DRA method[1]

Powerful tool for the calculation of the loop integrals.

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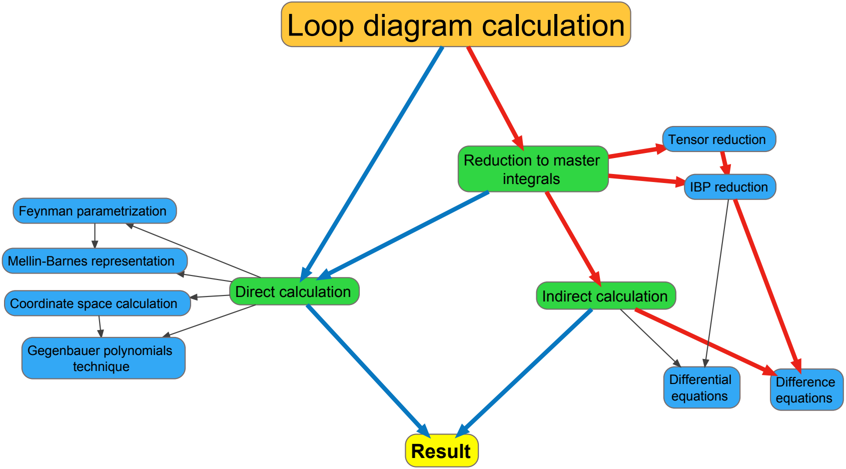
[1] Lee R.N., *Nucl. Phys. B* **830** (2010) 474

Outline

Calculation path



Loop Integral

L-loop diagram with E external momenta $p_1, \ldots p_E$:

Loop integral

$$J(\mathbf{n}) = J(n_1, \ldots, n_N) = \int d^{\mathscr{D}} l_1 \ldots d^{\mathscr{D}} l_L j(\mathbf{n}) = \int \frac{d^{\mathscr{D}} l_1 \ldots d^{\mathscr{D}} l_L}{D_1^{n_1} \ldots D_N^{n_N}}$$

where $D_1, \ldots, D_M$ are denominators of the diagram, and $D_{M+1}, \ldots, D_N$ are some additionally chosen numerators.

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Prerequisites

All denominators and numerators linearly depend on $l_i \cdot q_j$. Any product $l_i \cdot q_j$ can be expressed via D_k.

Notation

$$q_i = \begin{cases} l_i, & i \leqslant L \\ p_{i-L}, & i > L \end{cases}$$

The total number of denominators and numerators

$$N = L(L+1)/2 + LE, \quad N \geqslant M$$

Integration-by-part identities

The **integration-by-part identities** arise due to the fact, that, in dimensional regularization the integral of the total derivative is zero Tkachov (1981), Chetyrkin and Tkachov (1981)

IBP identities

$$\int d^{\mathscr{D}}l_1 \ldots d^{\mathscr{D}}l_L\, O_{ij}\, j(\mathbf{n}) = 0 \qquad \text{(IBP)}$$

IBP operators

$$O_{ij} = \frac{\partial}{\partial l_i} \cdot q_j$$

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The **Lorentz-invariance identities** arise due to the fact that loop integrals are scalar functions of the external momenta (Gehrmann and Remiddi 2000).

LI identities

$$p_{1\mu} p_{2\nu} M^{\mu\nu} J = 0 \qquad \text{(LI)}$$

Lorentz generators

$$M^{\mu\nu} = \sum_e p_e^{[\mu} \partial_e^{\nu]}$$

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Huge redundancy of IBP&LI identities. In particular, it can be shown that LI identities are linear combinations of the IBP identities (Lee 2008).

Calculation of master integrals

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Using several available methods we can reduce all loop integrals emerging in our problem to a small set of master integrals.

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Reduction vs Calculation

- Reduction to masters requires some **algebraic** methods.
- Calculation of master integrals requires **analytic** methods.

Differential equations

Differentiating with respect to external parameter and performing IBP reduction of the result, we obtain differential equation for a given master integral(Kotikov 1991, Remiddi 1997).

