A nonperturbative perspective on the light-front vacuum

Work done in collaboration with S.S. Chabysheva.
Supported in part by the Minnesota Supercomputing Institute and Research Computing and Data Services at the University of Idaho.

John R. Hiller

jhiller@d.umn.edu

Department of Physics

University of Idaho

Introduction

- the light-front (LF) vacuum is famously considered trivial, while the equal-time (ET) vacuum is not
- $\exists$ several indications that this is not really the case
  - critical coupling in ϕ_2^4 theory
  - nonzero one-loop bubble [Collins, arXiv:1801.03960]
  - zero-mode contributions for twist-3 GPDs [Aslan & Burkardt, PRD **101**, 016010 (2020); X. Ji, NPB **960**, 115181 (2020)]
- but what is missing?
- the key is that matrix elements of transitions to and from the Fock vacuum are not necessarily zero and terms usually dropped from LF Hamiltonians need to be kept
- this leads to vacuum bubbles and tadpoles, which we explore in a nonperturbative context

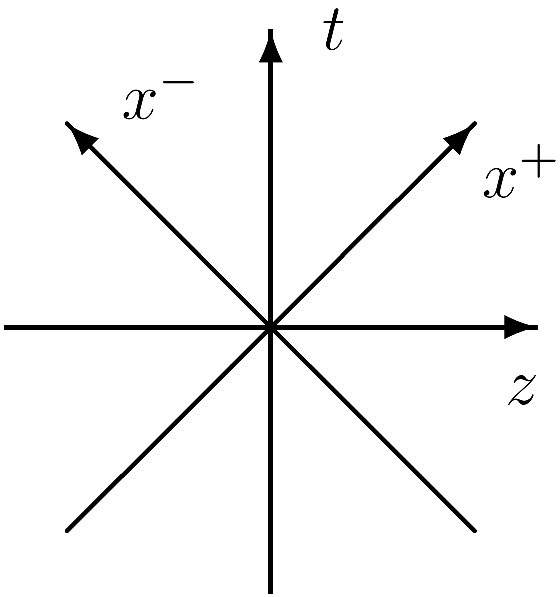
Outline

- light-front quantization
- evidence for a nontrivial vacuum
- restoration of vacuum transitions
- vacuum states for free & shifted scalars
- tadpoles and bubbles in ϕ^4 theory
- vacuum energy subtraction in scalar Yukawa theory
- summary

Light-front coordinates

Dirac, RMP **21**, 392 (1949).

- time: $x^+ = t + z$
- space: $\underline{x} = (x^-, \vec{x}_\perp), \;\; x^- \equiv t - z, \;\; \vec{x}_\perp = (x, y)$
- energy: $p^- = E - p_z$
- momentum: $\underline{p} = (p^+, \vec{p}_\perp), \;\; p^+ \equiv E + p_z, \;\; \vec{p}_\perp = (p_x, p_y)$
- mass-shell condition: $p^2 = m^2 \;\; \Rightarrow \;\; p^- = \frac{m^2 + p_\perp^2}{p^+}$



Mass eigenvalue problem

- Pauli and Brodsky, PRD **32**, 1993 (1985); 2001 (1985)

$$\mathcal{P}^-|\underline{P}\rangle = \frac{M^2+P_\perp^2}{P^+}|\underline{P}\rangle, \quad \underline{\mathcal{P}}|\underline{P}\rangle = \underline{P}|\underline{P}\rangle.$$

