SecDec: a versatile tool for multi-loop/leg calculations

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Outline

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New Physics?

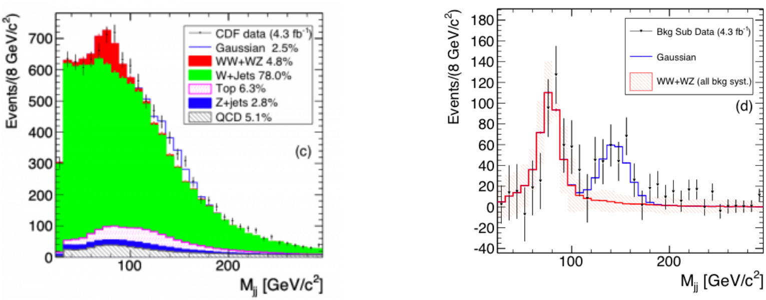


Figure: CDF collaboration, arXiv:1104.0699

In order to discover new physics, we need to know signal and background processes to a high degree of accuracy.

Higher Order Corrections

Knowledge of backgrounds to a high degree of accuracy requires calculations involving integrals (over loop momenta for virtual corrections, and over phase space). Thus multi-dimensional parameter integrals occur widely in these calculations.

Outline

Basic Concept (I)

Multi-dimensional parameter integrals can have complicated singularity structures.
Sector decomposition is an algorithmic approach to extracting regulated IR/UV singularities.
Consider

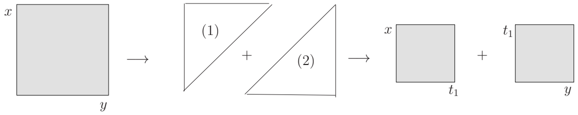
$$I = \int_0^1 dx \int_0^1 dy \; x^{-1-a\epsilon} y^{b\epsilon} (x+y)^{-1}$$

Here we have a singularity when x and y $\to$ 0 simultaneously. We split the integration region in two, with x>y in one half, y>x in the other

$$I = \left(\int_0^1 dx \int_0^x dy + \int_0^1 dy \int_0^y dx\right) \; x^{-1-a\epsilon} y^{b\epsilon} (x+y)^{-1}$$

In the first we set $y = xt_1$, and the second $x = yt_1$

Basic Concept (II)



$$I = \int_0^1 dx\; x^{-1-(a+b)\epsilon} \int_0^1 dt_1\; t_1^{-b\epsilon}(1+t_1)^{-1}$$

$$+ \int_0^1 dy\; y^{-1-(a+b)\epsilon} \int_0^1 dt_1\; t_1^{-1-a\epsilon}(1+t_1)^{-1}$$

The singularities are now factorised and can be read off from the powers of simple monomials in the integration variables. The polynomial in the denominator $\rightarrow$ constant as the integration variables $\rightarrow 0$

Outline

Components of an NLO calculation

For an n-particle observable, there are 2 types of processes to be considered at NLO

- 1-loop correction to n-particle process
- Tree-level (n+1)-particle process, where 1 particle can be unresolved

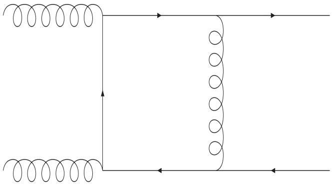


Figure: gg→ $t\bar{t}$

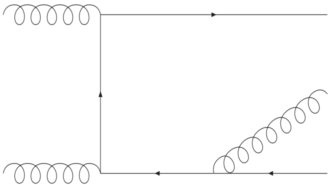
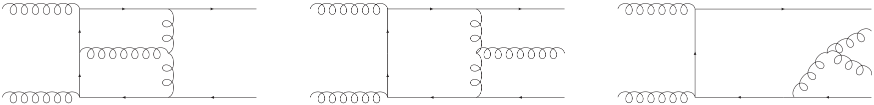


Figure: gg→ $t\bar{t}$+g

Components of an NNLO calculation

For an n-particle observable, there are 3 types of process to be considered at NNLO

- 2-loop correction to n-particle process
- 1-loop correction to (n+1)-particle process, where 1 particle can be unresolved
- Tree-level (n+2)-particle process, where up to 2 particles can be unresolved.



Each of these contains IR singularities. The soft and final-state collinear singularities cancel when the sum is taken.