Differential equation

$$\frac{\partial}{\partial a} J = f(a) J + h(a). \qquad \text{(DE)}$$

External parameter

$$s = \begin{cases} \text{mass} & \text{(Kotikov, 1991)} \\ \text{invariant of } p_e & \text{(Remiddi, 1997)} \end{cases}$$

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- **Scaleless integrals are zero in dimensional regularization.**
- ***n*-scale integrals ($n \geq 2$) can be investigated by the differential equation method.**
  Initial conditions for the differential equation are put in the point where the chosen parameter is expressed via the rest (or equal to $0, \infty$) $\Longrightarrow$ The problem is reduced to the calculation of integrals with $n-1$ parameter.

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  Initial conditions for the differential equation are put in the point where the chosen parameter is expressed via the rest (or equal to $0, \infty$) $\Longrightarrow$ The problem is reduced to the calculation of integrals with $n - 1$ parameter.
- **One-scale integrals have obvious dependence on this scale. Differential equations cannot help.**
  Example: Massless propagator-type integrals, massive vacuum-type integrals, onshell massless vertices, onshell massive propagator

Laporta's difference equations

One-scale multiloop ($L \geqslant 2$) integrals:

Conventional approach: either direct calculation (by Mellin-Barnes transformation) or by Laporta's difference equations (Laporta 2000).

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Derivation

Consider "generalized master" $J(x)$ obtained from the original $J(1)$ by rasing one denominator to power x. Perform Laporta algorithm near $J(x)$ in order to find the recurrence relation of the form

$$\sum_{k=0}^{n} c_k(x) J(x+k) = h(x) \qquad \text{(L}\Delta\text{E)}$$

The left-hand side contains simpler integrals which are assumed to be known.

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Solution

Factorial series or Laplace transform (Laporta 2000). Homogeneous part can be fixed from large-x asymptotics.

Laporta's difference equations

Weak points

- Order of difference equation can be high $n \sim 10$
- Slow convergence of the factorial series at small $x \implies$ Calculate at sufficiently large x and then use recurrence to reach $x = 1 \implies$ loss of precision.
- Does not work for all-massless cases.

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Results

- Numerical and analytical (using `pslq`) results for four-loop massive tadpoles (Laporta 2002, Schroder and Vuorinen 2003, 2005, Bejdakic and Schroder 2006)
- Numerical results for three-loop onshell massive operators. Numerical and analytical results for three and four-loop onshell sunrise.(Laporta 2001, 2008)

Dimensional recurrence relation

Another variant of difference equation for the master integrals: Dimensional recurrence relation (Tarasov 1996).

Advantages

- Small order of dimensional recurence. Topologies with only one master $\Longrightarrow$ first-order equation.
- Fast convergence. In many cases the convergence is exponential $\Longrightarrow$ easy to obtain precise results and then use `pslq`.

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Why not to use?

The homogeneous part of the solution depends on several (or one) periodic functions. Their determination appears to be extremely difficult!
Initial Tarasov's idea to fix them from the large-$\mathscr{D}$ asymptotics does not work for multiloop integrals.

Dimensional recurrence relation

Derivation from Baikov's formula

Baikov's approach (to reduction)

Pass from the integration over the loop momenta to the integration over loop-momenta dependent scalar products (or the denominators $\mathscr{D}$)

$$d^{\mathscr{D}}l_1 \dots d^{\mathscr{D}}l_L \longrightarrow ds_{11}ds_{12}\dots ds_{L,L+E}, \qquad s_{ij} = l_i \cdot q_j$$

Master formula

$$\int \frac{d^{\mathscr{D}}l_1 \dots d^{\mathscr{D}}l_L}{\pi^{L\mathscr{D}/2} D_1^{n_1} \dots D_N^{n_N}} = \frac{\mu^L \pi^{-LE/2 - L(L-1)/4}}{\Gamma\left[\left(\mathscr{D} - E - L + 1\right)/2, \dots, \left(\mathscr{D} - E\right)/2\right]}$$

$$\times \int \left(\prod_{i=1}^{L} \prod_{j=i}^{L+E} ds_{ij}\right) \frac{\left[V\left(l_1, \dots l_L, p_1, \dots, p_E\right)\right]^{(\mathscr{D}-E-L-1)/2}}{\left[V\left(p_1, \dots, p_E\right)\right]^{(\mathscr{D}-E-1)/2} D_1^{n_1} \dots D_N^{n_N}}$$

$$V\left(q_1, \dots\right) = \det\{q_i q_j\} = P\left(D_1, \dots D_N\right)$$

Operator representation

Let us introduce the operators, acting on the functions on $\mathbb{Z}^N$:

Operators $A_1,\ldots,A_N,B_1,\ldots,B_N$

$$\left(A_\alpha f\right)\left(n_1,\ldots,n_N\right)=n_\alpha f\left(n_1,\ldots,n_\alpha+1,\ldots,n_N\right),$$
$$\left(B_\alpha f\right)\left(n_1,\ldots,n_N\right)=f\left(n_1,\ldots,n_\alpha-1,\ldots,n_N\right).$$

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$$(B_\alpha f)(n_1,\ldots,n_N) = f(n_1,\ldots,n_\alpha-1,\ldots,n_N).$$

Commutator

$$[A_\alpha,B_\beta] = \delta_{\alpha\beta}$$

A and B well suited to sectors

For any polynomial $P(A,B)$ the result of action

$$P(A,B)J(\mathbf{n}) = \sum C_i J(\mathbf{n}_i)$$

contains only integrals of the same and lower sectors as $J(\mathbf{n})$.

Dimensional recurrence relation

Derivation from Baikov's formula

Lowering DRR

Separating one factor $V(l_1,\ldots l_L,p_1,\ldots,p_E) = P(D_1,\ldots D_N)$

$$J^{(\mathscr{D}+2)}(\mathbf{n}) = \frac{(2\mu)^L\left[V(p_1,\ldots,p_E)\right]^{-1}}{(\mathscr{D}-E-L+1)_L}\left(P(B_1,\ldots,B_N)\,J^{(\mathscr{D})}\right)(\mathbf{n})\,.$$

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Raising DRR

Recognising total derivative in Jacobian

$$J^{(\mathscr{D}-2)}(\mathbf{n}) = \mu^L \det\left[\sum_k \frac{\partial D_k}{\partial s_{ij}} A_k\bigg|_{i,j=1,\ldots L}\right] J^{(\mathscr{D})}(\mathbf{n}).$$

Solution of dimensional recurrence relation

Dimensional recurrence relation

$$J^{(\mathscr{D}-2)} = C(\mathscr{D})\, J^{(\mathscr{D})} + R(\mathscr{D}),$$

If there is no other master integrals of the same topology, $R(\mathscr{D})$ contain only integrals of the simpler topologies and is assumed to be known.

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  Summing factor permits multiplication by periodic function.
- The general solution reads:

$$J^{(\mathscr{D})} = \Sigma^{-1}(\mathscr{D})\left[\omega(z) - \sum_{k=0}^{\infty}\Sigma(\mathscr{D}-2k-2)\,R(\mathscr{D}-2k)\right],$$

where $\omega(z) = \omega(\exp[i\pi\mathscr{D}])$ is arbitrary function to be fixed.

Liouville&Mittag-Leffler's theorems

Liouville&Mittag-Leffler's theorems from complex analysis

Meromorphic function $f(z)$ can be restored from its singular parts up to the holomorphic function $h(z)$. If $f(z)$ is bounded at infinity, $h(z)$ is constant.

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Idea.

We can fix $\omega(z)$ by considering the analytical properties of $J^{(\mathscr{D})}$. Let us express $\omega(z)$ from the general solution

$$\omega(z) = \Sigma(\mathscr{D}) J^{(\mathscr{D})} + \sum_{k=0}^{\infty} \Sigma(\mathscr{D} - 2k - 2) R(\mathscr{D} - 2k),$$

Suppose that we know all singularities of $\Sigma(\mathscr{D}) J^{(\mathscr{D})}$ on some **basic stripe** $S = \{\mathscr{D}, \operatorname{Re}\mathscr{D} \in (d, d+2]\}$ and know that $\Sigma(\mathscr{D}) J^{(\mathscr{D})}$ behaves well when $\operatorname{Im}\mathscr{D} \to \pm\infty$. Then we can use Mittag-Leffler's theorem to fix $\omega(z)$.