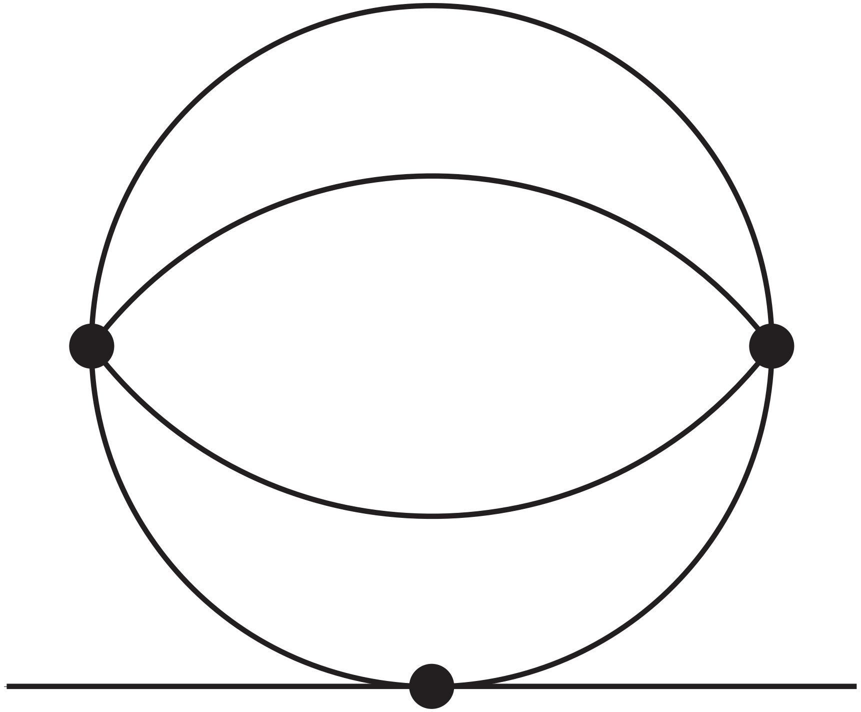
- no spurious vacuum contributions to eigenstates(?)
  - $p^+ = \sqrt{m^2+p_\perp^2+p_z^2}+p_z > 0$ for all massive particles
  - cannot produce particles from vacuum and still conserve p^+
  - Fock vacuum $|0\rangle$ is the physical vacuum
- boost-invariant separation of internal and external momenta
  - longitudinal momentum fractions $x_i \equiv p_i^+/P^+$
  - relative transverse momenta $\vec{k}_{i\perp} \equiv \vec{p}_{i\perp} - x_i\vec{P}_\perp$

ϕ_2^4 critical couplings compared

Method	$g_c \equiv \lambda_c/(24\mu^2)$	Reported by
LF sym. polys.	1.1$\pm$0.03	Burkardt, SSC & JRH
DLCQ	0.94$\pm$0.01	Vary *et al.*
Quasi-sparse		
eigenvector	2.5	Lee & Salwen
Density matrix	2.4954(4)	Sugihara
Lattice	2.70 $\begin{cases} +0.025 \\ -0.013 \end{cases}$	Schaich & Loinaz
	2.79$\pm$0.02	Bosetti *et al.*
Uniform matrix		
product	2.766(5)	Milsted *et al.*
Renorm. H trunc.	2.97(14)	Rychkov & Vitale

Systematic difference between LF (top) and ET (bottom).

Troublesome tadpole



Contributes to ET self-energy, but not standard LF.

Mass renormalization

Bare mass renormalized by tadpole contributions in ET quantization but not in LF quantization
[M. Burkardt, PRD **47**, 4628 (1993)]

$$\mu_{\rm LF}^2 = \mu_{\rm ET}^2 + \lambda\left[\langle 0|\frac{\phi^2}{2}|0\rangle - \langle 0|\frac{\phi^2}{2}|0\rangle_{\rm free}\right].$$

The vev's of ϕ^2 resum the tadpole contributions; the subscript *free* indicates the vev with $\lambda = 0$.
$\rightarrow$ need to calculate vev's
[M. Burkardt, SSC, and JRH, PRD **94**, 065006 (2016)]

$$\langle 0|\frac{\phi^2}{2}|0\rangle \rightarrow \frac{1}{2}\langle 0|\phi(\epsilon^+,\epsilon^-)\int_0^\infty dP\sum_n |\psi_n(P)\rangle\langle\psi_n(P)|\phi(0,0)|0\rangle.$$

$$\phi(\epsilon^+,\epsilon^-) = e^{i\mathcal{P}^-\epsilon^+/2}\phi(0,\epsilon^-)e^{-i\mathcal{P}^-\epsilon^+/2}.$$

