One-loop tensor Feynman integral reduction with signed minors

Jochem Fleischer[1], **Tord Riemann**[2] **and Valery Yundin**[3]

[1] Fakultät für Physik, Universität Bielefeld, Universitätsstr. 25, 33615 Bielefeld, Germany

[2] Deutsches Elektronen-Synchrotron, DESY, Platanenallee 6, 15738 Zeuthen, Germany

[3] Niels Bohr International Academy and Discovery Center, Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17, DK-2100, Copenhagen, Denmark

E-mail: `fleischer@physik.uni-bielefeld.de, Tord.Riemann@desy.de, yundin@nbi.dk`

Abstract. We present an algebraic approach to one-loop tensor integral reduction. The integrals are presented in terms of scalar one- to four-point functions. The reduction is worked out explicitly until five-point functions of rank five. The numerical C++ package PJFry evaluates tensor coefficients in terms of a basis of scalar integrals, which is provided by an external library, e.g. QCDLoop. We shortly describe installation and use of PJFry. Examples for numerical results are shown, including a special treatment for small or vanishing inverse four-point Gram determinants. An extremely efficient application of the formalism is the immediate evaluation of complete contractions of the tensor integrals with external momenta. This leads to the problem of evaluating sums over products of signed minors with scalar products of chords. Chords are differences of external momenta. These sums may be evaluated analytically in a systematic way. The final expressions for the numerical evaluation are then compact combinations of the contributing basic scalar functions.

1. Tensor reductions

The reduction of tensorial Feynman integrals to scalar Feynman integrals is an old technique. A systematic approach has been developed for one- to four-point functions in [1]. The simple case of a self-energy vector

$$\begin{aligned} I_2^\mu &= \int \frac{d^dk}{i\pi^{d/2}} \; \frac{k^\mu}{[k^2-M_1^2]\,[(k+p)^2-M_2^2]} \\ &= p^\mu B_1 \end{aligned} \tag{1}$$

is evaluated in terms of scalar functions A_0, B_0 as follows:

$$\begin{aligned} p_\mu I_2^\mu &= p^2 B_1(p,M_1,M_2) \\ &= \int \frac{d^dk}{i\pi^{d/2}} \; \frac{pk}{[k^2-M_1^2]\,[(k+p)^2-M_2^2]} \;\equiv\; \int \frac{d^dk}{i\pi^{d/2}} \; \frac{pk}{D_1\;D_2} \\ &= \frac{1}{2}\int \frac{d^dk}{i\pi^{d/2}} \; \left[\frac{D_2-(p^2-M_2^2-M_1^2)-D_1}{D_1\;D_2}\right], \end{aligned} \tag{2}$$

$$B_1(p,M_1,M_2) = \frac{1}{2p^2}\left[A_0(M_1)-A_0(M_2)-\frac{1}{2}(p^2-M_2^2-M_1^2)B_0(p,M_1,M_2)\right]. \tag{3}$$

This works fine for many situations, but for higher-point functions and for specific kinematical situations other approaches are more useful. We advocate here Davydychev's approach [2] and represent n-point tensors by n-point scalars in higher dimensions and with higher indices (powers of the propagators $D_i = 1/[(k-q_i)^2 - m_i^2]$). An example is:

$$\begin{aligned} I_5^{\mu\nu\lambda} &= \int \frac{d^dk}{i\pi^{d/2}} \frac{k^\mu k^\nu k^\lambda}{D_1 \cdots D_5} \\ &= -\sum_{i,j,k=1}^{4} q_i^\mu q_j^\nu q_k^\lambda n_{ijk} I_{5,ijk}^{[d+]^3} + \frac{1}{2}\sum_{i=1}^{n-1} (g^{\mu\nu} q_i^\lambda + g^{\mu\lambda} q_i^\nu + g^{\nu\lambda} q_i^\mu) I_{5,i}^{[d+]^2}, \end{aligned} \tag{4}$$

with $n_{ijk} = (1+\delta_{ij})(1+\delta_{ik}+\delta_{jk})$. The $I_{5,ijk}^{[d+]^L}$ is defined here in $D = d-2\epsilon+2L$ dimensions and propagators of lines i,j,k have a power raised by one unit. In a next step Tarasov's dimensional recurrences [3, 4] may be applied in order to diminish the dimensions of the scalar integrals:[1]

$$\nu_j \left(\mathbf{j}^+ I_5^{[d+]}\right) = \frac{1}{()_5}\left[-\binom{j}{0}_5 + \sum_{k=1}^{5}\binom{j}{k}_5 \mathbf{k}^-\right] I_5, \tag{5}$$

and

$$(d - \sum_{i=1}^{5}\nu_i + 1) I_5^{[d+]} = \frac{1}{()_5}\left[\binom{0}{0}_5 - \sum_{k=1}^{5}\binom{0}{k}_5 \mathbf{k}^-\right] I_5, \tag{6}$$

where the operators $\mathbf{i}^\pm, \mathbf{j}^\pm, \mathbf{k}^\pm$ act by shifting the indices ν_i, ν_j, ν_k by ± 1.

