Regularization Schemes and Higher Order Corrections

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Abstract. I apply commonly used regularization schemes to a multiloop calculation to examine the properties of the schemes at higher orders. I find complete consistency between the conventional dimensional regularization scheme and dimensional reduction, but I find that the four-dimensional helicity scheme produces incorrect results at next-to-next-to-leading order and singular results at next-to-next-to-next-to-leading order. It is not, therefore, a unitary regularization scheme.

1. Regularization Schemes

In recent years, there has been a great deal of progress in the calculation of higher-order corrections. At one loop, especially, there are many new techniques being developed. It is important to understand whether these new techniques are reliable tools of quantum field theory that can be applied to multi-loop calculations or if they are just short-cuts that are only valid at one loop.

One of the workhorses of the effort to compute one-loop helicity amplitudes in QCD is the Four Dimensional Helicity (FDH) regularization scheme [1, 2]. In a recent paper I have shown that the FDH is not a unitary regularization scheme (for non-supersymmetric theories) and that it generates incorrect results beyond one loop.

1.1. Dimensional Regularization

Dimensional Regularization [3] is the basis for most regularization schemes in use today. Among its many favorable properties are that it respects gauge invariance, respects Lorentz invariance and handles both UV and IR divergences.

The application of Dimensional Regularization to different kinds of problems has led to the development of a variety of regularization schemes which share the dimensional regularization of momentum integrals but differ in their handling of observed states and spin degrees of freedom.

I will be discussing four different regularization schemes which commonly appear in the literature, the 't Hooft-Veltman (HV) Scheme [3], the Conventional Dimensional Regularization (CDR) Scheme [4], the Dimensional Reduction (DRED) Scheme [5] and the FDH Scheme [1, 2].

The first two are closely related and yield identical results in the calculation that I will be describing. Superficially at least, the second two are also closely related in much the same way, but yield very different results.

1.2. The 't Hooft-Veltman Scheme

The original formulation of dimensional regularization (the HV scheme) specifies that external (observed) states are treated as four-dimensional, while internal states are to be treated as $D_m = 4 - 2\varepsilon$ dimensional. The D_m-dimensional vector space is *larger* than 4-dimensional space-time:

$$
\begin{aligned}
&g^{\mu\nu} g^{\alpha}_{\nu} = g^{\mu\alpha}\,, \qquad && g^{\mu\nu} \eta^{\alpha}_{\nu} = \eta^{\mu\alpha}\,, \qquad && \eta^{\mu\nu} \eta^{\alpha}_{\nu} = \eta^{\mu\alpha}\,, \\
&g^{\mu\nu} g_{\mu\nu} = D_m\,, && \eta^{\mu\nu} \eta_{\mu\nu} = 4\,. &&
\end{aligned} \tag{1}
$$

In HV, internal gluons have $D_m - 2 = 2 - 2\varepsilon$ spin degrees of freedom. Internal fermions, however, still have exactly 2 spin degrees of freedom.

1.3. The Conventional Dimensional Regularization Scheme

In the CDR scheme, all states (observed or internal) are continued to $D_m = 4 - 2\varepsilon$ dimensions. This is in many ways simpler than the HV scheme, especially when dealing with infrared sensitive theories like QCD. In HV, if external states have an infrared overlap, they must be treated as internal (D_m-dimensional). In CDR, all states are already D_m-dimensional, so the overlap is automatically treated properly.

The HV and CDR schemes are closely related. Their behaviors under the renormalization group (β-functions, anomalous dimensions) is identical and in the calculations I will present they give identical results.

1.4. The Dimensional Reduction Scheme

In the DRED scheme, one starts from 4-dimensional space-time and compactifies to a *smaller* vector space of dimension $D_m = 4 - 2\varepsilon$ in which momenta take values.

$$
g^{\mu\nu} g^{\alpha}_{\nu} = g^{\mu\alpha}\,, \qquad g^{\mu\nu} \eta^{\alpha}_{\nu} = g^{\mu\alpha}\,, \qquad \eta^{\mu\nu} \eta^{\alpha}_{\nu} = \eta^{\mu\alpha}\,. \tag{2}
$$

Particles in the spectrum retain their 2 spin degrees of freedom from 4 dimensions. This preserves supersymmetry.

BUT: The Ward Identity only applies to the vector subspace in which momenta are defined!

In non-SUSY theories, the "evanescent" (2ε-dimensional) gluons are independent degrees of freedom from the D_m-dimensional gluons. The fields and their couplings renormalize independently!

1.5. The Four Dimensional Helicity Scheme

The FDH takes the D_m-dimensional space where momenta take values to be *larger* than 4-dimensional space-time, but also defines a *still larger* D_s-dimensional vector space where spin degrees of freedom take values. D_s is taken to be equal to 4 so that particles have the same number of spin degrees of freedom as they have in 4 dimensions.

$$
\begin{aligned}
&g^{\mu\nu} g_{\mu\nu} = D_s\,, \qquad && \hat{g}^{\mu\nu} \hat{g}_{\mu\nu} = D_m\,, \qquad && \eta^{\mu\nu} \eta_{\mu\nu} = 4\,, \\
&g^{\mu\nu} \hat{g}^{\rho}_{\nu} = \hat{g}^{\mu\rho}\,, && g^{\mu\nu} \eta^{\rho}_{\nu} = \eta^{\mu\rho}\,, && \hat{g}^{\mu\nu} \eta^{\rho}_{\nu} = \eta^{\mu\rho}\,, \\
&g^{\mu\nu} \delta^{\rho}_{\nu} = \delta^{\mu\rho}\,, && \hat{g}^{\mu\nu} \delta^{\rho}_{\nu} = 0\,, && \eta^{\mu\nu} \delta^{\rho}_{\nu} = 0\,.
\end{aligned} \tag{3}
$$

One might expect that my remarks about the Ward Identity and evanescent states for DRED would apply to FDH, but that is not the way the scheme has been used.