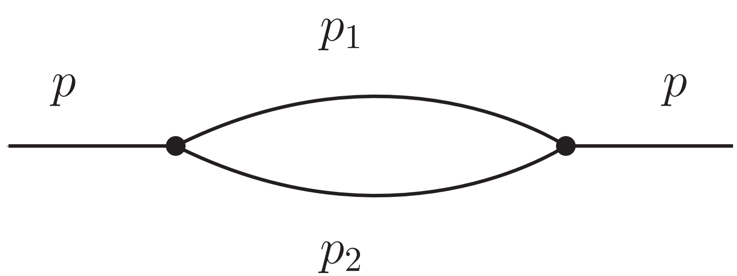
ET to LF interpolation

K. Hornbostel, Phys. Rev. D **45**, 3781 (1992);
B. Ma and C.R. Ji, PRD **104**, 036004 (2021).

- $x^\pm = \frac{1}{\sqrt{2}}[\sqrt{1 \pm c}t \pm \sqrt{1 \mp c}z] \rightarrow$ ET: $c = 1$, $x^\pm = t, -z$
  - LF: $c = 0$, $x^\pm = (t \pm z)/\sqrt{2}$ as usual, modulo $\sqrt{2}$
- $p_\pm = \frac{1}{\sqrt{2}}[\sqrt{1 \pm c}E \mp \sqrt{1 \mp c}p_z]$
- $p \cdot x = p_+ x^+ + p_- x^-$
- $\mu^2 = E^2 - p_z^2 = cp_+^2 - cp_-^2 + 2sp_+p_-$, $s \equiv \sqrt{1 - c^2}$
  - positive root $\rightarrow p_+ = [\sqrt{p_-^2 + c\mu^2} - sp_-]/c$
- compute for arbitrary c and study limit
  - SSC and JRH, Phys. Rev. D **102**, 116010 (2020)
  - obtain consistency with equal-time
  - LF appears inequivalent to $c \rightarrow 0$ limit

Nontrivial vacuum bubbles

- Collins considered the one-loop self-energy in ϕ^3 theory and the related invariant function [arXiv:1801.03960] $\Pi(p^2) = -\frac{1}{8\pi^2}\int \frac{d^2k}{[k^2-\mu^2+i\epsilon][(p-k)^2-\mu^2+i\epsilon]}$.
- the one-loop bubble is obtained in the $p^2 \to 0$ limit; $\Pi(0) = -i/8\pi\mu^2$.



- naive LF approach gives zero for the bubble.
- ET to LF transition agrees with ET [SSC & JRH].