After dedicated simplifications [5, 6], the tensor of rank $R=3$ may be written as follows:

$$I_5^{\mu\,\nu\,\lambda} = \sum_{i,j,k=1}^{4} q_i^\mu\, q_j^\nu\, q_k^\lambda E_{ijk} + \sum_{k=1}^{4} g^{[\mu\nu} q_k^{\lambda]} E_{00k}, \tag{7}$$

with the tensor coefficients

$$E_{00j} = \sum_{s=1}^{5} \frac{1}{\binom{0}{0}_5}\left[\frac{1}{2}\binom{0s}{0j}_5 I_4^{[d+],s} - \frac{d-1}{3}\binom{s}{j}_5 I_4^{[d+]^2,s}\right], \tag{8}$$

$$E_{ijk} = -\sum_{s=1}^{5} \frac{1}{\binom{0}{0}_5}\left\{\left[\binom{0j}{sk}_5 I_{4,i}^{[d+]^2,s} + (i \leftrightarrow j)\right] + \binom{0s}{0k}_5 \nu_{ij} I_{4,ij}^{[d+]^2,s}\right\}. \tag{9}$$

At this stage, we have reached:

- All scalar 5-point integrals are eliminated.
- No inverse Gram determinants $()_5$ have appeared.
- Scalar 4-point integrals in higher dimensions and/or with higher indices are present yet, e.g. $I_{4,ij}^{[d+]^2,s}$.
- Inverse Cayley determinants $\binom{0}{0}_5$ have appeared.

The integrals $I_{4,ij}^{[d+]^2,s}$ arise from 5-point integrals by shrinking line s.

[1] The explanation of notations may be found in any of the articles [5, 3, 4, 6].For Gram determinants and signed minors see Appendix A.

By further, nontrivial manipulations [6] we have avoided shifted indices and isolated the inverse Gram determinants $()_4$ in the higher-dimensional scalar 4-point functions; see e.g. in (9):

$$I_{4,i}^{[d+],s} = \frac{1}{\binom{0s}{0s}_5}\left[-\binom{0s}{is}_5 (d-3) I_4^{[d+],s} + \sum_{t=1}^{5}\binom{0st}{0si}_5 I_3^{st}\right], \tag{10}$$

$$\begin{aligned}\nu_{ij} I_{4,ij}^{[d+]^2} &= \frac{\binom{0}{i}_4}{\binom{0}{0}_4}\frac{\binom{0}{j}_4}{\binom{0}{0}_4}(d-2)(d-1)I_4^{[d+]^2} + \frac{\binom{0i}{0j}_4}{\binom{0}{0}_4} I_4^{[d+]} \\ &\quad - \frac{\binom{0}{j}_4}{\binom{0}{0}_4}\frac{d-2}{\binom{0}{0}_4}\sum_{t=1}^{4}\binom{0t}{0i}_4 I_3^{[d+],t} + \frac{1}{\binom{0}{0}_4}\sum_{t=1}^{4}\binom{0t}{0j}_4 I_{3,i}^{[d+],t}.\end{aligned} \tag{11}$$

These equations contain yet the generic 4-point and (partly indexed) 3-point functions in higher dimensions, $I_4^{[d+],s}$, $I_{3,i}^{[d+],t}$, etc. Several strategies are now possible:

- Evaluate them analytically in $d+2l-2\epsilon$ dimensions – if you may do that.
- Evaluate them numerically in $d+2l-2\epsilon$ dimensions.
- Reduce them further by recurrences → apply (6) – but buy some towers of $1/()_4$ at this stage.
- Make a small Gram determinant expansion → apply (6) the other way round

The last two items are done here.

For non-small 4-point Gram determinants, the direct, iterative use of (6) yields

$$I_4^{[d+]^l} = \left[\frac{\binom{0}{0}_4}{()_4} I_4^{[d+]^{l-1}} - \sum_{t=1}^{4}\frac{\binom{t}{0}_4}{()_4} I_3^{[d+]^{l-1},t}\right]\frac{1}{d+2l-5}, \tag{12}$$

$$I_3^{[d+]^l,t} = \left[\frac{\binom{0t}{0t}_4}{\binom{t}{t}_4} I_3^{[d+]^{l-1},t} - \sum_{u=1,u\neq t}^{4}\frac{\binom{ut}{0t}_4}{\binom{t}{t}_4} I_2^{[d+]^{l-1},tu}\right]\frac{1}{d+2l-4}, \tag{13}$$

and we are done. This works fine if $()_4$ is not small [and also the $\binom{t}{t}_4$].

