{k} \to \binom{0s}{ks}_5$ etc.

# Contractions with external momenta [or with CHORDS] I

A chord is the momentum shift of an internal line due to external momenta, $D_i = (k - q_i)^2 - m_i^2 + i\epsilon$, and $q_i = (p_1 + p_2 + + p_i)$, with $q_n = 0$.

The tensor 5-point integral of rank $R = 1$ yields, when contracted with a chord,

$$q_{a\mu} I_5^{\mu} = -\frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \left[ \sum_{i=1}^{4} (q_a \cdot q_i) \binom{0i}{0s}_5 \right] I_4^s. \tag{29}$$

In fact, the sum over $i$ may be performed explicitly:

$$\Sigma_a^{1,s} \equiv \sum_{i=1}^{4} (q_a \cdot q_i) \binom{0s}{0i}_5 = +\frac{1}{2} \left\{ \binom{s}{0}_5 (Y_{a5} - Y_{55}) + \binom{0}{0}_5 (\delta_{as} - \delta_{5s}) \right\},$$

## Contractions with external momenta I

We get immediately

$$q_{a\mu} I_5^{\mu} = - \frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \Sigma_a^{1,s} \quad I_4^s. \tag{30}$$

## Contractions with external momenta I

### The tensor 5-point integral of rank $R = 2$

$$I_5^{\mu\,\nu} = \sum_{i,j=1}^{4} q_i^{\mu}\, q_j^{\nu} E_{ij} + g^{\mu\nu} E_{00}, \tag{31}$$

has the following tensor coefficients free of $1/()_5$:

$$E_{00} = -\sum_{s=1}^{5} \frac{1}{2} \frac{1}{\binom{0}{0}_5} \binom{s}{0}_5 I_4^{[d+],s}, \tag{32}$$

$$E_{ij} = \sum_{s=1}^{5} \frac{1}{\binom{0}{0}_5} \left[ \binom{0i}{sj}_5 I_4^{[d+],s} + \binom{0s}{0j}_5 I_{4,i}^{[d+],s} \right]. \tag{33}$$

# Contractions with external momenta I

Equation (31) yields for the contractions with chords:

$$q_{a\mu} q_{b\nu} I_5^{\mu\,\nu} = \sum_{i,j=1}^{4} (q_a \cdot q_i)(q_b \cdot q_j) E_{ij} + (q_a \cdot q_b) E_{00}. \tag{34}$$

and finally (34) simply reads

$$\begin{aligned} q_{a\mu} q_{b\nu} I_5^{\mu\,\nu} \quad &= \quad \frac{1}{4} \sum_{s=1}^{5} \left\{ \frac{\binom{s}{0}_5}{\binom{0s}{0s}_5} (\delta_{ab}\delta_{as} + \delta_{5s}) + \frac{\binom{s}{s}_5}{\binom{0s}{0s}_5} \Big[ (\delta_{as} - \delta_{5s}) \left( Y_{b5} - Y_{55} \right) \right. \\ &+ \left. (\delta_{bs} - \delta_{5s}) \left( Y_{a5} - Y_{55} \right) + \frac{\binom{s}{0}_5}{\binom{0}{0}_5} \left( Y_{a5} - Y_{55} \right) \left( Y_{b5} - Y_{55} \right) \Big] \right\} I_4^{[d+],s} \\ &+ \frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \frac{\Sigma_b^{1,s}}{\binom{0s}{0s}_5} \sum_{t=1}^{5} \Sigma_a^{2,st} I_3^{st}, \end{aligned}$$

# Contractions with external momenta I

with

$$\Sigma_a^{2,st} \equiv \sum_{i=1}^{4} (q_a \cdot q_i) \binom{0st}{0si}_5$$
$$= \frac{1}{2}(1-\delta_{st}) \left\{ \binom{ts}{0s}_5 (Y_{a5} - Y_{55}) + \binom{0s}{0s}_5 (\delta_{at} - \delta_{5t}) - \binom{0s}{0t}_5 (\delta_{as} - \delta_{5s}) \right\}$$

This has been extended also to higher ranks.
We need at most double sums, e.g.:

$$\Sigma_{ab}^{2,s} \equiv \sum_{i,j=1}^{4} (q_a \cdot q_i)(q_b \cdot q_j) \binom{si}{sj}_5 \frac{1}{2}(q_a \cdot q_b) \binom{s}{s}_5$$
$$= -\frac{1}{4}()_5 (\delta_{ab}\delta_{as} + \delta_{5s}), \tag{35}$$

# Contractions with external momenta I

Many of the sums over signed minors, weighted with scalar products of chords are given in PLB 2011 [[11]], and an almost complete list may be obtained on request from J. Fleischer, T.R.

## Summary

- Recursive treatment of hexagon and pentagon tensor integrals of rank $R$ in terms of pentagons and boxes of rank $R-1$
- Systematic derivation of expressions which are explicitely free of inverse Gram determinants $()_5$ until pentagons of rank $R=5$
- Proper isolation of inverse Gram determinants of subdiagrams of the type $\binom{s}{s}_n 4$, which cannot be completely avoided
- Numerical C++ package PJFry (V. Yundin, open source) for C, c++, Mathematica, Fortran
- Perform multiple sums with signed minors and scalar products after contractions with chords or external momenta

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