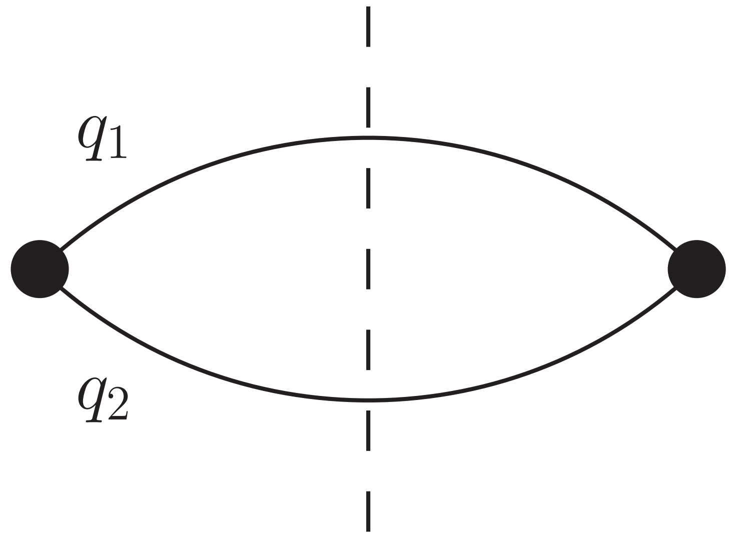
Proposed resolution

- restore vacuum transitions to LF quantization
  - $\mathcal{P}^-_{\rm free} = \int dx^- : \frac{1}{2}\mu^2\phi^2 := \int dp^+ \frac{\mu^2}{p^+} a^\dagger(p^+)a(p^+)$
    $$+\frac{\mu^2}{2}\int \frac{dp_1^+ dp_2^+}{\sqrt{p_1^+ p_2^+}}\delta(p_1^+ + p_2^+)\left[a(p_1^+)a(p_2^+) + a^\dagger(p_1^+)a^\dagger(p_2^+)\right]$$
  - standard practice is to drop the last two terms, because p_i^+ forced to zero
- instead keep; then have tadpoles and bubbles
  - matrix elements of such terms need not be zero
- need to regulate bubbles
  - perturbative calculations allow classification and exclusion of such contributions
  - nonperturbative calculations do not
  - $\delta \to \delta_\epsilon$ with ϵ a width to be taken to zero
    - ‘ephemeral’ modes

Free vacuum

- generalized coherent state: $|\mathrm{vac}\rangle = \sqrt{Z} e^{A^\dagger}|0\rangle$,
  where $A^\dagger \equiv \int \frac{dp_1 dp_2}{\sqrt{p_1 p_2}} \frac{f(p_1,p_2)}{\frac{1}{p_1}+\frac{1}{p_2}} a^\dagger(p_1) a^\dagger(p_2)$.
  (have dropped plus superscript for brevity)
- solves $\mathcal{P}^-_{\mathrm{free}}|\mathrm{vac}\rangle = P^-_{\mathrm{vac}}|\mathrm{vac}\rangle$
  if $f(p_1,p_2) = -\frac{1}{2}\delta_\epsilon(p_1+p_2)$
- with $P^-_{\mathrm{vac}} = -\frac{\mu^2}{2}\int dQ \delta_\epsilon(Q)^2 = -\frac{\mu^2 L}{16\pi}$
- where $\int dQ \delta_\epsilon(Q)^2 \to \delta(0)\int_0^\infty \delta(Q) dQ = \frac{1}{2}\frac{L}{4\pi}$
  and $L = \int dx^- = 4\pi\delta(0)$ is the light-front volume
- a massive state $a^\dagger(P)|\mathrm{vac}\rangle$ is then an eigenstate of $\mathcal{P}^-_{\mathrm{free}} + \frac{\mu^2 L}{16\pi}$ with eigenvalue $\frac{M^2}{P}$
  - with $\epsilon \ll P$, the ephemeral modes are disjoint

One-loop vacuum bubble



- P^-_{vac} is proportional to the one-loop vacuum bubble computed by Collins
- the equivalent perturbative calculation, corresponding to this loop is

$$2\frac{\mu^2}{2}\int\frac{dq_1dq_2}{\sqrt{q_1q_2}}\frac{\delta_\epsilon(q_1+q_2)}{\frac{M^2}{P}-\frac{\mu^2}{q_1}-\frac{\mu^2}{q_2}}\frac{\mu^2}{2}\frac{\delta_\epsilon(q_1+q_2)}{\sqrt{q_1q_2}}=2\frac{\mu^4}{4}\int\frac{dq_1dq_2}{q_1q_2}\frac{\delta_\epsilon(q_1+q_2)}{-\frac{\mu^2}{q_1}-\frac{\mu^2}{q_2}}$$

$$=-\frac{\mu^2}{2}\int\frac{QdQdx}{Q^2x(1-x)}\frac{\delta_\epsilon(Q)^2}{\frac{1}{Q}\frac{1}{x(1-x)}}=-\frac{\mu^2}{2}\int dQ\delta_\epsilon(Q)^2=P^-_{\text{vac}}$$