Instead, FDH calculations are performed using the following rules.

(i) All momentum integrals are D_m dimensional.

(ii) All "observed" external states are taken to be four-dimensional.

(iii) All "unobserved" or internal states are treated as D_s dimensional, and the D_s dimensional vector space is taken to be larger than the D_m dimensional vector space.

(iv) Both the D_s and D_m dimensional vector spaces are larger than the standard four-dimensional space-time.

All degrees of freedom that originate from the gauge symmetry are treated as parts of the gauge bosons, NOT as independent degrees of freedom with independent couplings.

The claim is that the crucial difference between FDH and DRED that allows this treatment of the evanescent components is that $D_s > 4$.

2. Test Calculation

I will test the reliability of computing high-order corrections in these schemes by recalculating a physical quantity that is known to very high order: the inclusive cross section for an electron and positron to annihilate and produce hadrons.

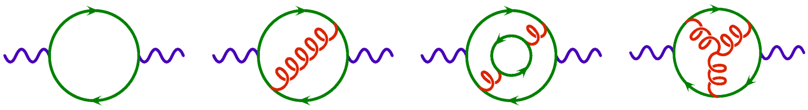
I will perform these calculations by means of the optical theorem, taking the imaginary part of the forward scattering amplitudes. This means taking the imaginary part of the photon vacuum polarization tensor sandwiched between external states.

Since the optical theorem is a direct consequence of the unitarity of the S-matrix, any unitary regularization scheme must give the same result, once one expands in terms of a standard coupling.

The basic Lagrangian (4-dimensional) is

$$\begin{aligned}\mathscr{L} = & -\frac{1}{2} A^a_\mu \left(\partial^\mu \partial^\nu (1-\xi^{-1}) - g^{\mu\nu}\Box\right) A^a_\nu \\ & - g f^{abc} (\partial^\mu A^{a\nu}) A^b_\mu A^c_\nu - \frac{g^2}{4} f^{abc} f^{ade} A^{b\mu} A^{c\nu} A^d_\mu A^e_\nu \\ & + i \sum_f \overline{\psi}^i_f \left(\delta_{ij} \not\partial - i g t^a_{ij} \not A^a - i g_V Q_f \not V\right) \psi^j_f - \bar{c}^a \Box c^a + g f^{abc} \left(\partial_\mu \bar{c}^a\right) A^{b\mu} c^c \,.\end{aligned} \tag{4}$$

Some sample diagrams at 1, 2, and 3 loops are



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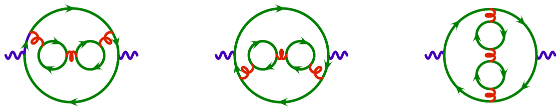
I will also compute the N_f^2 terms at 4 loops,



.

The full result through order α_s^3 is well known [6, 7, 8, 9, 10]

$$\begin{aligned}
\sigma^{e^+e^-\to\ \mathrm{had}}(Q^2) &= \frac{4\pi\alpha^2}{3Q^2} N_c \sum_f Q_f^2 \Bigg\{ 1 + \left(\frac{\alpha_s^{\overline{\mathrm{MS}}}}{\pi}\right) C_F \frac{3}{4} \\
&+\left(\frac{\alpha_s^{\overline{\mathrm{MS}}}}{\pi}\right)^2 \Bigg[\bigg(-C_F^2 \frac{3}{32} + C_F C_A \left(\frac{123}{32} - \frac{11}{4}\zeta_3\right) + C_F N_f \left(-\frac{11}{16} + \frac{1}{2}\zeta_3\right)\bigg) \\
&+\left(\frac{\alpha_s^{\overline{\mathrm{MS}}}}{\pi}\right)^3 \Bigg[-C_F^3 \frac{69}{128} + C_F^2 C_A \left(-\frac{127}{64} - \frac{143}{16}\zeta_3 + \frac{55}{4}\zeta_5\right) \\
&\qquad + C_F C_A^2 \left(\frac{90445}{3456} - \frac{2737}{144}\zeta_3 - \frac{55}{24}\zeta_5\right) \\
&\qquad + C_F^2 N_f \left(-\frac{29}{128} + \frac{19}{8}\zeta_3 - \frac{5}{2}\zeta_5\right) + C_F C_A N_f \left(-\frac{485}{54} + \frac{56}{9}\zeta_3 + \frac{5}{12}\zeta_5\right) \\
&\qquad + C_F N_f^2 \left(\frac{151}{216} - \frac{19}{36}\zeta_3\right) - \frac{1}{4}\pi^2 C_F \beta_0^{\overline{\mathrm{MS}}\,2} \Bigg] \Bigg\} \,.
\end{aligned} \tag{5}$$

3. Renormalization

In order to obtain the correct result, it is essential that we properly renormalize the theory. In CDR, this just means carrying out the standard $\overline{\mathrm{MS}}$ renormalization.