Analytical properties from parametric representation

Parametric representation

If I is the number of internal lines of the integral, parametric representation reads

$$J^{(\mathscr{D})} = \Gamma\left(I - L\mathscr{D}/2\right) \int dx_1 \dots dx_I \delta\left(1 - \sum x_i\right) \frac{\left[Q\left(x\right)\right]^{\mathscr{D}L/2-I}}{\left[P\left(x\right)\right]^{\mathscr{D}(L+1)/2-I}}$$

$P(x) > 0$ and $Q(x) > 0$ are determined in terms of trees and 2-trees of the graph. Dependence on $\mathscr{D}$ is explicit here.

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Observations

- If the integral converges on the real interval $\mathscr{D} \in (d_1, d_2]$, it is a holomorphic function on the whole stripe $\{\mathscr{D}, \operatorname{Re}\mathscr{D} \in (d_1, d_2]\}$.
- When $\operatorname{Im}\mathscr{D} \to \pm\infty$ the integral can be estimated as

$$J^{(\mathscr{D})} \lesssim \text{const} \times e^{-\pi L|\operatorname{Im}\mathscr{D}|/4} \left|\operatorname{Im}\mathscr{D}\right|^{I-1/2-L\operatorname{Re}(\mathscr{D})/2}$$

Path of calculations

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6. If needed, fix the remaining constants by conventional methods.

Comparison with MB

Resulting expressions both in DRA and MB methods are multiple sums.
Is there any advantage in DRA result?

DRA method

Geometric multiple sums of the form

$$\sum_{\infty > k_1 \geqslant \cdots \geqslant k_n \geqslant 0} f_1(k_1) \ldots f_n(k_n)$$

MB representation

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$$\sum_{k_1} \ldots \sum_{k_n} f(k_1, \ldots, k_n)$$

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Complexity scales linearly with n.

do $k = 0..k_{max}$

do $i = 1..n$

$S_i = S_i + S_{i-1} f_i(k)$

end do

end do

return S_n

MB representation

Harmonic multiple sums of the form

$$\sum_{k_1} \dots \sum_{k_n} f(k_1, \dots, k_n)$$

Complexity scales exponentially.

do $k_1 = 0..k_{max}$

... //n-fold

do $k_n = 0..k_{max}$

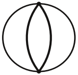
$S = S + f(k_1, \dots)$

end do

return S

Example 1

Three-loop sunrise tadpole

$$J^{(\mathscr{D})} = $$  $$= \int \frac{d^{\mathscr{D}}k\, d^{\mathscr{D}}l d^{\mathscr{D}}r}{\pi^{3\mathscr{D}/2}\left[k^2+1\right]\left[l^2+1\right]\left[r^2+1\right]\left[(k+l+r)^2+1\right]}$$

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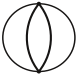
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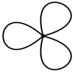
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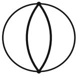
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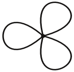
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2. The integral $J^{(\mathscr{D})}$ is a holomorphic function in the stripe $S = \{\mathscr{D}, \operatorname{Re}\mathscr{D} \in [-2,0)\}$ which can be deduced from its parametric representation. It is easy to check that any Euclidean integral with all lines massive is holomorphic in the whole half-plane $\operatorname{Re}\mathscr{D} < 0$.

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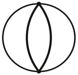
$$J^{(\mathscr{D})} =$$  $$= \int \frac{d^{\mathscr{D}}k\, d^{\mathscr{D}}l\, d^{\mathscr{D}}r}{\pi^{3\mathscr{D}/2}\left[k^2+1\right]\left[l^2+1\right]\left[r^2+1\right]\left[(k+l+r)^2+1\right]}$$

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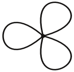
$$J_a^{(\mathscr{D})} =$$  $$= \Gamma^3\left(1-\mathscr{D}/2\right)$$

2. The integral $J^{(\mathscr{D})}$ is a holomorphic function in the stripe $S = \{\mathscr{D}, \operatorname{Re}\mathscr{D} \in [-2,0)\}$ which can be deduced from its parametric representation. It is easy to check that any Euclidean integral with all lines massive is holomorphic in the whole half-plane $\operatorname{Re}\mathscr{D} < 0$.
3. Dimensional recurrence reads

$$J^{(\mathscr{D}-2)} = -\frac{(3\mathscr{D}-10)_3\,(\mathscr{D}-2)}{128(\mathscr{D}-4)}J^{(\mathscr{D})} - \frac{(11\mathscr{D}-38)(\mathscr{D}-2)^3}{64(\mathscr{D}-4)}J_a^{(\mathscr{D})}$$