Benefits of Sector Decomposition

- Doesn't require analytic integration
- Can be used for real corrections
- Can be used for virtual corrections to resolve endpoint singularities

Current Status of Sector Decomposition

A number of public programs exist for virtual corrections, including

- FIESTA2 (A.V Smirnov et al, arXiv:0912.0518)
- sector_decomposition (C. Bogner & S. Weinzierl arXiv:0709.4092), written in C++
- CSectors (Gluza, Kajda, Riemann, Yundin, arXiv:1010.1667), a Mathematica interface to sector_decomposition
- SecDec (JC & Heinrich,arXiv:1011.5493)

Current Status of Sector Decomposition

A few examples of calculations of cross sections using sector decomposition, and further developments

- Hadronic $t\bar{t}+$ double real radiation(Czakon arXiv:1101.0642), using sector decomposition guided by knowledge of the singularity structure
- NNLO QCD predictions for $gg \to H \to WW \to l\nu l\nu$ (Anastasiou et al. arXiv:0707.2373)
- Fully differential EW gauge boson production at hadron colliders to NNLO (Melnikov & Petriello, hep-ph/0609070)
- Second order QCD corrections to inclusive semileptonic $b \to X_c l \bar{\nu}_l$ (Biswas & Melnikov, arXiv:0911.4142)
- Non-linear transformations have been used alongside sector decomposition to reduce the number of produced sectors (Anastasiou et al, arXiv:1011.4867)

What are Multi-Dimensional Parameter Integrals?

In general, we wish to calculate integrals of the form

$I = \int_0^1 d^N\mathbf{x} \prod_{j=1}^k f_j(\mathbf{x})^{a_j+b_j\epsilon}$

where the f_j are polynomials, and ϵ is the regulator

(cf dimensional regularisation $\epsilon = \frac{4-D}{2}$)

Some of the f_j may contain overlapping singularities such that

$f_j(\mathbf{x}) \to 0$ as some subset of the $x_i \to 0$

Outline

Feynman Parameters

- Write down amplitude using Feynman rules
- Employ reduction methods to give the amplitude as a sum of certain master integrals with various coefficients
- Use Feynman parameters and integrate over loop momenta
- These master integrals are of the form $I = \int_0^1 (\prod_{i=1}^{N} dx_i)\delta(1 - \sum_{i=1}^{N} x_i)\mathcal{U}(\mathbf{x})^{a+b\epsilon}\mathcal{F}(\mathbf{x})^{c+d\epsilon}$
- N is the number of propagators, $\mathcal{U}$ is a function of $\mathbf{x}$, and $\mathcal{F}$ is a function of $\mathbf{x}$ and external invariants (s,m^2,...). Both can have zeroes when all or some of $x_i \to 0$

Behaviour of $\mathcal{F}$ and $\mathcal{U}$

- $\mathcal{U}$ is positive semi-definite.
- We require $\mathcal{F}$ to also be (positive or negative) semi-definite, such that the integrand has no singularities in the interior of the integration region.
- This is satisfied in the 'Euclidean region', defined to be the region where all invariants are negative, and all masses are positive.

  This is not physical, as clearly $\left(\sum_i^N p_i\right)^2 = 0$ cannot be satisfied.

  However, results from this region can be used to check analytic results.
- $\mathcal{F}$ can be semi-definite in some parts of the physical region, but this needs to be checked case by case.

Example

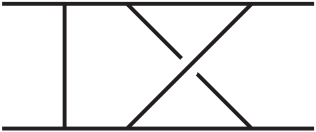


Figure: Non-Planar 2-loop massless box

$$\mathcal{F}(\mathbf{x}) = -x_1(s_{13}x_5x_6 + s_{23}x_4x_7) - s_{12}(x_3x_4x_5 + x_2(x_6x_7 + x_3(x_4 + x_5 + x_6 + x_7)))$$

$$\mathcal{U}(\mathbf{x}) = x_5x_6 + x_7x_6 + x_4(x_5 + x_7) + x_1(x_4 + x_5 + x_6 + x_7) + x_2(x_4 + x_5 + x_6 + x_7) + x_3(x_4 + x_5 + x_6 + x_7)$$