In DRED, we must follow a more elaborate program. Naïve application of minimal subtraction to the scattering amplitudes does not properly renormalize the evanescent terms [11]. Instead we must renormalize so that the evanescent Green functions are finite before we sum over the spin degrees of freedom [12, 13].

3.1. CDR Renormalization

In the CDR scheme, the Lagrangian has the same form as in 4 dimensions and the needed renormalizations are

$$\begin{aligned}
\Gamma^{(B)}_{AAA} &= Z_1 \Gamma_{AAA}\,, & \psi_f^{(B)\,i} &= Z_2^{\frac{1}{2}}\, \psi_f^i\,, & A_\mu^{(B)\,a} &= Z_3^{\frac{1}{2}}\, A_\mu^a \\
\Gamma^{(B)}_{c\bar{c}A} &= \widetilde{Z}_1 \Gamma_{q\bar{q}A}\,, & c^{(B)\,a} &= \widetilde{Z}_3^{\frac{1}{2}}\, c^a\,, & \bar{c}^{(B)\,a} &= \widetilde{Z}_3^{\frac{1}{2}}\, \bar{c}^a\,, \\
\Gamma^{(B)}_{q\bar{q}A} &= Z_{1F} \Gamma_{q\bar{q}A}\,, & \xi^{(B)} &= Z_3\, \xi\,.
\end{aligned} \tag{6}$$

To remove sub-divergences in the calculation of the photon vacuum polarization, the QCD coupling needs to be renormalized, which requires the self-energy and vertex renormalization constants.

$$\alpha_s^B = \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\epsilon} Z_{\alpha_s^{\overline{\mathrm{MS}}}}\, \alpha_s^{\overline{\mathrm{MS}}}\,, \qquad Z_{\alpha_s^{\overline{\mathrm{MS}}}} = \frac{Z_1^2}{Z_3^3} = \frac{Z_{1F}^2}{Z_2^2 Z_3} = \frac{\widetilde{Z}_1^2}{\widetilde{Z}_3^2 Z_3}\,. \tag{7}$$

3.2. DRED Renormalization

Because the evanescent gauge bosons and their couplings are independent, the DRED Lagrangian and the resulting renormalization is far more complicated.

$$
\begin{aligned}
\Gamma^{(B)}_{AAA} &= Z_1 \Gamma_{AAA}\,, & \psi_f^{(B)\,i} &= Z_2^{\frac{1}{2}}\, \psi_f^i\,, & A_\mu^{(B)\,a} &= Z_3^{\frac{1}{2}}\, A_\mu^a\,, \\
\Gamma^{(B)}_{c\bar{c}A} &= \widetilde{Z}_1 \Gamma_{q\bar{q}A}\,, & c^{(B)\,a} &= \widetilde{Z}_3^{\frac{1}{2}}\, c^a\,, & \bar{c}^{(B)\,a} &= \widetilde{Z}_3^{\frac{1}{2}}\, \bar{c}^a\,, \\
\Gamma^{(B)}_{q\bar{q}A} &= Z_{1\,F} \Gamma_{q\bar{q}A}\,, & \xi^{(B)} &= Z_3\, \xi\,, & & \\
\Gamma^{(B)}_{q\bar{q}e} &= Z_{1\,e} \Gamma_{q\bar{q}e}\,, & A^{(B)\,a}_{e\,\mu} &= Z_{3\,e}^{\frac{1}{2}} A^a_{e\,\mu}\,, & \Gamma^{(B)\,i}_{eeee} &= Z^i_{1\,eeee} \Gamma^i_{eeee}\,, \\
\Gamma^{(B)}_{q\bar{q}V_e} &= Z_{1\,Ve} \Gamma_{q\bar{q}V_e}\,, & V^{(B)}_{e\,\mu} &= Z_{3\,Ve}^{\frac{1}{2}} V_{e\,\mu}\,. & &
\end{aligned}
\tag{8}
$$

Note that we also need to compute the wavefunction and vertex corrections of the evanescent photon!

As in CDR, subdivergences are removed through coupling constant renormalizations, but in DRED, there are many couplings to renormalize.

$$
\begin{aligned}
\alpha_s^B &= \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\varepsilon} Z_{\alpha_s^{\overline{\mathrm{DR}}}}\, \alpha_s^{\overline{\mathrm{DR}}}\,, & Z_{\alpha_s^{\overline{\mathrm{DR}}}} &= \frac{Z_1^2}{Z_3^3} = \frac{Z_{1\,F}^2}{Z_2^2 Z_3} = \frac{\widetilde{Z}_1^2}{\widetilde{Z}_3^2 Z_3}\,, \\
\alpha_e^B &= \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\varepsilon} Z_{\alpha_e^{\overline{\mathrm{DR}}}}\, \alpha_e^{\overline{\mathrm{DR}}}\,, & Z_{\alpha_e^{\overline{\mathrm{DR}}}} &= \frac{Z_{1\,e}^2}{Z_2^2 Z_{3\,e}}\,, \\
\eta_i^B &= \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\varepsilon} Z_{\eta_i^{\overline{\mathrm{DR}}}}\, \eta_i^{\overline{\mathrm{DR}}}\,, & Z_{\eta_i^{\overline{\mathrm{DR}}}} &= \frac{\left(Z^i_{1\,eeee}\right)^2}{Z_{3\,e}^4}\,, \\
\alpha_{Ve}^B &= \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\varepsilon} Z_{\alpha_{Ve}^{\overline{\mathrm{DR}}}}\, \alpha_{Ve}^{\overline{\mathrm{DR}}}\,, & Z_{\alpha_{Ve}^{\overline{\mathrm{DR}}}} &= \frac{Z_{1\,Ve}^2}{Z_2^2 Z_{3\,Ve}}\,.
\end{aligned}
\tag{9}
$$