Example 1

Three-loop sunrise tadpole

4. We choose the summing factor as

$$\Sigma(\mathscr{D}) = \frac{4^{-\mathscr{D}}\Gamma(2-\mathscr{D}/2)}{\Gamma(3/2-\mathscr{D}/2)\Gamma(3-3\mathscr{D}/2)}$$

The general solution has the form

$$\Sigma(\mathscr{D})J^{(\mathscr{D})} = \omega(z) + \sum_{k=1}^{\infty} t(\mathscr{D}-2k), \quad t(\mathscr{D}) = \frac{4^{-\mathscr{D}-2}(11\mathscr{D}-16)\Gamma^4(1-\mathscr{D}/2)}{\Gamma(3/2-\mathscr{D}/2)\Gamma(3-3\mathscr{D}/2)}$$

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5. Both $\Sigma(\mathscr{D})J^{(\mathscr{D})}$ and $\sum_{k=1}^{\infty} t(\mathscr{D}-2k)$ are holomorphic on $S \implies$ the function $\omega(z)$ is also holomorphic in the whole plane except, maybe $z=0$.
   Both $\Sigma(\mathscr{D})J^{(\mathscr{D})}$ and $\sum_{k=1}^{\infty} t(\mathscr{D}-2k)$ grow slower than $|z|^{\mp 1}$ when $\mathrm{Im}\,D \to \pm\infty \implies$ the function $\omega(z)$ is constant!

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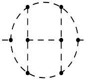
6. From $J^{(0)} = 1$ we obtain

$$\omega(z) = -\sum_{k=0}^{\infty} t(-2k) \stackrel{\text{pslq}}{=} \frac{3\pi^{3/2}}{16}$$

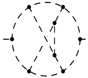
How does it work in practice?

Some new results

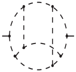
Baikov&Chetyrkin's MIs up to 12th t.w. (Lee, Smirnov and Smirnov 2011b)