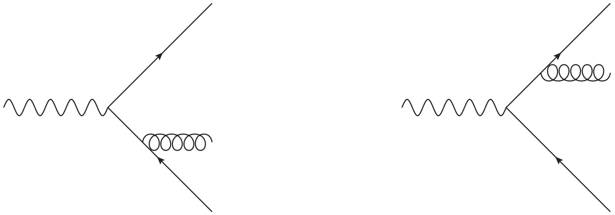
Outline

Simple Example (I)

The phase space of a $1 \to n$ process is given by

$$d\Phi_{1\to n} = \prod_{j=1}^{n}\left(\frac{d^D p_j}{(2\pi)^{D-1}}\delta^+(p_j^2 - m_j^2)\right)(2\pi)^D\delta^{(D)}\left(q - \sum_{k=1}^{n} p_k\right)$$

1 or more particles becoming unresolved (soft or collinear) can lead to singularities, which must be extracted so that we can compute them numerically.
As a simple example, consider $\gamma^*(q) \to u(p_1)\bar{u}(p_2)g(p_3)$

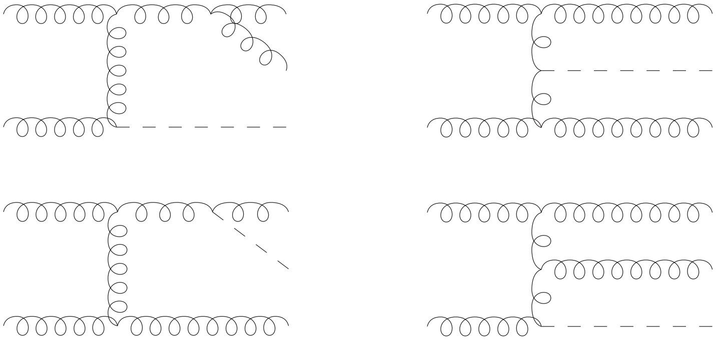


Simple Example (II)

After integrating over most degrees of freedom, including the matrix element squared and factoring out constants, we are left with the integral $\int_0^1 dz_1 dz_2 z_1^{-1-2\epsilon}(1-z_1)^{-\epsilon} z_2^{-1-\epsilon}(1-z_2)^{-1-\epsilon}\left(Ms_0+\epsilon Ms_1\right)$
Where we have used the parametrisation
$s_{12}=s(1-z_1), s_{23}=s(z_1(1-z_2)), s_{13}=s(z_1 z_2)$, and Ms_0, Ms_1 are the finite and $O(\epsilon)$ matrix elements squared, with singular factors pulled out. In this case all singularities are factorised immediately, but for more complicated examples this will not be the case.

More Complicated Example

When 2 final particles can become unresolved the singularity structure is significantly more complex. Eg
$g(p_1)g(p_2) \to H(p_h)g(p_3)g(p_4)$



Can have singular denominators s_{34}, s_{ih} and s_{if}

Outline

Iterative Decomposition

Consider the integral $I = \int_0^1 d^N\mathbf{x} \prod_{j=1}^k f_j(\mathbf{x})^{a_j+b_j\epsilon}$

- Find an f_j which requires decomposition. If no such f_j exists then the iteration terminates.
- Find a minimal subset of integration variables $\mathcal{S} = \{x_{i_1}, ..., x_{i_m}\}$ such that $f_j(\mathbf{x}) \to 0$ as $\mathcal{S} \to \mathbf{0}$
- Split $I = I_1 + ... + I_m$ where I_p has $x_{i_p} > x_{i_q} \forall x_{i_q} \in \mathcal{S}$
- For each I_p, remap the variables of $\mathcal{S}$ such that $x_{i_q} \to x_{i_p} x_{i_q} \forall p \neq q$
- Factorize x_{i_p} from all polynomials
- Repeat the iteration until the process terminates

Termination of Iteration

In general the choice of subset $\{x_{i_1}, ...x_{i_p}\}$ is not unique.

It has been shown (arXiv:0709.4092 Bogner & Weinzierl, arXiv:0812.4700 Smirnov & Smirnov) that strategies exist which guarantee that the iteration terminates, but such strategies tend to lead to a large number of subsectors.

A deterministic algorithm based on methods from computational geometry has been devised by Kaneko & Ueda (arXiv:0908.2897)

SecDec follows certain heuristic rules which leads to fewer subsectors, and has worked well in a multitude of practical applications.