3.3. FDH Renormalization

As in the CDR scheme, the Lagrangian in FDH has the same form as in 4 dimensions and the needed renormalizations are

$$
\begin{aligned}
\Gamma^{(B)}_{AAA} &= Z_1 \Gamma_{AAA}\,, & \psi_f^{(B)\,i} &= Z_2^{\frac{1}{2}}\, \psi_f^i\,, & A_\mu^{(B)\,a} &= Z_3^{\frac{1}{2}}\, A_\mu^a\,, \\
\Gamma^{(B)}_{c\bar{c}A} &= \widetilde{Z}_1 \Gamma_{q\bar{q}A}\,, & c^{(B)\,a} &= \widetilde{Z}_3^{\frac{1}{2}}\, c^a\,, & \bar{c}^{(B)\,a} &= \widetilde{Z}_3^{\frac{1}{2}}\, \bar{c}^a\,, \\
\Gamma^{(B)}_{q\bar{q}A} &= Z_{1\,F} \Gamma_{q\bar{q}A}\,, & \xi^{(B)} &= Z_3\, \xi\,, & &
\end{aligned}
\tag{10}
$$

Again as in CDR, it would seem that only the QCD coupling needs to be renormalized.

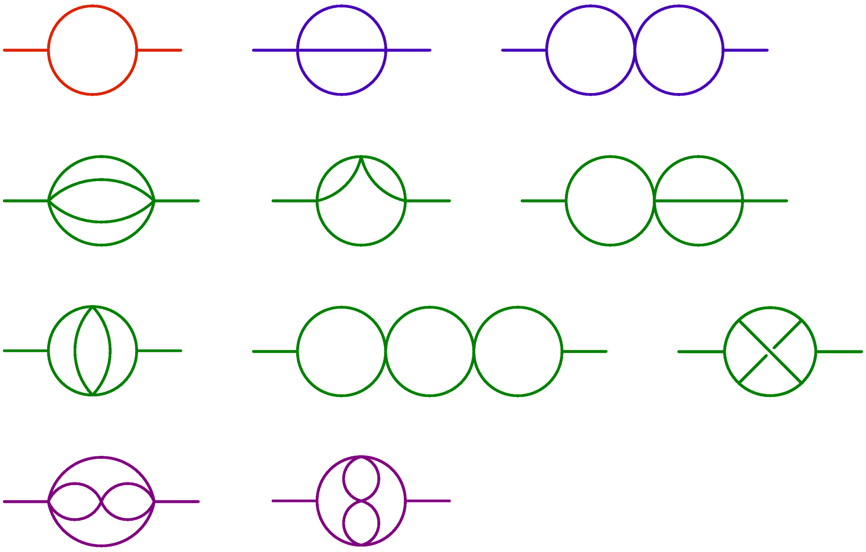
$$
\alpha_s^B = \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\varepsilon} Z_{\alpha_s^{\overline{\mathrm{FDH}}}}\, \alpha_s^{\overline{\mathrm{FDH}}}\,, \qquad Z_{\alpha_s^{\overline{\mathrm{FDH}}}} = \frac{Z_1^2}{Z_3^3} = \frac{Z_{1\,F}^2}{Z_2^2 Z_3} = \frac{\widetilde{Z}_1^2}{\widetilde{Z}_3^2 Z_3}\,.
\tag{11}
$$

4. Results

4.1. Methods

For each of the regularization schemes, I generate the necessary Feynman diagrams using QGRAF [14]. I then use the symbolic algebra program FORM [15] to identify the integral topology and implement the Feynman rules. The resulting Feynman integrals are reduced to master integrals using the program REDUZE [16].

The master integrals that occur in this calculation are:



Most of the master integrals are trivial iterated-bubble diagrams and the others were evaluated long ago [17, 18].