$$
\begin{aligned}
= \frac{(1-2\varepsilon)^3}{(1+3\varepsilon)^3}\Bigg\{ & -\frac{10\zeta_5}{\varepsilon} - \left(100\zeta_5 + 10\zeta_3^2 + 25\zeta_6\right) - \left(210\zeta_5 + 100\zeta_3^2 + 250\zeta_6 + 30\zeta_3\zeta_4 - \frac{19\zeta_7}{2}\right)\varepsilon + \Big(-66\zeta_3^2 \\
& - 525\zeta_6 - 300\zeta_3\zeta_4 + 1257\zeta_7 + 1564\zeta_3\zeta_5 - 567\zeta_8 + 162\zeta_{2,6}\Big)\varepsilon^2 + \Big(-198\zeta_3\zeta_4 + \frac{11151\zeta_7}{2} + 19344\zeta_3\zeta_5 + 1043\zeta_8 + 972\zeta_{2,6} + \frac{3440\zeta_3^3}{3} \\
& + 1374\zeta_4\zeta_5 + 3150\zeta_3\zeta_6 + \frac{21637\zeta_9}{3}\Big)\varepsilon^3 + \Big(43092\zeta_3\zeta_5 + 28455\zeta_8 - 1782\zeta_{2,6} + \frac{40792\zeta_3^3}{3} + 23184\zeta_4\zeta_5 + 44000\zeta_3\zeta_6 + \frac{339727\zeta_9}{3} \\
& + 5160\zeta_3^2\zeta_4 + \frac{287486\zeta_5^2}{7} - \frac{286\zeta_3\zeta_7}{7} + \frac{316935\zeta_{10}}{7} + \frac{6642\zeta_{3,7}}{7}\Big)\varepsilon^4 + \Big(17848\zeta_3^3 + 75330\zeta_4\zeta_5 + 116970\zeta_3\zeta_6 + \frac{1061648\zeta_9}{3} + 61188\zeta_3^2\zeta_4 \\
& + \frac{807571\zeta_5^2}{2} + \frac{313421\zeta_3\zeta_7}{2} + \frac{2509185\zeta_{10}}{4} + \frac{17127\zeta_{3,7}}{2} - 196432\zeta_3^2\zeta_5 + 158884\zeta_5\zeta_6 + 141174\zeta_4\zeta_7 + 133604\zeta_3\zeta_8 + 682428\zeta_2\zeta_9 \\
& - \frac{17560877\zeta_{11}}{24} - 78360\zeta_3\zeta_{2,6} + 36888\zeta_{2,1,8}\Big)\varepsilon^5 + \Big(80316\zeta_3^2\zeta_4 + \frac{13986207\zeta_5^2}{14} + \frac{4863417\zeta_3\zeta_7}{14} + \frac{55755309\zeta_{10}}{28} + \frac{490491\zeta_{3,7}}{14} \\
& - 1439456\zeta_3^2\zeta_5 + 1489454\zeta_5\zeta_6 + \frac{3397821\zeta_4\zeta_7}{2} + 925922\zeta_3\zeta_8 + 7235868\zeta_2\zeta_9 - \frac{330988949\zeta_{11}}{48} - 583584\zeta_3\zeta_{2,6} + 391128\zeta_{2,1,8} \\
& - \frac{720896\zeta_3^4}{9} - 777104\zeta_3\zeta_4\zeta_5 - \frac{3772008}{7}\zeta_2\zeta_5^2 - \frac{1498208}{9}\zeta_3^2\zeta_6 - \frac{23004592}{21}\zeta_2\zeta_3\zeta_7 + \frac{32644190\zeta_5\zeta_7}{9} + \frac{42933380\zeta_3\zeta_9}{27} \\
& + \frac{530242449679\zeta_{12}}{298512} - 219440\zeta_4\zeta_{2,6} + \frac{279408}{7}\zeta_2\zeta_{3,7} - \frac{2444290\zeta_{3,9}}{27} - \frac{186272}{3}\zeta_{2,1,1,8}\Big)\varepsilon^6 + O\left(\varepsilon^7\right)\Bigg\}
\end{aligned}
$$