Outline

Subtraction (I)

- After the iteration terminates and the subsectors are relabelled we have $I = \sum_{m=1}^{\#subsectors} I_m$
- Each I_m is of the form $\int_0^1 (\prod_{i=1}^{N} dx_i x_i^{\alpha_i+\beta_i\epsilon}) \prod_{j=1}^{k} \tilde{f}_j(\mathbf{x})^{a_j+b_j\epsilon}$
- Each $\tilde{f}_j$ is non-zero as $\mathbf{x} \to \mathbf{0}$ so we rewrite $\prod_{j=1}^{k} \tilde{f}_j(\mathbf{x})^{a_j+b_j\epsilon} \equiv g(\mathbf{x},\epsilon)$

Subtraction (II)

- All the singularities are contained in the $\prod_{i=1}^{N} dx_i x_i^{\alpha_i+\beta_i\epsilon}$
- If $\alpha_i > -1$ then there is no singularity in x_i
- If $\alpha_i = -1$, subtraction is needed.

  Write $g(x,\epsilon) \equiv g(0,\epsilon) + (g(x,\epsilon) - g(0,\epsilon))$

  $\int_0^1 x^{-1+\beta\epsilon} g(0,\epsilon) dx = \frac{g(0,\epsilon)}{\beta\epsilon} \int_0^1 dx$

  $\int_0^1 x^{\beta\epsilon} \frac{g(x,\epsilon)-g(0,\epsilon)}{x} = O(1)$
- If $\alpha_i <= -2$ then more terms of the Taylor expansion are required

Outline

Numerical Integration

- $I(\epsilon) = \sum_m I_m(\epsilon)$
- Perform the Laurent Expansion in ϵ
- For each order of ϵ the coefficient is a sum of well-behaved integrals over the N dimensional unit hypercube, each of which can be integrated numerically to yield the full result

Singularities for $x_i \neq 0$

The above method requires the f_j to only have zeros for some $x_i \to 0$. Often this is not the case. Trivially if a variable x_i causes a singularity at $x_i = 1$ only, one can remap $x_i \to 1 - x_i$. If x_i can cause a singularity at both 0 and 1, we split the integration region into $x_i < \frac{1}{2}$ and $x_i > \frac{1}{2}$.

There can also be line singularities of the form $\frac{1}{|f(x)-g(x)|}$. These f and g are non-negative functions, and have $f(x) = g(x)$ on a line $x_i = x_i(x_j)$. We split the integration region into $x_i < x_i(x_j)$ and $x_i > x_i(x_j)$, and now the singularity lies on the boundary of the integration.

Singularities for $x_i \neq 0$

For multi-scale loop integrals the kinematics determine the singularity structure of the integral. Eg a term in the integrand might be $\frac{1}{(-s_{12} t_1 t_2 - s_{13})^2}$, which can only be integrated numerically if the denominator is non-zero throughout the integration region.

This cannot be solved in general by splitting the hypercube, as the splitting would depend on the kinematic point. A solution can be contour integration in the complex plane (see e.g. work by Soper, Nagy, Binoth/Guillet/Heinrich, Anastasiou, Weinzierl, ...).

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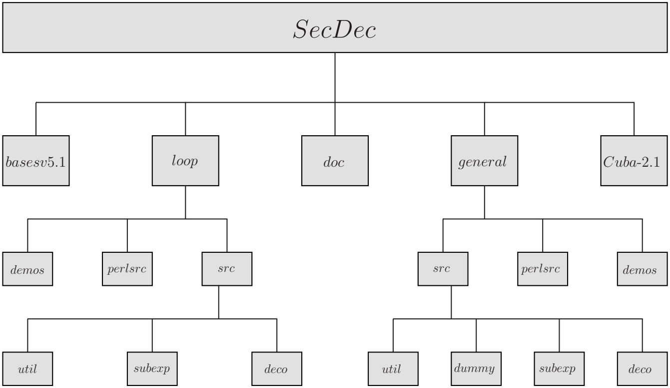
- The program can be downloaded from http://projects.hepforge.org/secdec/
- Documentation and examples can be found at Comput. Phys. Commun. 182(2011)1566 (JC & Heinrich), and in the program package itself.
- Review article on Sector Decomposition Int.J.Mod.Phys.A23:1457-1486 (2008) (Heinrich)
- Original article on multi-loop integrals Nucl.Phys.B585:741 (2000) (Binoth & Heinrich)