For small $()_4$, one may apply the recurrence relation the other way round and get:

$$\begin{aligned} I_n^D &= \sum_k^n \frac{\binom{0}{k}_n}{\binom{0}{0}_n} I_{n-1}^{D,k} \\ &\quad - \frac{()_n}{\binom{0}{0}_n}[(D+1) - \sum_i^n \nu_i] \\ &\quad \times \left[\sum_k^n \frac{\binom{0}{k}_n}{\binom{0}{0}_n} I_{n-1}^{D+2,k} - \frac{()_n}{\binom{0}{0}_n}[(D+3) - \sum_i^n \nu_i] I_n^{D+4}\right] + \cdots \end{aligned} \tag{14}$$

This works in fact fine, especially when improved by Pade approximations. For explicit examples see [6] and sec. 2 on the PJFry package here. The approach has been independently implemented and used by Reina et al. [7].

2. PJFry

The goal of the C++ package PJFry is a stable and fast open-source implementation of tensor reduction, suitable for any physically relevant kinematics.[2] It performs the reduction of 5-point 1-loop tensor integrals up to rank 5. The 4- and 3-point tensor integrals are obtained as a by-product. Main features are:

- Any combination of (real) internal or external masses
- Automatic selection of optimal formula for each coefficient
- Leading $()_5$ are eliminated in the reduction
- Small $()_4$ are avoided using asymptotic expansions where appropriate
- Cache system for tensor coefficients and signed minors
- Interfaces for C, C++, FORTRAN and Mathematica
- Uses QCDLoop [8, 9] or OneLOop [10] for 4-dim scalar integrals
- Available from the project webpage `https://github.com/Vayu/PJFry/` [11, 6]

The installation of PJFry may be performed following the instructions given at the project webpage.
The project subdirectories are
./src - the library source code
./mlink - the MathLink interface
./examples - the FORTRAN examples of library use, built with make check

A build on Unix/Linux and similar systems is done in a standard way by sequential performing ./configure, make, make install. See the INSTALL file for a detailed description of the ./configure options.

The PJFry is used as one option of the GoSam package [12].

The functions for tensor coeffcients for up to rank-5 pentagon integrals are declared in the Mathematica interface:

```
In:= Names["PJFry‘*"]Names[PJFry]

{A0v0, B0v0, B0v1, B0v2, C0v0, C0v1, C0v2, C0v3, \
D0v0, D0v1, D0v2, D0v3, D0v4, E0v0, E0v1, E0v2, \
E0v3, E0v4, E0v5, GetMu2, SetMu2}

Out= {A0v0, B0v0, B0v1, B0v2, C0v0, C0v1, C0v2, C0v3, \
D0v0, D0v1, D0v2, D0v3, D0v4, E0v0, E0v1, E0v2, \
E0v3, E0v4, E0v5, GetMu2, SetMu2}
```

The syntax is very close to that of e.g. LoopTools/FF:

```
E0v3[i,j,k,p1s,p2s,p3s,p4s,p5s,s12,s23,s34,s45,s15,m1s,m2s,m3s,m4s,m5s,ep=0]
```

where:
i, j, k are indices of the tensor coefficient $(0 < i \leq j \leq k < n)$,
$p1s, p2s, ...$ are squared external masses p_i^2,
$s12, s23, ...$ are Mandelstam invariants $(p_i + p_j)^2$,
$m1s, m2s, ...$ are squared internal masses m_i^2,
$ep = 0, -1, -2$ selects the coefficient of the ϵ-expansion.

[2] The presentation relies partly on: V. Yundin, One loop tensor reduction program PJFRY, talk at meeting of SFB/TR9, 15 Nov. 2011, Aachen, Germany.