$$
\begin{aligned}
&= \frac{(1-2\varepsilon)^3}{(1+3\varepsilon)^3(1+4\varepsilon)}\Bigg\{-\frac{10\zeta_5}{\varepsilon} - \left(60\zeta_5 + 10\zeta_3^2 + 25\zeta_6 + 70\zeta_7\right) - \Bigg(50\zeta_5 - 780\zeta_3^2 + 150\zeta_6 + 30\zeta_3\zeta_4 + \frac{1829\zeta_7}{2} \\
&- 2560\zeta_3\zeta_5 + 4655\zeta_8 - 1080\zeta_{2,6}\Bigg)\varepsilon + \Bigg(120\zeta_5 + 6046\zeta_3^2 - 125\zeta_6 + 2340\zeta_3\zeta_4 + 7575\zeta_7 + 29776\zeta_3\zeta_5 - 53193\zeta_8 \\
&+ 12258\zeta_{2,6} + \frac{4528\zeta_3^3}{3} - 2640\zeta_4\zeta_5 + 1000\zeta_3\zeta_6 - \frac{58460\zeta_9}{9}\Bigg)\varepsilon^2 + \Bigg(10704\zeta_3^2 + 300\zeta_6 + 18138\zeta_3\zeta_4 + \frac{156783\zeta_7}{2} \\
&+ 214920\zeta_3\zeta_5 - 215670\zeta_8 + 59940\zeta_{2,6} + \frac{73244\zeta_3^3}{3} - 28884\zeta_4\zeta_5 + 13200\zeta_3\zeta_6 - \frac{1510937\zeta_9}{18} + 6792\zeta_3^2\zeta_4 + \frac{143960\zeta_5^2}{7} \\
&+ \frac{495583\zeta_3\zeta_7}{7} - \frac{1038970\zeta_{10}}{7} - \frac{88260\zeta_{3,7}}{7}\Bigg)\varepsilon^3 + \Bigg(32112\zeta_3\zeta_4 + 146088\zeta_7 + 908472\zeta_3\zeta_5 - 395297\zeta_8 + 167994\zeta_{2,6} \\
&+ 140784\zeta_3^3 - 37260\zeta_4\zeta_5 + 233700\zeta_3\zeta_6 + \frac{166790\zeta_9}{3} + 109866\zeta_3^2\zeta_4 + \frac{8380551\zeta_5^2}{28} + \frac{26770505\zeta_3\zeta_7}{28} - \frac{108421795\zeta_{10}}{56} \\
&- \frac{4631493\zeta_{3,7}}{28} + 390194\zeta_3^2\zeta_5 - \frac{2438995\zeta_5\zeta_6}{6} + 488786\zeta_4\zeta_7 - \frac{1567351\zeta_3\zeta_8}{2} + 3792685\zeta_2\zeta_9 - \frac{100223975\zeta_{11}}{16} - 11366\zeta_3\zeta_{2,6} \\
&+ 205010\zeta_{2,1,8}\Bigg)\varepsilon^4 + \Bigg(1388592\zeta_3\zeta_5 - 324240\zeta_8 + 211248\zeta_{2,6} + \frac{1085348\zeta_3^3}{3} + 354744\zeta_4\zeta_5 + 1400980\zeta_3\zeta_6 + \frac{45002569\zeta_9}{18} \\
&+ 633528\zeta_3^2\zeta_4 + \frac{15929910\zeta_5^2}{7} + \frac{43860237\zeta_3\zeta_7}{7} - \frac{54746115\zeta_{10}}{7} - \frac{6015150\zeta_{3,7}}{7} + 5634364\zeta_3^2\zeta_5 - \frac{16554412\zeta_5\zeta_6}{3} \\
&+ \frac{28467331\zeta_4\zeta_7}{4} - 11209448\zeta_3\zeta_8 + 53532932\zeta_2\zeta_9 - \frac{2821800989\zeta_{11}}{32} - 143668\zeta_3\zeta_{2,6} + 2893672\zeta_{2,1,8} - \frac{696554\zeta_3^4}{9} \\
&- 4241552\zeta_3\zeta_4\zeta_5 - \frac{42064920}{7}\zeta_2\zeta_5^2 + \frac{7951810}{9}\zeta_3^2\zeta_6 - \frac{256544080}{21}\zeta_2\zeta_3\zeta_7 + \frac{254930897\zeta_5\zeta_7}{9} + \frac{714442631\zeta_3\zeta_9}{27} \\
&- \frac{12873185340379\zeta_{12}}{597024} - 843884\zeta_4\zeta_{2,6} + \frac{3115920}{7}\zeta_2\zeta_{3,7} - \frac{43891225\zeta_{3,9}}{27} - \frac{2077280}{3}\zeta_{2,1,1,8}\Bigg)\varepsilon^5 + O\left(\varepsilon^6\right)\Bigg\}
\end{aligned}
$$