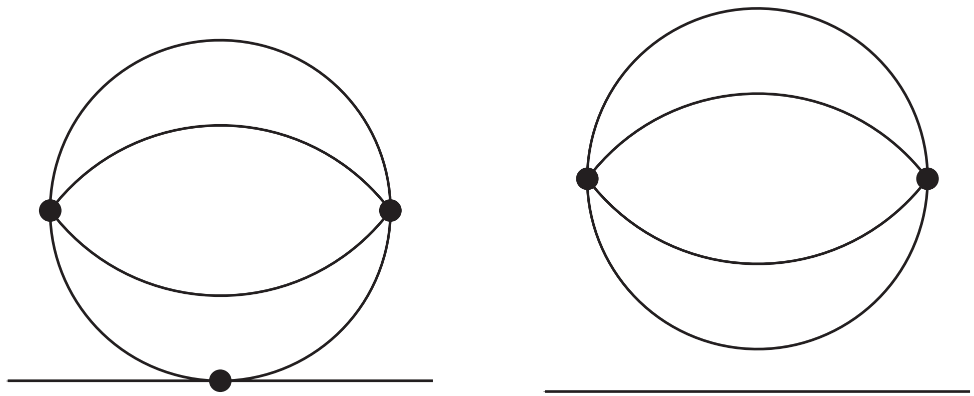
Shifted scalar

- $\phi \to \phi + v$
- $\mathcal{L} = \mathcal{L}_0 - \mu^2 v\phi - \frac{1}{2}\mu^2 v^2$
- $\phi(x^-) = \int \frac{dp}{\sqrt{4\pi p}} \left\{ a(p) e^{-ipx^-/2} + a^\dagger(p) e^{ipx^-/2} \right\}$
- $$\begin{aligned} \mathcal{P}^-_{\rm int} &= \int dx^- [\mu^2 v\phi + \tfrac{1}{2}\mu^2 v^2] \\ &= \sqrt{4\pi}\mu^2 v \int \frac{dp}{\sqrt{p}} \delta_\epsilon(p)[a(p) + a^\dagger(p)] + \tfrac{1}{2}\mu^2 v^2 L \end{aligned}$$
- define $B = \int dp v\sqrt{4\pi p k}\delta_\epsilon(p)[a^\dagger(p) - a(p)]$;
  then $e^B \phi(x^-) e^{-B} = \phi(x^-) + v$
  and $e^B \mathcal{P}^-_{\rm free} e^{-B} = \mathcal{P}^-_{\rm free} + \mathcal{P}^-_{\rm int}$
- vacuum of the shifted Hamiltonian: $|{\rm vac}\rangle_v = e^B |{\rm vac}\rangle$
- vev: ${}_v\langle{\rm vac}|\phi(x^-)|{\rm vac}\rangle_v = -v$

Tadpoles and bubbles in ϕ^4 theory

- light-front Hamiltonian density: $\mathcal{H} = \frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4$
- light-front Hamiltonian: $\mathcal{P}^- = \mathcal{P}^-_{\rm free} + \mathcal{P}^-_{\rm eph} + \cdots$

  $\mathcal{P}^-_{\rm eph} = \frac{\lambda}{24}\int \frac{\prod_i^4 dp_i}{4\pi\sqrt{\prod_i^4 p_i}}\delta_\epsilon(\sum_i^4 p_i)[\prod_i^4 a(p_i) + \prod_i^4 a^\dagger(p_i)]$
- focus on leading tadpole and bubble contributions



- Fock-state expansion: $|\psi(P)\rangle = \psi_1 a^\dagger(P)|0\rangle + \cdots +$
  $\int \prod_i^5 dp_i \delta(P - \sum_i^5 p_i)\psi_5(p_1,\ldots,p_5)\frac{1}{5!}\prod_i^5 a^\dagger(p_i)|0\rangle + \cdots$

Equations for wave functions

- the eigenvalue problem:

  $$(\mathcal{P}^-_{\rm free}+\mathcal{P}^-_{\rm int})|\psi(P)\rangle=\left(\frac{M^2}{P}+P^-_{\rm vac}\right)|\psi(P)\rangle$$