Outline

Structure


SecDec
basesv5.1
loop
doc
general
Cuba-2.1
demos
perlsrc
src
src
perlsrc
demos
util
subexp
deco
util
dummy
subexp
deco

Loop Template File

To perform the calculation, SecDec requires 2 files: a template file and a parameter file.
The template file contains

- A list of propagators
- A list of loop momenta
- A list containing the numerator (eg $\{1\}$ or $\{p_1 k_1, k_1 k_2\}$)
- A list of replacement rules - ‘onshell’ conditions
- The dimension (usually Dim=4-2eps)

Template Example

For the 1-loop box:

momlist= $\{k1\}$;

proplist= $\{k1^2, (k1+p1)^2, (k1+p1+p3)^2, (k1-p2)^2\}$;

numerator= $\{1\}$

onshell= $\{ssp[\_] \to 0\}$;

Dim= $4-2*eps$;

Loop Parameter File

The parameter file contains

- The name of the graph
- Number of legs
- Number of loops
- Number of propagators
- Required order in the ϵ expansion
- Further parameters, all of which can be unassigned and take default values

Parameter Example

For the 1-loop box:

graph=box

legs=4

loops=1

propagators=4

epsord=2

Template for General Parameter Integrals

The template file contains

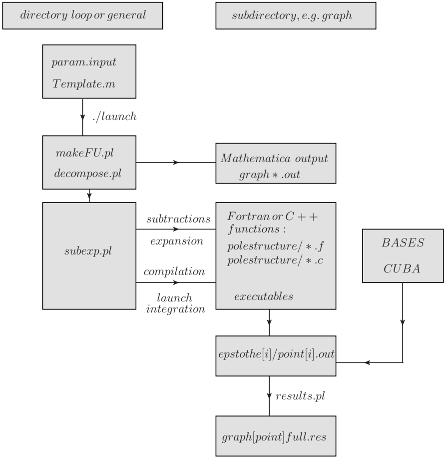
- A list of integration variables
- The integrand, in the form $\{\{f_1, a_1 + b_1 eps\}, ...\{f_k, a_k + b_k eps\}\}$ (which represents $\prod_{j=1}^{k} f_j^{a_j + b_j \epsilon}$)
- A list of variables which can cause singularities at both 0 and 1

General Parameter File

The parameter file contains

- The name of the integrand
- Required order in the ϵ expansion
- Further parameters, all of which can be unassigned and take default values

Flowchart


directory loop or general
subdirectory, e.g. graph
param.input
Template.m
./launch
makeFU.pl
decompose.pl
Mathematica output
graph * .out
subexp.pl
subtractions
expansion
compilation
launch
integration
Fortran or C ++
functions :
polestructure/ * .f
polestructure/ * .c
executables
BASES
CUBA
epstothe[i]/point[i].out
results.pl
graph[point]full.res

Outline

Original Features

SecDec includes a number of useful features

- Parameters (eg invariant masses) can be left implicit up until the numerical integration. This means that the algebraic part needs to only be run once, and then various numerical points can be calculated
- Contracted tensors in the numerator and non-integer powers of propagators in the denominator allowed for loop integrals
- Choice of integrators - BASES (Kawabata, Comput. Phys. Commun. 88 (1995) 309) and the Cuba library (Hahn, Comput.Phys.Commun.168(2005)78) are available, with full control over parameters used (eg desired accuracy)
- Simple to use with Portable Batch System for parallel processing, and readily adjusted to work for different batch syntax
- Subtraction with non-integer powers of variables

New Features

Since SecDec-1.0, a number of new features have been added:

- Functions can be left implicit for the algebraic stage. This is particularly useful for dealing with complicated but finite functions, where quantitative knowledge of these functions is not required to guide the decomposition, or for including measurement functions (currently `general` only)
- Choice of Fortran or C++ for numerics (currently `loop` only)
- Automation of calculating a set of different numerical points (currently `general` only)

Future Features

There are a number of features currently being planned/developed:

- Contour deformation to calculate loop integrals in the physical region. This has been implemented, and is in the testing stage. It has proved successful in a number of cases, see later for an example.
- Interfacing SecDec with a matrix element generator to calculate real radiation contributions

Outline

Non-planar 2-loop Corrections for $t\bar{t}$ (I)

Analytic results for these diagrams were not available until an hour ago...