4.2. Conventional Dimensional Regularization

The imaginary part of the unrenormalized vacuum polarization tensor in the CDR scheme is

$$
\begin{aligned}
\Im\left[\Pi^{(B)}_{\mu\nu}(Q)\Big|_{CDR}\right] = \frac{-Q^2 g_{\mu\nu}+Q_\mu Q_\nu}{3}\,\alpha_V^B N_c \sum_f Q_f^2 \left(\frac{4\pi}{Q^2 e^{\gamma_E}}\right)^{\varepsilon} \Bigg\{ \\
1+\left(\frac{\alpha_s^B}{\pi}\right)\left(\frac{4\pi}{Q^2 e^{\gamma_E}}\right)^{\varepsilon} C_F \left[\frac{3}{4}+\varepsilon\left(\frac{55}{8}-6\,\zeta_3\right)+\varepsilon^2\left(\frac{1711}{48}-\frac{15}{4}\,\zeta_2-19\,\zeta_3-9\,\zeta_4\right)+\mathscr{O}(\varepsilon^3)\right] \\
+\left(\frac{\alpha_s^B}{\pi}\right)^2\left(\frac{4\pi}{Q^2 e^{\gamma_E}}\right)^{2\varepsilon}\left[\frac{1}{\varepsilon}\left(\frac{11}{16}C_F C_A-\frac{1}{8}C_F N_f\right)\right. \\
-\frac{3}{32}C_F^2+C_F C_A\left(\frac{487}{48}-\frac{33}{4}\zeta_3\right)+C_F N_f\left(-\frac{11}{6}+\frac{3}{2}\,\zeta_3\right) \\
+\varepsilon\left(C_F^2\left(-\frac{143}{32}-\frac{111}{8}\,\zeta_3+\frac{45}{2}\,\zeta_5\right)+C_F C_A\left(\frac{50339}{576}-\frac{231}{32}\,\zeta_2-\frac{109}{2}\,\zeta_3-\frac{99}{8}\,\zeta_4-\frac{15}{4}\,\zeta_5\right)\right. \\
\left.\left.+C_F N_f\left(-\frac{4417}{288}+\frac{21}{16}\,\zeta_2+\frac{19}{2}\,\zeta_3+\frac{9}{4}\,\zeta_4\right)\right)+\mathscr{O}(\varepsilon^2)\right] \\
+\left(\frac{\alpha_s^B}{\pi}\right)^3\left(\frac{4\pi}{Q^2 e^{\gamma_E}}\right)^{3\varepsilon} C_F N_f^2\left[\frac{1}{48\,\varepsilon^2}+\frac{1}{\varepsilon}\left(\frac{121}{288}-\frac{1}{3}\zeta_3\right)+\frac{2777}{576}-\frac{3}{8}\zeta_2-\frac{19}{6}\zeta_3-\frac{1}{2}\zeta_4\right]+\ldots\Bigg\}\,. \qquad (12)
\end{aligned}
$$

Upon renormalizing the couplings, I find

$$\Im\left[\Pi_{\mu\nu}(Q)\big|_{CDR}\right] = \frac{-Q^2\, g_{\mu\nu} + Q_\mu Q_\nu}{3}\alpha_V\, N_c \sum_f Q_f^2 \left\{ 1 + \left(\frac{\alpha_s^{\overline{\mathrm{MS}}}}{\pi}\right) C_F\,\frac{3}{4} \right.$$
$$+ \left(\frac{\alpha_s^{\overline{\mathrm{MS}}}}{\pi}\right)^2 \left[-C_F^2\,\frac{3}{32} + C_F\, C_A\left(\frac{123}{32} - \frac{11}{4}\zeta_3\right) + C_F\, N_f\left(-\frac{11}{16} + \frac{1}{2}\zeta_3\right)\right] \tag{13}$$
$$\left. + \left(\frac{\alpha_s^{\overline{\mathrm{MS}}}}{\pi}\right)^3 C_F\, N_f^2\left[\frac{151}{216} - \frac{1}{24}\zeta_2 - \frac{19}{36}\zeta_3\right] + \ldots \right\}.$$

Sandwiching the vacuum polarization between external states, I obtain the expected result that I showed in Eq. (5).

4.3. Dimensional Reduction

In DRED, there are two independent vacuum polarization tensors to compute, corresponding to the photon and the evanescent photon.

$$\Im\left[\Pi_{\mu\nu}^{(B)}(Q)\Big|_{DRED}\right] = \frac{-Q^2\,\hat{g}_{\mu\nu} + Q_\mu Q_\nu}{3}\,\Im\left[\Pi_A^{(B)}(Q)\Big|_{DRED}\right] - Q^2\,\frac{\delta_{\mu\nu}}{2\varepsilon}\,\Im\left[\Pi_B^{(B)}(Q)\Big|_{DRED}\right]. \tag{14}$$

Before renormalization, both components are singular and depend on the QCD coupling and the various evanescent couplings.