  [$P^-_{\rm vac}$ obtained from solving $(\mathcal{P}^-_{\rm free}+\mathcal{P}^-_{\rm int})|{\rm vac}\rangle=P^-_{\rm vac}|{\rm vac}\rangle$]
- projection onto Fock sectors

  $$\frac{\mu^2}{P}\psi_1+\frac{\lambda}{\sqrt{24}}\int\frac{\prod_i^4 dp_i}{4\pi\sqrt{\prod_i^4 p_i}}\delta_\epsilon(\textstyle\sum_i^4 p_i)\psi_5(p_1,\ldots,p_5)$$
  $$=\left(\frac{M^2}{P}+P^-_{\rm vac}\right)\psi_1,$$

  $$\left(\sum_i^5\frac{\mu^2}{p_i}\right)\psi_5+\frac{\lambda}{24}\frac{1}{5}\left[\frac{\delta_\epsilon(\sum_i^4 p_i)}{4\pi\sqrt{\prod_i^4 p_i}}+(p_5\leftrightarrow p_1,p_2,p_3,p_4)\right]\psi_1$$
  $$+20\frac{\lambda}{4}\int\frac{dp_1'dp_2'}{4\pi\sqrt{p_1p_2p_1'p_2'}}\delta(p_1+p_2-p_1'-p_2')\psi_5(p_1',p_2',p_3,p_4,p_5)$$
  $$=\frac{M^2}{P}\psi_5$$
- sector-dependent energy shift: no $P^-_{\rm vac}$ in the top sector

Iterative solution

- second equation solved iteratively w.r.t. self-coupling of the five-constituent Fock state
  - this corresponds to a diagrammatic expansion
- substitute into first equation
- leading term generates the vacuum bubble

$$\int \frac{\prod_i^5 dp_i}{\prod_i^4 p_i}\delta(P-\sum_i^5 p_i)\frac{\delta_\epsilon(\sum_i^4 p_i)^2}{\frac{M^2}{P}-\sum_i^5 \frac{\mu^2}{p_i}}$$

$$\sim -\int \delta_\epsilon(Q)^2\frac{dQ}{\mu^2}\int \frac{\prod_i^4 dx_i}{\prod_i^4 x_i}\delta(1-\sum_i^4 x_i)$$

  - diverges as $\epsilon\to 0$, proportional to $\delta(0)=L/4\pi$
  - canceled by bubble in $P^-_{\rm vac}$
- second term, where the self interaction acts once, produces the tadpole (next slide)

Evaluation of tadpole

$$\int \frac{\prod_i^5 dp_i \delta_\epsilon(\sum_i^4 p_i)}{\sqrt{\prod_i^4 p_i}} \frac{\delta(P - \sum_i^5 p_i)}{\frac{M^2}{P} - \sum_i^5 \frac{\mu^2}{p_i}} \int \frac{dp_1' dp_2'}{\sqrt{p_4 p_5 p_1' p_2'}} \frac{\delta(p_4 + p_5 - p_1' - p_2')}{\frac{M^2}{P} - \sum_i^3 \frac{\mu^2}{p_i} - \frac{\mu^2}{p_1'} - \frac{\mu^2}{p_2'}} \frac{\delta_\epsilon(\sum_i^3 p_i + p_1')}{\sqrt{\prod_i^3 p_i p_1'}}$$

- $\delta(P - \sum_i^5 p_i)$ reduces to $\delta(P - p_5)$
  - used to do the p_5 integral
- $\delta(p_4 + p_5 - p_1' - p_2')$ becomes $\delta(p_4 + P - p_1' - p_2')$
  - used to do the p_2' integral
- $\delta_\epsilon(\sum_i^3 p_i + p_1')$ can be written $\delta_\epsilon(p_4 - p_1')$
  - used to do the p_1' integral
- these leave $\int \frac{\prod_i^4 dp_i}{\prod_i^4 p_i} \frac{1}{p_4 P} \frac{\delta_\epsilon(\sum_i^4 p_i)}{\left[\frac{M^2}{P} - \sum_i^4 \frac{\mu^2}{p_i} - \frac{\mu^2}{P}\right]^2}$

$$\sim \frac{1}{P} \int_0^\infty \delta(Q) dQ \int \frac{\prod_i^4 dx_i}{(\prod_i^4 x_i) x_4 \left(\sum_i^4 \frac{\mu^2}{x_i}\right)^2}$$