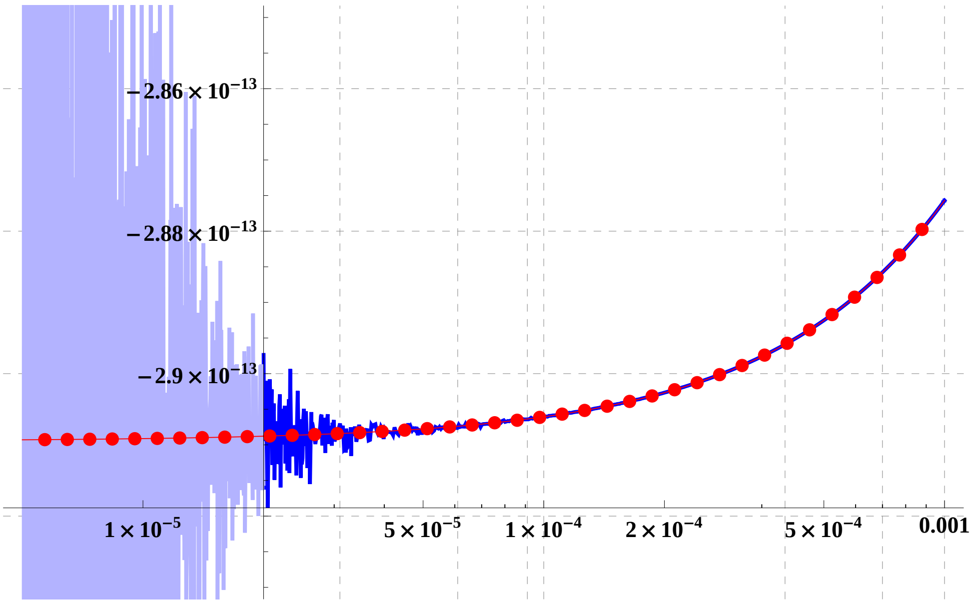

Figure 1. Absolute accuracy of E_{3333} near the region of small Gram determinant. Blue curve: conventional Passarino-Veltman reduction, red curve: PJFry.

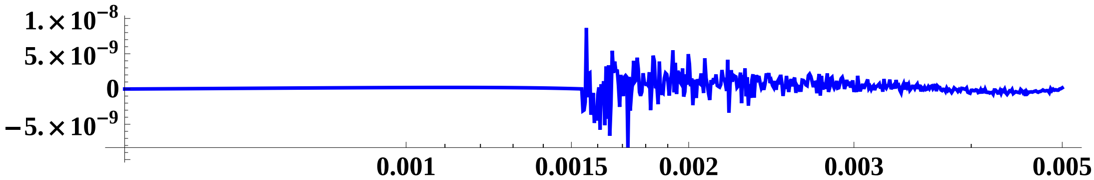

Figure 2. Relative accuracy of E_{3333} near the region of small Gram determinant. At $x \sim 0.0015$ PJFry switched to the asymtotic expansion.

The average evaluation time per phase-space point on a 2 GHz Core 2 laptop for the evaluation of all 81 rank 5 tensor form-factors amounts to 2 ms.

Numerical examples are shown in figures 1 and 2 for a rank $R = 4$ tensor coefficient in a region, where one of the 4-point sub-Gram determinants becomes small

$$E_{3333}(0, 0, -6{\times}10^4(x+1), 0, 0, 10^4, -3.5{\times}10^4, 2{\times}10^4, -4{\times}10^4, 1.5{\times}10^4, 0, 6550, 0, 0, 8315)$$

When $x = 0$, the 4-point Gram determinant vanishes; see Appendix A for details.

3. Contractions of tensor integrals with external momenta

The contents of this section is worked out in detail in [13]. A starting point are representations of tensors in terms of basis functions I_n^D, which are independent of the tensor indices $\{i, j, k, ...\}$. Then, one may use contractions with external momenta in order to perform all the sums over $\{i, j, k, ...\}$. This will evidently lead to a significant simplification and shortening of calculations.

To demonstrate the idea, let us look at the most involved 5-point tensor studied:

$$I_5^{\mu\nu\lambda\rho\sigma} = \sum_{s=1}^{5}\left[\sum_{i,j,k,l,m=1}^{5} q_i^{\mu} q_j^{\nu} q_k^{\lambda} q_l^{\rho} q_m^{\sigma} E_{ijklm}^{s} + \sum_{i,j,k=1}^{5} g^{[\mu\nu} q_i^{\lambda} q_j^{\rho} q_k^{\sigma]} E_{00ijk}^{s} + \sum_{i=1}^{5} g^{[\mu\nu} g^{\lambda\rho} q_i^{\sigma]} E_{0000i}^{s}\right]. \quad (15)$$

In the approach avoiding inverse Gram determinants $1/()_5$, we have

$$E^s_{ijklm} = -\frac{1}{\binom{0}{0}_5}\left\{\left[\binom{0l}{sm}_5 n_{ijk} I^{[d+]^4,s}_{4,ijk} + (i\leftrightarrow l) + (j\leftrightarrow l) + (k\leftrightarrow l)\right] + \binom{0s}{0m}_5 n_{ijkl} I^{[d+]^4,s}_{4,ijkl}\right\} \tag{16}$$

and simpler expressions. Now, in a next step, one may avoid the appearance of inverse sub-Gram determinants $()_4$ and eliminate the higher indices of propagators $\{i,j,k,\ldots\}$. Then, the complete dependence on the indices i of the tensor coefficients is contained in the pre-factors with signed minors: *the tensor indices decouple from the integrals.*