$$\Im\left[\Pi_A^{(B)}(Q)\Big|_{DRED}\right] = \alpha_V^B\, N_c \sum_f Q_f^2 \left(\frac{4\pi}{Q^2\, e^{\gamma_E}}\right)^{\varepsilon} \left\{ \vphantom{\frac{1}{1}} \right.$$
$$1 + \left(\frac{\alpha_s^B}{\pi}\right)\left(\frac{4\pi}{Q^2\, e^{\gamma_E}}\right)^{\varepsilon} C_F\left[\frac{3}{4} + \varepsilon\left(\frac{51}{8} - 6\,\zeta_3\right) + \varepsilon^2\left(\frac{497}{16} - \frac{15}{4}\zeta_2 - 15\,\zeta_3 - 9\,\zeta_4\right) + \mathscr{O}(\varepsilon^3)\right]$$
$$+ \left(\frac{\alpha_e^B}{\pi}\right)\left(\frac{4\pi}{Q^2\, e^{\gamma_E}}\right)^{\varepsilon} C_F\left[-\varepsilon\,\frac{3}{4} - \varepsilon^2\,\frac{29}{8} + \mathscr{O}(\varepsilon^3)\right]$$
$$+ \left(\frac{\alpha_s^B}{\pi}\right)^2\left(\frac{4\pi}{Q^2\, e^{\gamma_E}}\right)^{2\varepsilon}\left[\frac{1}{\varepsilon}\left(\frac{11}{16}C_F\, C_A - \frac{1}{8}C_F\, N_f\right) - \frac{3}{32}C_F^2 + \left(\frac{77}{8} - \frac{33}{4}\zeta_3\right) C_F\, C_A - \left(\frac{7}{4} - \frac{3}{2}\zeta_3\right) C_F\, N_f\right.$$
$$+ \varepsilon\left(C_F^2\left(-\frac{141}{32} - \frac{111}{8}\zeta_3 + \frac{45}{2}\zeta_5\right) + C_F\, C_A\left(\frac{15301}{192} - \frac{231}{32}\zeta_2 - \frac{193}{4}\zeta_3 - \frac{99}{8}\zeta_4 - \frac{15}{4}\zeta_5\right)\right.$$
$$\left.\left. + C_F\, N_f\left(-\frac{1355}{96} + \frac{21}{16}\zeta_2 + \frac{17}{2}\zeta_3 + \frac{9}{4}\zeta_4\right)\right) + \mathscr{O}(\varepsilon^2)\right]$$
$$+ \left(\frac{\alpha_e^B}{\pi}\right)^2\left(\frac{4\pi}{Q^2\, e^{\gamma_E}}\right)^{2\varepsilon}\left[\frac{3}{4}C_F^2 - \frac{3}{8}C_F\, C_A + \frac{3}{16}C_F\, N_f - \varepsilon\left(\frac{47}{8}C_F^2 - \frac{11}{4}C_F\, C_A + \frac{7}{4}C_F\, N_f\right) + \mathscr{O}(\varepsilon^2)\right]$$
$$+ \left(\frac{\alpha_s^B}{\pi}\right)\left(\frac{\alpha_e^B}{\pi}\right)\left(\frac{4\pi}{Q^2\, e^{\gamma_E}}\right)^{2\varepsilon}\left[-\frac{9}{8}C_F^2 - \varepsilon\left(\frac{141}{16}C_F^2 + \frac{21}{16}C_F\, C_A\right) + \mathscr{O}(\varepsilon^2)\right]$$
$$+ \left(\frac{\alpha_s^B}{\pi}\right)^3\left(\frac{4\pi}{Q^2\, e^{\gamma_E}}\right)^{3\varepsilon} C_F\, N_f^2\left[\frac{1}{48\,\varepsilon^2} + \frac{1}{\varepsilon}\left(\frac{13}{32} - \frac{1}{3}\zeta_3\right) + \frac{7847}{1728} - \frac{3}{8}\zeta_2 - \frac{53}{18}\zeta_3 - \frac{1}{2}\zeta_4\right]$$
$$\left. + \left(\frac{\alpha_e^B}{\pi}\right)^3\left(\frac{4\pi}{Q^2\, e^{\gamma_E}}\right)^{3\varepsilon} C_F\, N_f^2\left[-\frac{1}{\varepsilon}\frac{3}{64} - \frac{83}{128}\right] + \ldots \right\}, \tag{15}$$