Tadpole self-energy correction

$$\sim \frac{1}{P}\int_0^\infty \delta(Q)dQ \int \frac{\prod_i^4 dx_i}{(\prod_i^4 x_i)x_4\left(\sum_i^4 \frac{\mu^2}{x_i}\right)^2} = \frac{1}{2P}\int \frac{\prod_i^4 dx_i}{(\prod_i^4 x_i)x_4\left(\sum_i^4 \frac{\mu^2}{x_i}\right)^2}$$

- finite and inversely proportional to P
  - a light-front self-energy correction
- in a nonperturbative calculation, these contributions cannot be separated.
  - regulate bubbles via $\delta \to \delta_\epsilon$
  - solve separate eigenproblems for the vacuum and the massive states
  - carry out the $P^-_{\rm vac}$ subtraction
  - take the width parameter ϵ to zero.
- with interactions, ephemeral modes no longer disjoint
  - for strong coupling, wave functions are broad
  - may have edge effects (aka zero modes)

Quenched scalar Yukawa theory

- simplest nonperturbative case (beyond shift)
- Lagrangian:
  $\mathcal{L} = |\partial_\mu \chi|^2 - m^2|\chi|^2 + \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}\mu^2\phi^2 - g\phi|\chi|^2$
- quenched light-front 2D Hamiltonian:
  $\mathcal{P}^- = \int dx^- \mathcal{H} = \mathcal{P}^-_{\text{free}} + \mathcal{P}^-_{\text{int}}$

$$\mathcal{P}^-_{\text{free}} = \int dp \frac{m^2}{p} \left[c_+^\dagger(p)c_+(p) + c_-^\dagger(p)c_-(p)\right] + \int dq \frac{\mu^2}{q} a^\dagger(q)a(q)$$
$$+ \frac{\mu^2}{2} \int \frac{dq_1 dq_2}{\sqrt{q_1 q_2}} \delta_\epsilon(q_1+q_2) \left[a(q_1)a(q_2) + a^\dagger(q_1)a^\dagger(q_2)\right],$$

$$\mathcal{P}^-_{\text{int}} =$$
$$g \int \frac{dp dq}{\sqrt{4\pi pq(p+q)}} \left\{\left[c_+^\dagger(p+q)c_+(p) + c_-^\dagger(p+q)c_-(p)\right] a(q) + \text{h.c.}\right\}$$

- quenching suppresses ephemeral modes for the complex scalar
  - leaving only those of the neutral scalar

Eigenproblem and Fock expansion

- charge-zero sector corresponds to the free scalar
  - provides the needed subtraction of $P^-_{\rm vac}$ for calculations in the charge-one sector
  - $P^-_{\rm vac} = -\frac{\mu^2 L}{16\pi} = -\frac{\mu^2}{2}\int_0^\infty dQ\delta_\epsilon(Q)^2$
- charge-one sector: $\mathcal{P}^-|\psi(P)\rangle = \left(\frac{M^2}{P} + P^-_{\rm vac}\right)|\psi(P)\rangle$
  - eigenstate is a single complex scalar dressed by a cloud of neutrals
  - keeping only the first three Fock sectors

$$|\psi(P)\rangle = \psi_0 c_+^\dagger(P)|0\rangle + \int dqdp\,\delta(P-q-p)\psi_1(q)a^\dagger(q)c_+^\dagger(p)|0\rangle$$
$$+\int dq_1dq_2dp\,\delta(P-q_1-q_2-p)\psi_2(q_1,q_2)\frac{1}{\sqrt{2}}a^\dagger(q_1)a^\dagger(q_2)c_+^\dagger(p)|0\rangle$$