As an example, we reproduce the 4-point part of $I^{[d+]^4}_{4,ijkl}$:

$$\begin{aligned} n_{ijkl} I^{[d+]^4}_{4,ijkl} &= \frac{\binom{0}{i}}{\binom{0}{0}}\frac{\binom{0}{j}}{\binom{0}{0}}\frac{\binom{0}{k}}{\binom{0}{0}}\frac{\binom{0}{l}}{\binom{0}{0}} d(d+1)(d+2)(d+3) I^{[d+]^4}_4 \\ &+ \frac{\binom{0i}{0j}\binom{0}{k}\binom{0}{l} + \binom{0i}{0k}\binom{0}{j}\binom{0}{l} + \binom{0j}{0k}\binom{0}{i}\binom{0}{l} + \binom{0i}{0l}\binom{0}{j}\binom{0}{k} + \binom{0j}{0l}\binom{0}{i}\binom{0}{k} + \binom{0k}{0l}\binom{0}{i}\binom{0}{j}}{\binom{0}{0}^3} d(d+1) I^{[d+]^3}_4 \\ &+ \frac{\binom{0i}{0l}\binom{0j}{0k} + \binom{0j}{0l}\binom{0i}{0k} + \binom{0k}{0l}\binom{0i}{0j}}{\binom{0}{0}^2} I^{[d+]^2}_4 + \cdots \end{aligned} \tag{17}$$

In (17), the 3-point terms are not reproduced, and one has to understand the 4-point integrals to carry the corresponding index s and the signed minors are $\binom{0}{k} \to \binom{0s}{ks}_5$ etc.

A chord is the momentum shift of an internal line due to external momenta, $D_i = (k-q_i)^2 - m_i^2 + i\epsilon$, and $q_i = (p_1 + p_2 + \ + p_i)$, with $q_n = 0$. The tensor 5-point integral of rank $R = 1$ yields, when contracted with a chord,

$$q_{a\mu} I^{\mu}_5 = -\frac{1}{\binom{0}{0}_5}\sum_{s=1}^{5}\left[\sum_{i=1}^{4}(q_a\cdot q_i)\binom{0i}{0s}_5\right] I^s_4. \tag{18}$$

In fact, the sum over i may be performed explicitly:

$$\Sigma^{1,s}_a \equiv \sum_{i=1}^{4}(q_a\cdot q_i)\binom{0s}{0i}_5 = +\frac{1}{2}\left\{\binom{s}{0}_5 (Y_{a5} - Y_{55}) + \binom{0}{0}_5 (\delta_{as} - \delta_{5s})\right\}, \tag{19}$$

and we get immediately the desired compact result for the contraction of chords (or external momenta) and tensor integrals:

$$q_{a\mu} I^{\mu}_5 = -\frac{1}{\binom{0}{0}_5}\sum_{s=1}^{5}\Sigma^{1,s}_a \ I^s_4. \tag{20}$$

Similarly, the tensor 5-point integral of rank $R = 2$ may be treated:

$$I^{\mu\,\nu}_5 = \sum_{i,j=1}^{4} q^{\mu}_i\, q^{\nu}_j E_{ij} + g^{\mu\nu} E_{00}, \tag{21}$$

It has the following tensor coefficients free of $1/()_5$:

$$E_{00} = -\sum_{s=1}^{5}\frac{1}{2}\frac{1}{\binom{0}{0}_5}\binom{s}{0}_5 I^{[d+],s}_4, \tag{22}$$

$$E_{ij} = \sum_{s=1}^{5}\frac{1}{\binom{0}{0}_5}\left[\binom{0i}{sj}_5 I^{[d+],s}_4 + \binom{0s}{0j}_5 I^{[d+],s}_{4,i}\right]. \tag{23}$$

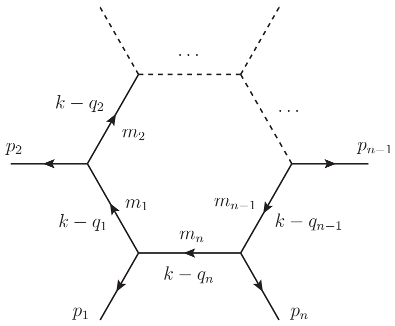

Figure 3. Momenta definitions.

Equation (21) yields for the contractions with chords:

$$q_{a\mu} q_{b\nu} I_5^{\mu\,\nu} = \sum_{i,j=1}^{4} (q_a \cdot q_i)(q_b \cdot q_j) E_{ij} + (q_a \cdot q_b) E_{00}, \tag{24}$$

and finally (24) simply reads

$$\begin{aligned} q_{a\mu} q_{b\nu} I_5^{\mu\,\nu} \quad = \quad & \frac{1}{4} \sum_{s=1}^{5} \Bigg\{ \frac{\binom{s}{0}_5}{\binom{0s}{0s}_5} (\delta_{ab}\delta_{as} + \delta_{5s}) + \frac{\binom{s}{s}_5}{\binom{0s}{0s}_5} \Big[(\delta_{as} - \delta_{5s}) \left(Y_{b5} - Y_{55} \right) \\ & + \; (\delta_{bs} - \delta_{5s}) \left(Y_{a5} - Y_{55} \right) + \frac{\binom{s}{0}_5}{\binom{0}{0}_5} \left(Y_{a5} - Y_{55} \right) \left(Y_{b5} - Y_{55} \right) \Big] \Bigg\} I_4^{[d+],s} \\ & + \; \frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \frac{\Sigma_b^{1,s}}{\binom{0s}{0s}_5} \sum_{t=1}^{5} \Sigma_a^{2,st} I_3^{st}, \end{aligned} \tag{25}$$