An overall factor of $\Gamma(1+\epsilon)^2$ is extracted.

Figure: Non-planar graphs ocurring in the calculation of $gg \to t\bar{t}$ at NNLO. Blue (solid) lines denote massive particles.

Non-planar 2-loop Corrections for $t\bar{t}$ (II)

ggtt1		
(s,t,u,m_1^2,m_2^2)	(-0.5,-0.4,-0.1,0.17,0.17)	(-1.5,-0.3,-0.2,3,1)
P_0	-38.0797±0.0027	-0.19904±1.5 $\times 10^{-5}$
P_1	-263.22± 0.015	-0.71466±6 $\times 10^{-5}$
P_2	-936.86± 0.06	-1.45505± 0.0002
ggtt2		
(s,t,u,m_1^2,m_2^2)	(-0.5,-0.4,-0.1,0.17,0)	(-1.5,-0.3,-0.2,3,0)
P_{-4}	-10.9159 ± 0.0006	-0.13678±1.46 $\times 10^{-5}$
P_{-3}	-43.5213 ± 0.0075	-0.2087 ±0.00024
P_{-2}	165.384 ± 0.048	3.3417 ±0.0014
P_{-1}	20.842±0.268	-6.593±0.007
P_0	2117.5 ± 1. 57	20.42±0.04

3-loop Form Factors (I)

9-propagator master integrals for massless 3-loop form factors have been calculated using Mellin-Barnes representation and sector decomposition (Heinrich, Huber, Kosower & Smirnov '09, Baikov, Chetyrkin, Smirnov, Smirnov & Steinhauser '09, Lee, Smirnov & Smirnov '10). These result are reproduced using SecDec.

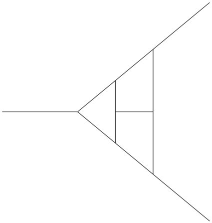


Figure: $A_{9,1}$

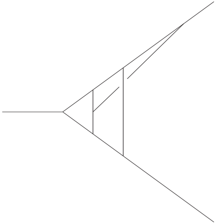


Figure: $A_{9,2}$

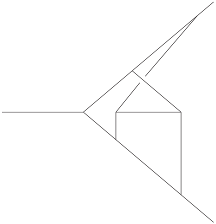


Figure: $A_{9,4}$

3-loop Form Factors (II)

$$A_{9,*} = i\Gamma(1-\epsilon)^{-3}(-q^2-i\eta)^{-3-3\epsilon}(C_{-6}/\epsilon^6 + C_{-5}/\epsilon^5 + C_{-4}/\epsilon^4 + C_{-3}/\epsilon^3 + C_{-2}/\epsilon^2 + C_{-1}/\epsilon + C_0)$$

$A_{9,4}$	Analytic	SecDec	Longest time(s)	Total time(s)
C_{-6}	0.111111	0.11111	8	17
C_{-5}	0.888889	0.8889	60	154
C_{-4}	-4.65541	-4.652	529	1397
C_{-3}	-33.1607	-33.14	5705	16297
C_{-2}	-42.8359	-42.83	7879	93598
C_{-1}	117.400	117.5	15375	382138
C_0	1948.17	1948	20330	1371382

Longest algebraic calculation took 40173 seconds, total algebraic time was 108190 seconds.

4-loop Propagator (I)

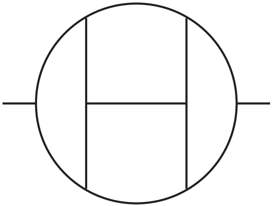


Figure: A four-loop two-point master integral

4-loop Propagator (II)

Stage	Longest time	Total time (secs)
Sector Decomposition	615	2160
Subtraction & ϵ-expansion	809	3767
Numerical integration ϵ^{-1}	156	508
Numerical integration ϵ^{0}	422	4720
Numerical integration ϵ^{1}	492	5946
Numerical integration ϵ^{2}	2172	8123
Full Calculation	-	25224

4-loop Propagator (III)

Order	Analytical result	Numerical result
ϵ^{-1}	-10.3692776	-10.371$\pm$0.002
ϵ^{0}	-70.99081719	-71.002$\pm$0.013
ϵ^{1}	-21.663005	-21.65$\pm$0.12
ϵ^{2}	Unknown	2833.79$\pm$0.92

Analytical results calculated by Baikov & Chetyrkin (arXiv:1004.1153v2)

This has also been calculated using FIESTA 2, with a lower stated error 0.17 for ϵ^2, however this takes almost 9 days to run, compared to SecDec's 7 hours on a single core.