$$
\begin{aligned}
\Im\left[\Pi_B^{(B)}(Q)\Big|_{DRED}\right] = {}& \alpha_{Ve}^B N_c \sum_f Q_f^2 \left(\frac{4\pi}{Q^2 e^{\gamma_E}}\right)^{\varepsilon} \Bigg\{ \varepsilon + 2\varepsilon^2 + \left(4 - \frac{3}{2}\zeta_2\right)\varepsilon^3 + \mathscr{O}(\varepsilon^4) \\
& + \left(\frac{\alpha_s^B}{\pi}\right)\left(\frac{4\pi}{Q^2 e^{\gamma_E}}\right)^{\varepsilon} C_F \left[\frac{3}{2} + \varepsilon\frac{29}{4} + \varepsilon^2\left(\frac{227}{8} - \frac{15}{2}\zeta_2 - 6\zeta_3\right) + \mathscr{O}(\varepsilon^3)\right] \\
& + \left(\frac{\alpha_e^B}{\pi}\right)\left(\frac{4\pi}{Q^2 e^{\gamma_E}}\right)^{\varepsilon} C_F \left[-1 - 4\varepsilon - \varepsilon^2\left(\frac{27}{2} - 5\zeta_2\right) + \mathscr{O}(\varepsilon^3)\right] \\
& + \left(\frac{\alpha_s^B}{\pi}\right)^2 \left(\frac{4\pi}{Q^2 e^{\gamma_E}}\right)^{2\varepsilon} \left[\frac{1}{\varepsilon}\left(\frac{9}{8}C_F^2 + \frac{11}{16}C_F C_A - \frac{1}{8}C_F N_f\right) + \frac{279}{32}C_F^2 + \frac{199}{32}C_F C_A - \frac{17}{16}C_F N_f\right. \\
& \quad + \varepsilon\left(C_F^2\left(\frac{3139}{64} - \frac{189}{16}\zeta_2 - \frac{45}{4}\zeta_3\right) + C_F C_A\left(\frac{2473}{64} - \frac{231}{32}\zeta_2 - \frac{75}{8}\zeta_3\right)\right. \\
& \qquad \left.\left. + C_F N_f\left(-\frac{207}{32} + \frac{21}{16}\zeta_2 + \frac{3}{2}\zeta_3\right)\right) + \mathscr{O}(\varepsilon^2)\right] \\
& + \left(\frac{\alpha_s^B}{\pi}\right)\left(\frac{\alpha_e^B}{\pi}\right)\left(\frac{4\pi}{Q^2 e^{\gamma_E}}\right)^{2\varepsilon} \left[-\frac{1}{\varepsilon}\frac{9}{4}C_F^2 - \frac{129}{8}C_F^2 - \frac{3}{8}C_F C_A\right. \\
& \quad \left. - \varepsilon\left(\left(\frac{671}{8} - \frac{189}{8}\zeta_2 - 9\zeta_3\right)C_F^2 + \frac{53}{16}C_F C_A\right) + \mathscr{O}(\varepsilon^2)\right] \\
& + \left(\frac{\alpha_e^B}{\pi}\right)^2 \left(\frac{4\pi}{Q^2 e^{\gamma_E}}\right)^{2\varepsilon} \left[\frac{1}{\varepsilon}\left(C_F^2 - \frac{1}{4}C_F C_A + \frac{1}{8}C_F N_f\right) + \frac{13}{2}C_F^2 - \frac{3}{2}C_F C_A + \frac{15}{16}C_F N_f\right. \\
& \left. + \varepsilon\left(\left(31 - \frac{21}{2}\zeta_2 - \frac{3}{4}\zeta_3\right)C_F^2 - \left(\frac{53}{8} - \frac{21}{8}\zeta_2 - \frac{3}{8}\zeta_3\right)C_F C_A + \left(\frac{157}{32} - \frac{21}{16}\zeta_2\right)C_F N_f\right) + \mathscr{O}(\varepsilon^2)\right] \\
& + \left(\frac{\alpha_s^B}{\pi}\right)^3 \left(\frac{4\pi}{Q^2 e^{\gamma_E}}\right)^{3\varepsilon} C_F N_f^2 \left[\frac{1}{72\varepsilon^2} + \frac{1}{\varepsilon}\frac{73}{432} + \frac{3595}{2592} - \frac{1}{4}\zeta_2 - \frac{1}{3}\zeta_3\right] \\
& + \left(\frac{\alpha_e^B}{\pi}\right)^3 \left(\frac{4\pi}{Q^2 e^{\gamma_E}}\right)^{3\varepsilon} C_F N_f^2 \left[-\frac{1}{48\varepsilon^2} - \frac{1}{\varepsilon}\frac{11}{48} - \frac{155}{96} + \frac{3}{8}\zeta_2\right] \qquad + \ldots \qquad \Bigg\}.
\end{aligned}
\tag{16}
$$

Renormalizing the many couplings, including that of the evanescent photon, including the finite renormalization [19, 20] which shifts $\alpha_s^{\overline{\mathrm{DR}}}$ to $\alpha_s^{\overline{\mathrm{MS}}}$, I obtain

$$
\begin{aligned}
\Im\left[\Pi_A(Q)|_{DRED}\right] = {}& \alpha_V N_c \sum_f Q_f^2 \Bigg\{ 1 + \left(\frac{\alpha_s^{\overline{\mathrm{MS}}}}{\pi}\right) C_F \frac{3}{4} \\
& + \left(\frac{\alpha_s^{\overline{\mathrm{MS}}}}{\pi}\right)^2 \left[-C_F^2\frac{3}{32} + C_F C_A\left(\frac{123}{32} - \frac{11}{4}\zeta_3\right) + C_F N_f\left(-\frac{11}{16} + \frac{1}{2}\zeta_3\right)\right] \\
& + \left(\frac{\alpha_s^{\overline{\mathrm{MS}}}}{\pi}\right)^3 C_F N_f^2 \left[\frac{151}{216} - \frac{1}{24}\zeta_2 - \frac{19}{36}\zeta_3\right] \Bigg\}, \\
\Im\left[\Pi_B(Q)|_{DRED}\right] = {}& \mathscr{O}(\varepsilon).
\end{aligned}
\tag{17}
$$

The evanescent vacuum polarization does not contribute to the cross section, while the gluon vacuum polarization produces exactly the expected result.

4.4. Four Dimensional Helicity

In FDH, however, the calculation is in trouble from the very beginning. The calculation is term-by-term identical to the DRED calculation except that evanescent terms are identified with gauge terms. So, as in DRED, the vacuum polarization tensor splits into two independent components; a D_m-dimensional component and a D_x-dimensional ($D_x = D_s - D_m$) component. For the photon vacuum polarization, the demand that external states be 4-dimensional means that we only need the D_m-dimensional component.