with $\Sigma_b^{1,s}$ from (19) and

$$\begin{aligned} \Sigma_a^{2,st} \quad \equiv \quad & \sum_{i=1}^{4} (q_a \cdot q_i) \binom{0st}{0si}_5 \\ = \quad & \frac{1}{2} \left(1 - \delta_{st} \right) \left\{ \binom{ts}{0s}_5 \left(Y_{a5} - Y_{55} \right) + \binom{0s}{0s}_5 \left(\delta_{at} - \delta_{5t} \right) - \binom{0s}{0t}_5 \left(\delta_{as} - \delta_{5s} \right) \right\}, \end{aligned} \tag{26}$$

This can be extended also to higher ranks. We need at most double sums, e.g.:

$$\begin{aligned} \Sigma_{ab}^{2,s} \quad \equiv \quad & \sum_{i,j=1}^{4} (q_a \cdot q_i)(q_b \cdot q_j) \binom{si}{sj}_5 \\ = \quad & \frac{1}{2} (q_a \cdot q_b) \binom{s}{s}_5 - \frac{1}{4} ()_5 \left(\delta_{ab}\delta_{as} + \delta_{5s} \right). \end{aligned} \tag{27}$$

Many of the sums over signed minors, weighted with scalar products of chords are given in [13], and an almost complete list may be obtained on request from the authors.

4. Summary

We gave an introductory to the one-loop tensor reduction package PJFry and the underlying algorithms. PJFry may be used for Feynman integrals with up to five external legs. The treatment of 6-point functions is straightforward, while the expressions for contracted tensor integrals, exemplified in section 3, have not yet been completely worked out. Both 6-point functions and contractions are foreseen to be included into PJFry in the near future. The n-point functions with $n > 6$ can be conveniently treated following the approach advocated in [14, 15]. This was worked out after ACAT 2011 in [16].

We gave a presentation of tensor reduction with some focus on recent develoments by our group. Of course, several other approaches had been worked out in recent years which also deserve notice, usually treating reductions for arbitrary n. Let us mention few of them, just as an introduction to the literature. The authors of [17] construct combinations of n-point and $(n-1)$-point scalar integrals that are finite in the limit of vanishing Gram determinant. The representations are obtained by differentiation with respect to external parameters or by expanding the scalar integrals in $D = 6 - 2\epsilon$ or higher dimensions. An algorithm proposed in [18], based on the so-called generalized Bernstein functional relation, is applied and worked out in [19] for one-loop (tensor) Feynman integrals with n legs. In [20], a set of algorithms is worked out which are a basis for a fast and reliable numerical calculation of one-loop multi-leg Feynman diagrams with up to 6 external legs; also with a special attention to potentially singular points in phase space. In the approach in [21], six 4-point functions in $D = 6$, one 3-point function in $D = 4$, and one 2-point function in D dimensions are used for a general one-loop tensor reduction. All the soft and collinear singularities of general one-loop n-point integrals are worked out explicitly in [22]. The singular parts are expressed in terms of 3-point functions. Finally, we like to mention the approach in [23] for n-point functions. The infrared and ultraviolet divergences are separated analytically from the finite one-loop contributions, and the latter can then be evaluated numerically using recursion relations.

To summarize, there are a lot of competing algorithms available in principle and worked out in detail. One may only hope that most of them will be realized in publicly available open-source software packages for the use by others.

Appendix A. Determinants

The Gram determinant G_N of the diagram shown in 3 is

$$G_N \quad = \quad |2q_iq_j|, i, j = 1, \ldots N. \tag{A.1}$$

We are also using the modified Cayley determinant $()_N$ of a diagram with N internal lines. For the choice $q_N = 0$, both determinants are related:

$$()_N \quad \equiv \quad \begin{vmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & Y_{11} & Y_{12} & \ldots & Y_{1N} \\ 1 & Y_{12} & Y_{22} & \ldots & Y_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & Y_{1N} & Y_{2N} & \ldots & Y_{NN} \end{vmatrix} = -G_{N-1}, \tag{A.2}$$

where the matrix elements are defined by

$$Y_{ij} = -(q_i - q_j)^2 + m_i^2 + m_j^2, \quad (i, j = 1 \ldots N). \tag{A.3}$$

Evidently, the Gram determinant G_N does not depend on the masses, and so doesn't $()_N$.