4-loop Propagator (IV)

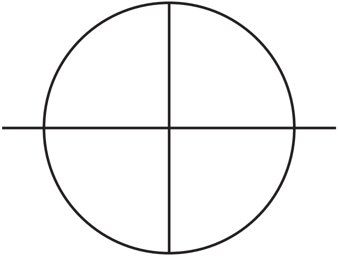


Figure: M_{36}, $O(\epsilon^2)$
Total time (s)
SecDec: 976, FIESTA2: 10565

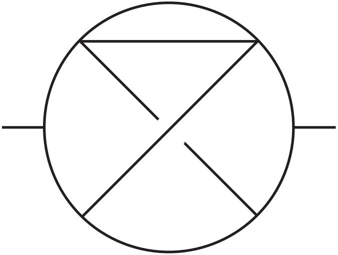
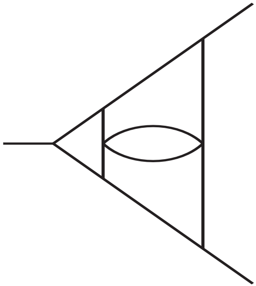


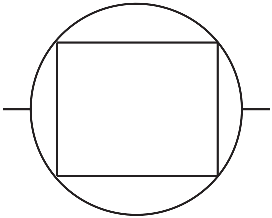
Figure: M_{41}, $O(\epsilon^2)$
Total time (s)
SecDec: 10595, FIESTA2: 22778

Calculation performed by M. Zentile

Higher loop calculations

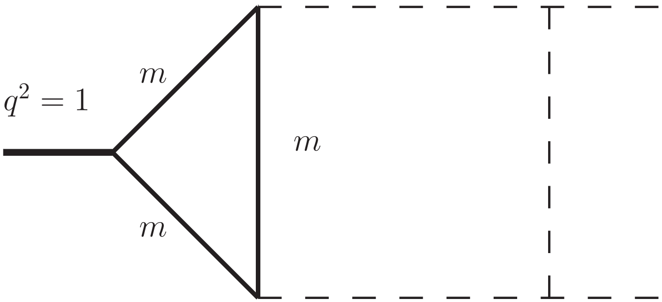
SecDec has recently been applied to a number of 4-loop vertex and 5-loop propagator master integrals (G. Welsh & M. Zentile), many of which have not been calculated analytically.





Contour Deformation: 2-Loop Massive Triangle

Contour deformation is being implemented in SecDec. Here I present the result for a 2-loop massive triangle with one leg offshell.



2-Loop Massive Triangle

m=0.2	SecDec	Analytic
ϵ^{-2}	4.47+3.02*I	4.47167+3.02354*I
ϵ^{-1}	-16.6+2.98*I	-16.54938+2.98071*I
finite	10.98-63.57*I	10.9682- 63.63681*I

Table: A factor of $-m^{-2\epsilon}Exp[-2\epsilon\gamma_E]$ has been extracted from the result. Analytic result is that of P_{126} from hep-th/0303162, Davydychev & Kalmykov

Hypergeometric ${}_4F_3$ (I)

This example demonstrates half-integer exponents. The analytic result was calculated using HypExp 2
(Huber & Maitre, arXiv:0708.2443)

$$\frac{\Gamma\left(2\epsilon-\frac{1}{2}\right)\Gamma\left(4\epsilon-\frac{1}{2}\right)\Gamma\left(6\epsilon+\frac{1}{2}\right)(1-x_1)^{3\epsilon-1}x_1^{-\epsilon-\frac{3}{2}}(1-x_2)^{6\epsilon}x_2^{-2\epsilon-\frac{5}{2}}(1-x_3)^{9\epsilon-1}x_3^{-3\epsilon-\frac{1}{2}}(1-\beta x_1x_2x_3)^{4\epsilon}}{\Gamma\left(\frac{1}{2}-3\epsilon\right)\Gamma\left(-2\epsilon-\frac{3}{2}\right)\Gamma\left(-\epsilon-\frac{1}{2}\right)\Gamma(3\epsilon)\Gamma(9\epsilon)\Gamma(6\epsilon+1)}$$