Coupled equations for wave functions

$$\frac{m^2}{P}\psi_0 + \frac{g}{\sqrt{4\pi}}\int_0^P \frac{dq\,\psi_1(q)}{\sqrt{qP(P-q)}} + \frac{\mu^2}{\sqrt{2}}\int \frac{dq_1 dq_2}{\sqrt{q_1 q_2}}\delta_\epsilon(q_1+q_2)\psi_2(q_1,q_2)$$
$$= \left(\frac{M^2}{P} + P^-_{\rm vac}\right)\psi_0,$$

$$\left(\frac{\mu^2}{q} + \frac{m^2}{P-q}\right)\psi_1(q) + \frac{g}{\sqrt{4\pi}}\frac{\psi_0}{\sqrt{qP(P-q)}}$$
$$+\sqrt{2}\frac{g}{\sqrt{4\pi}}\int_0^{P-q}\frac{dq'\,\psi_2(q,q')}{\sqrt{q'(P-q)(P-q-q')}} = \frac{M^2}{P}\psi_1(q),$$

and

$$\left(\frac{\mu^2}{q_1} + \frac{\mu^2}{q_2} + \frac{m^2}{P-q_1-q_2}\right)\psi_2(q_1,q_2) + \frac{\mu^2}{\sqrt{2}}\delta_\epsilon(q_1+q_2)\frac{\psi_0}{\sqrt{q_1 q_2}}$$
$$+\frac{1}{\sqrt{2}}\frac{g}{\sqrt{4\pi}}\left[\frac{\psi_1(q_1)}{\sqrt{q_2(P-q_1)(P-q_1-q_2)}} + \frac{\psi_1(q_2)}{\sqrt{q_1(P-q_2)(P-q_1-q_2)}}\right]$$
$$= \frac{M^2}{P}\psi_2(q_1,q_2)$$

Again, the $P^-_{\rm vac}$ shift is taken to be sector dependent.

Energy subtraction

- ψ_2 contains a piece proportional to $\delta_\epsilon(q_1+q_2)$

$$\psi_2 \sim \left[\frac{M^2}{P} - \frac{\mu^2}{q_1} - \frac{\mu^2}{q_2} - \frac{m^2}{P-q_1-q_2}\right]^{-1} \frac{\mu^2}{\sqrt{2}}\delta_\epsilon(q_1+q_2)\frac{\psi_0}{\sqrt{q_1 q_2}}$$

$$\sim -\frac{1}{\sqrt{2}}\frac{\sqrt{q_1 q_2}}{q_1+q_2}\delta_\epsilon(q_1+q_2)\psi_0$$

  - it represents the underlying vacuum
- substitution into the 3rd term of the 1st equation yields

$$\frac{\mu^2}{\sqrt{2}}\int \frac{dq_1 dq_2}{\sqrt{q_1 q_2}}\delta_\epsilon(q_1+q_2)\psi_2(q_1,q_2) = -\frac{\mu^2}{2}\int \frac{dq_1 dq_2}{q_1+q_2}\delta_\epsilon(q_1+q_2)^2\psi_0$$

- this is just $P^-_{\rm vac}\psi_0$, which then cancels from both sides
  - this severe truncation is simple enough to allow this subtraction by hand
  - in general, this would not be the case
- the δ_ϵ are best interpreted in matrix-element integrals w.r.t. a suitable basis

Summary

- the LF vacuum is nontrivial in general
  - matrix elements of vacuum transition operators are not guaranteed to be zero
  - the vacuum must then be computed and its energy subtracted
  - Fock-state wave functions can have vacuum contributions
- vacuum transitions induce bubble and tadpole contributions
  - bubbles can be regulated and subtracted
  - tadpoles provide the ‘missing link’ between ET and LF values of the ϕ_2^4 critical coupling
    - a calculation needs to be done to check