The gluon vacuum polarization, however, is a problem, since we need to extract the renormalization constant to determine the β-function. At one loop, averaging over degrees of freedom means that only the D_m-dimensional piece contributes, and we get the usual QCD β-function. At two loops, the D_x-dimensional piece is still singular after spin averaging. Only by dropping the D_x term do I get the usual two-loop contribution to the QCD β-function,

$$\begin{aligned}
\Im\left[\Pi_A^{(B)}(Q)\Big|_{FDH}\right] &= \alpha_V^B N_c \sum_f Q_f^2 \left(\frac{4\pi}{Q^2 e^{\gamma_E}}\right)^{\varepsilon} \Bigg\{ \\
& 1 + \left(\frac{\alpha_s^B}{\pi}\right)\left(\frac{4\pi}{Q^2 e^{\gamma_E}}\right)^{\varepsilon} C_F \left[\frac{3}{4} + \varepsilon\left(\frac{45}{8} - 6\zeta_3\right) + \varepsilon^2\left(\frac{439}{16} - \frac{15}{4}\zeta_2 - 15\zeta_3 - 9\zeta_4\right) + \mathscr{O}(\varepsilon^3)\right] \\
& + \left(\frac{\alpha_s^B}{\pi}\right)^2 \left(\frac{4\pi}{Q^2 e^{\gamma_E}}\right)^{2\varepsilon} \Bigg[\frac{1}{\varepsilon}\left(\frac{11}{16}C_F C_A - \frac{1}{8}C_F N_f\right) - \frac{15}{32}C_F^2 + \left(\frac{37}{4} - \frac{33}{4}\zeta_3\right) C_F C_A \\
& \quad - \left(\frac{25}{16} - \frac{3}{2}\zeta_3\right) C_F N_f + \varepsilon\Bigg(C_F^2\left(-\frac{235}{32} - \frac{111}{8}\zeta_3 + \frac{45}{2}\zeta_5\right) \\
& \quad + C_F C_A\left(\frac{14521}{192} - \frac{231}{32}\zeta_2 - \frac{193}{4}\zeta_3 - \frac{99}{8}\zeta_4 - \frac{15}{4}\zeta_5\right) \\
& \quad + C_F N_f\left(-\frac{1187}{96} + \frac{21}{16}\zeta_2 + \frac{17}{2}\zeta_3 + \frac{9}{4}\zeta_4\right)\Bigg) + \mathscr{O}(\varepsilon^2)\Bigg] \\
& + \left(\frac{\alpha_s^B}{\pi}\right)^3 \left(\frac{4\pi}{Q^2 e^{\gamma_E}}\right)^{3\varepsilon} C_F N_f^2 \left[\frac{1}{48\varepsilon^2} + \frac{1}{\varepsilon}\left(\frac{23}{64} - \frac{1}{3}\zeta_3\right) + \frac{13453}{3456} - \frac{3}{8}\zeta_2 - \frac{53}{18}\zeta_3 - \frac{1}{2}\zeta_4\right]\Bigg\} .
\end{aligned} \tag{18}$$

I only need the leading term in the β-function to renormalize these terms.

$$\begin{aligned}
\Im\left[\Pi_A(Q)|_{FDH}\right] = \alpha_V N_c \sum_f Q_f^2 \Bigg\{ & 1 + \left(\frac{\alpha_s^{\overline{\mathrm{FDH}}}}{\pi}\right)\frac{3}{4}C_F \\
& + \left(\frac{\alpha_s^{\overline{\mathrm{FDH}}}}{\pi}\right)^2 \left[-C_F^2\frac{15}{32} + C_F C_A\left(\frac{131}{32} - \frac{11}{4}\zeta_3\right) + C_F N_f\left(-\frac{5}{8} + \frac{1}{2}\zeta_3\right)\right] \\
& + \left(\frac{\alpha_s^{\overline{\mathrm{FDH}}}}{\pi}\right)^3 C_F N_f^2 \left[-\frac{1}{192\varepsilon} + \frac{1843}{3456} - \frac{1}{24}\zeta_2 - \frac{19}{36}\zeta_3\right]\Bigg\} .
\end{aligned} \tag{19}$$

Even after the finite transformation of $\alpha_s^{\overline{\mathrm{FDH}}} \to \alpha_s^{\overline{\mathrm{MS}}}$, the NNLO term is incorrect and no finite transformation can repair the fact that the N^3LO term is singular.

One must conclude that the renormalization program of the FDH scheme has failed, resulting in the violation of unitarity.

Is this the end of the FDH?

No! The FDH greatly simplifies loop calculations and is therefore a useful tool. The fact that it is non-unitary and generates spurious terms beyond one-loop simply means that we must find ways to correct for the errors [21].

Because the breakdown is a failure of the renormalization program, the spurious terms are proportional to the lower-loop contributions and can be determined from them. Boughezal, Melnikov and Petriello have proposed a method that they call "dimensional reconstruction" which involves calculating the lower-loop terms in various integer dimensions, allowing one to solve for the correct renormalizations.

5. Conclusions

I have studied the behavior of several regularization schemes in high-order radiative corrections. I find that the CDR and DRED schemes are correct and equivalent ways of performing QCD calculations through N^3LO. The FDH scheme, however, has been shown to be incorrect and to violate unitarity beyond NLO when applied to nonsupersymmetric theories.

The FDH scheme is not a unitary regularization scheme because its renormalization program fails to remove all of the ultraviolet singularities. This failure can be overcome by solving for the terms which must multiply the lower-loop results to remove the uncanceled ultraviolet singularities.

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