Signed minors of $()_N$ are constructed by deleting m rows and m columns from $()_N$, and multiplying it with a sign factor:

$$\begin{pmatrix} j_1 & j_2 & \cdots & j_m \\ k_1 & k_2 & \cdots & k_m \end{pmatrix}_N \equiv (-1)^{\sum_l (j_l+k_l)} \mathrm{sgn}_{\{j\}}\, \mathrm{sgn}_{\{k\}} \begin{vmatrix} \text{rows } j_1 \cdots j_m \text{ deleted} \\ \text{columns } k_1 \cdots k_m \text{ deleted} \end{vmatrix}, \tag{A.4}$$

where $\mathrm{sgn}_{\{j\}}$ and $\mathrm{sgn}_{\{k\}}$ are the signs of permutations that sort the deleted rows $j_1 \cdots j_m$ and columns $k_1 \cdots k_m$ into ascending order.

Example:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}_N \equiv \begin{vmatrix} Y_{11} & Y_{12} & \ldots & Y_{1N} \\ Y_{12} & Y_{22} & \ldots & Y_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1N} & Y_{2N} & \ldots & Y_{NN} \end{vmatrix}, \tag{A.5}$$

To be definite, take fig. A1 as a starting point. The example is taken from [24]. The corresponding 4-point tensor integrals are, in LoopTools/FF [25, 26, 9] notation:

$$\mathrm{D0i}(\mathrm{id}, 0, 0, s_{\bar{\nu}u}, t_{ed}, t_{\bar{e}\mu}, s_{\mu\bar{\nu}u}, 0, M_Z^2, 0, 0). \tag{A.6}$$

The Gram determinant is

$$()_4 \quad = \quad -\, 2t_{\bar{e}\mu}[s_{\mu\bar{\nu}u}^2 + s_{\bar{\nu}u}t_{ed} - s_{\mu\bar{\nu}u}(s_{\bar{\nu}u} + t_{ed} - t_{\bar{e}\mu})], \tag{A.7}$$

and it vanishes if:

$$t_{ed} \to t_{ed,\mathrm{crit}} = \frac{s_{\mu\bar{\nu}u}(s_{\mu\bar{\nu}u} - s_{\bar{\nu}u} + t_{\bar{e}\mu})}{s_{\mu\bar{\nu}u} - s_{\bar{\nu}u}}. \tag{A.8}$$

In terms of a dimensionless scaling parameter x,

$$t_{ed} \quad = \quad (1+x)t_{ed,\mathrm{crit}}, \tag{A.9}$$

the Gram determinant becomes:

$$()_4 = 2\ x\ s_{\mu\bar{\nu}u}t_{\bar{e}\mu}(s_{\mu\bar{\nu}u} - s_{\bar{\nu}u} + t_{\bar{e}\mu}). \tag{A.10}$$

The modified Cayley determinant

$$\begin{aligned} \binom{0}{0}_4 &= \begin{pmatrix} 2M_Z^2 & M_Z^2 & M_Z^2 - s_{\mu\bar{\nu}u} & M_Z^2 \\ M_Z^2 & 0 & -s_{\bar{\nu}u} & M_Z^2 \\ M_Z^2 - s_{\mu\bar{\nu}u} & -s_{\bar{\nu}u} & 0 & -t_{ed} \\ M_Z^2 & -t_{\bar{e}\mu} & -t_{ed} & 0 \end{pmatrix} \\ &= s_{\mu\bar{\nu}u}^2 t_{\bar{e}\mu}^2 + 2\ M_Z^2 t_{\bar{e}\mu}[-2s_{\bar{\nu}u}t_{ed} + s_{\mu\bar{\nu}u}(s_{\bar{\nu}u} + t_{ed} - t_{\bar{e}\mu})] \\ &\quad + M_Z^4(s_{\bar{\nu}u}^2 + (t_{ed} - t_{\bar{e}\mu})^2 - 2s_{\bar{\nu}u}(t_{ed} + t_{\bar{e}\mu})) \end{aligned} \tag{A.11}$$

has another dimension. From (14) we see that a small Gram determinant expansion will be useful when the following dimensionless parameter becomes small:

$$R \quad = \quad \frac{()_4}{\binom{0}{0}_4} \times S, \tag{A.12}$$

where S is a typical scale of the process, e.g. we will choose $S = s_{\mu\bar{\nu}u}$.