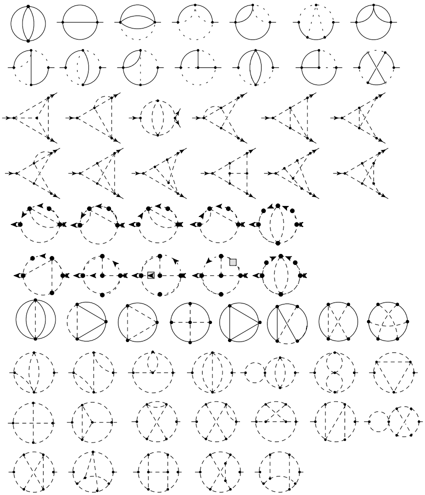


$$
\begin{aligned}
&= \frac{(1-2\varepsilon)^3}{(1+3\varepsilon)(1+4\varepsilon)}\Bigg\{-\frac{5\zeta_5}{\varepsilon}-\left(20\zeta_5+41\zeta_3^2+\frac{25\zeta_6}{2}-\frac{161\zeta_7}{2}\right)+\Bigg(-308\zeta_3^2-50\zeta_6-123\zeta_3\zeta_4+514\zeta_7+4862\zeta_3\zeta_5\\
&-\frac{24451\zeta_8}{4}+1566\zeta_{2,6}\Bigg)\varepsilon+\Bigg(-924\zeta_3\zeta_4-1500\zeta_7+68636\zeta_3\zeta_5-\frac{744639\zeta_8}{8}+23220\zeta_{2,6}+\frac{1526\zeta_3^3}{3}-2103\zeta_4\zeta_5+4325\zeta_3\zeta_6\\
&+\frac{111709\zeta_9}{36}\Bigg)\varepsilon^2+\Bigg(235200\zeta_3\zeta_5-\frac{710311\zeta_8}{2}+85536\zeta_{2,6}+\frac{22048\zeta_3^3}{3}-36366\zeta_4\zeta_5+55695\zeta_3\zeta_6+\frac{237103\zeta_9}{12}+2289\zeta_3^2\zeta_4\\
&+\frac{1341143\zeta_5^2}{56}+\frac{3816969\zeta_3\zeta_7}{56}-\frac{7815019\zeta_{10}}{112}-\frac{500565\zeta_{3,7}}{56}\Bigg)\varepsilon^3+\Bigg(\frac{61040\zeta_3^3}{3}-160416\zeta_4\zeta_5+161860\zeta_3\zeta_6-\frac{460411\zeta_9}{9}+33072\zeta_3^2\zeta_4\\
&+\frac{60035137\zeta_3\zeta_7}{56}+\frac{12859479\zeta_5^2}{56}-\frac{134815227\zeta_{10}}{112}-\frac{7724781\zeta_{3,7}}{56}-453668\zeta_3^2\zeta_5+\frac{280574047\zeta_{11}}{64}+\frac{1346777\zeta_5\zeta_6}{6}-\frac{4654793\zeta_4\zeta_7}{8}\\
&+1309878\zeta_3\zeta_8-2749211\zeta_2\zeta_9-87752\zeta_3\zeta_{2,6}-148606\zeta_{2,1,8}\Bigg)\varepsilon^4+\Bigg(91560\zeta_3^2\zeta_4+\frac{5319415\zeta_5^2}{14}+\frac{55472425\zeta_3\zeta_7}{14}-\frac{705626\zeta_3^4}{9}\\
&-\frac{7239597\zeta_{3,7}}{14}-5183674\zeta_3^2\zeta_5+\frac{3338505\zeta_5\zeta_6}{2}-1597650\zeta_{2,1,8}-\frac{1752859}{18}\zeta_3^2\zeta_6+\frac{62741559\zeta_3\zeta_8}{4}-29556525\zeta_2\zeta_9+\frac{2990096591\zeta_{11}}{64}\\
&-\frac{29828681054659\zeta_{12}}{2388096}-\frac{53503881\zeta_4\zeta_7}{8}-\frac{142150835\zeta_{10}}{28}-1729457\zeta_3\zeta_4\zeta_5-\frac{25832277}{7}\zeta_2\zeta_5^2-\frac{157544998}{21}\zeta_2\zeta_3\zeta_7+\frac{311533051\zeta_5\zeta_7}{18}\\
&+\frac{1792012205\zeta_3\zeta_9}{108}-1085340\zeta_3\zeta_{2,6}-227636\zeta_4\zeta_{2,6}+\frac{1913502}{7}\zeta_2\zeta_{3,7}-\frac{105698899\zeta_{3,9}}{108}-\frac{1275668}{3}\zeta_{2,1,1,8}\Bigg)\varepsilon^5+O\left(\varepsilon^6\right)\Bigg\}
\end{aligned}
$$

No 6$^{\text{th}}$ roots of unity, only MZVs!

Summary

- In < 2 year the DRA method has become an extremely practical tool
  - Three-loop $g-2$ master integrals (Lee and Smirnov 2011)
  - Three-loop quark and gluon form factors(Lee, Smirnov and Smirnov 2010, Lee and Smirnov 2011).
  - Work on three-loop static quark potential is in progress.
  - Four-loop QED-type tadpoles (Lee and Terekhov 2011).
  - Four-loop massless propagator master integrals.(Lee, Smirnov and Smirnov 2011a,b)



Outlook

- Making the evaluation program (`SummerTime`) public.
- Automatization of the multi-master case.
- Application to the multi-scale case.

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