Hypergeometric ${}_4F_3$ (II)

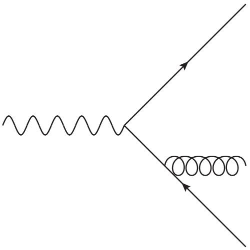
ϵ order	analytic result	numerical result	time taken (secs)
ϵ^0	1	$1.000000 \pm 1.6 \times 10^{-5}$	0.01
ϵ^1	-4.27969	-4.2823$\pm$0.0013	0.33
ϵ^2	-26.6976	-26.734$\pm$0.075	1.32

Table: ${}_4F_3(-4\epsilon, -\frac{1}{2}-\epsilon, -\frac{3}{2}-2\epsilon, \frac{1}{2}-3\epsilon; -\frac{1}{2}+2\epsilon, -\frac{1}{2}+4\epsilon, \frac{1}{2}+6\epsilon; \beta)$ at $\beta = 0.5$.

Algebraic part of the calculation took 4.8 seconds.

Implicit Functions Example

Again we consider $\gamma^* \to u\bar{u}g$. If one parton is unresolved, then this is an NLO correction to $\gamma^* \to 2j$. To this end we include a jet measurement function in the integrand, which is left implicit during the algebraic calculation. The following results were obtained with an implementation of the JADE algorithm, ie if $min(s_{12}, s_{23}, s_{13}) < yq^2$ then one parton is considered unresolved.



y :	0.05	0.1
ϵ^{-2}	2	2
ϵ^{-1}	3	3
finite	-9.02	-4.06

Summary

- Higher order calculations require the evaluation of multi-dimensional parameter integrals.
- Sector decomposition algorithm can be used to compute these integrals.
- I presented the public program SecDec, and demonstrated how it performs very well, both for complicated loop diagrams and general parameter integrals.

Numerical Stability (I)

Consider the integral $I = \int_0^1 x^{-2+\epsilon} f(x) dx$ After subtraction, we have an integral of the form $\int_0^1 x^{\epsilon} \frac{f(x)-f(0)-f'(0)x}{x^2} \equiv \int_0^1 x^{-2+\epsilon} h(x)$

This is integrable, but as $x \to 0$, we get something which is $O(\frac{0^2}{0^2})$, which is numerically unstable.

In practice this example is coped with easily by numerical integrators, for complicated problems with many subtractions, the problem is magnified.

Numerical Stability (II)

One can use integration by parts to explicitly remove the $\frac{1}{x^2}$.

$I = [\frac{x^{-1+\epsilon}}{-1+\epsilon} h(x)]_0^1 + \frac{1}{1-\epsilon} \int_0^1 x^{-1+\epsilon} h'(x) dx$

By construction, $h(x)$ is at least $O(x^2)$ as $x \to 0$, so $h'(x)$ is at least $O(x)$

$I = \frac{h(1)}{-1+\epsilon} + \frac{1}{1-\epsilon}([\frac{x^\epsilon}{\epsilon} h'(x)]_0^1 - \frac{1}{\epsilon} \int_0^1 x^\epsilon h''(x) dx)$

We expand $x^\epsilon = 1 + \epsilon log(x) + ...$ and explicitly integrate the first term:

$I = \frac{h(1)}{-1+\epsilon} + \frac{1}{\epsilon(1-\epsilon)} \int_0^1 (1 - x^\epsilon) h''(x) dx$

Numerical Stability (III)

The downside to this is that expressions can become large. However the method gives a solution to this problem:
If we consider $h(x) = \sum_m g_m(x)$, then the integral after IBP is

$$I = \frac{\sum_m g_m(1)}{-1+\epsilon} + \frac{1}{\epsilon(1-\epsilon)} \int_0^1 (1 - x^\epsilon)(\sum_m g_m(x))'' dx$$

$$I = \sum_m (\frac{g_m(1)}{-1+\epsilon}) + \sum_m \frac{1}{\epsilon(1-\epsilon)} \int_0^1 (1 - x^\epsilon) g_m''(x) dx$$

where each of these integrals is finite, and so can be performed separately, thus large expressions can be split down into many smaller ones.