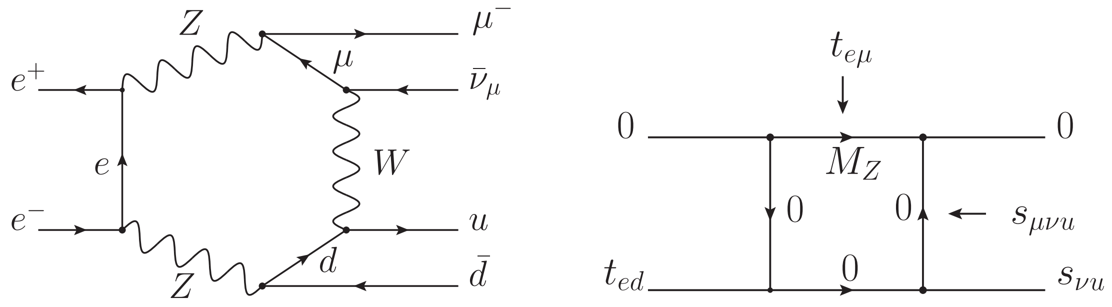

Figure A1. A six-point topology (a) leading to four-point functions (b) with realistically vanishing Gram determinants.

Acknowledgements

J.F. thanks DESY for kind hospitality. Work is supported in part by Sonderforschungsbereich/Transregio SFB/TRR 9 of DFG "Computergestützte Theoretische Teilchenphysik" and European Initial Training Network LHCPHENOnet PITN-GA-2010-264564.

References

[1] Passarino G and Veltman M 1979 *Nucl. Phys.* **B160** 151 doi:10.1016/0550-3213(79)90234-7
[2] Davydychev A I 1991 *Phys. Lett.* **B263** 107–111 doi:10.1016/0370-2693(91)91715-8
[3] Tarasov O 1996 *Phys.Rev.* **D54** 6479–6490 (*Preprint* hep-th/9606018)
[4] Fleischer J, Jegerlehner F and Tarasov O 2000 *Nucl. Phys.* **B566** 423–440 (*Preprint* hep-ph/9907327)
[5] Diakonidis T, Fleischer J, Gluza J, Kajda K, Riemann T and Tausk J 2009 *Phys. Rev.* **D80** 036003 (*Preprint* arXiv:0812.2134)
[6] Fleischer J and Riemann T 2011 *Phys. Rev.* **D83** 073004 (*Preprint* arXiv:1009.4436)
[7] Reina L and Schutzmeier T 2011 (*Preprint* 1110.4438)
[8] Ellis R K and Zanderighi G 2008 *JHEP* **02** 002 (*Preprint* arXiv:0712.1851)
[9] van Oldenborgh G J 1991 *Comput. Phys. Commun.* **66** 1–15 doi:10.1016/0010-4655(91)90002-3, scanned version at http://ccdb3fs.kek.jp/cgi-bin/img_index?9004168
[10] van Hameren A 2011 *Comput. Phys. Commun.* **182** 2427–2438 (*Preprint* arXiv:1007.4716)
[11] Yundin V c++ package PJFry, available at the webpage https://github.com/Vayu/PJFry/
[12] Cullen G, Greiner N, Heinrich G, Luisoni G, Mastrolia P, Ossola G, Reiter T and Tramontano F 2011 (*Preprint* 1111.2034)
[13] Fleischer J and Riemann T 2011 *Phys.Lett.* **B701** 646–653 (*Preprint* arXiv:1104.4067)
[14] Bern Z, Dixon L J and Kosower D A 1994 *Nucl. Phys.* **B412** 751–816 (*Preprint* hep-ph/9306240)
[15] Binoth T, Guillet J, Heinrich G, Pilon E and Schubert C 2005 *JHEP* **10** 015 (*Preprint* hep-ph/0504267)
[16] Fleischer J and Riemann T 2012 *Phys.Lett.* **B707** 375–380 (*Preprint* 1111.5821)
[17] Campbell J, Glover E W N and Miller D 1997 *Nucl. Phys.* **B498** 397–442 (*Preprint* hep-ph/9612413)
[18] Tkachov F V 1997 *Nucl.Instrum.Meth.* **A389** 309–313 (*Preprint* hep-ph/9609429)
[19] Passarino G 2001 *Nucl.Phys.* **B619** 257–312 (*Preprint* hep-ph/0108252)
[20] Ferroglia A, Passera M, Passarino G and Uccirati S 2003 *Nucl.Phys.* **B650** 162–228 (*Preprint* hep-ph/0209219)
[21] Duplancic G and Nizic B 2004 *Eur. Phys. J.* **C35** 105–118 (*Preprint* hep-ph/0303184)
[22] Dittmaier S 2003 *Nucl. Phys.* **B675** 447–466 (*Preprint* hep-ph/0308246)
[23] Giele W T and Glover E W N 2004 *JHEP* **04** 029 (*Preprint* hep-ph/0402152)
[24] Denner A Introductory Lecture at DESY Theory Workshop on Collider Phenomenology, Hamburg, 29 Sep - 2 Oct 2009
[25] Hahn T and Perez-Victoria M 1999 *Comput.Phys.Commun.* **118** 153–165 (*Preprint* hep-ph/9807565)
[26] Hahn T LT25Guidepdf