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Elementary example of exact effective-Hamiltonian computation

Stanisław D. Głazek

Institute of Theoretical Physics, Faculty of Physics, University of Warsaw

An exact computation of effective Hamiltonians in an elementary model akin to quantum field theory is carried out by solving equations of the renormalization group procedure for effective particles (RGPEP) [1]. The computation exhibits the mechanism by which the weak-coupling expansion and Tamm-Dancoff approximation increase in accuracy along the RGPEP evolution. The model computational pattern can be followed in perturbative computations of effective Hamiltonians in realistic theories.

[1] SDG, Phys. Rev. D **103**, 014021 (2021).

stglazek@fuw.edu.pl

Motivation

- goal: presenting a universal method for computing effective Hamiltonians in QFT

- method: renormalization group procedure for effective particles (**RGPEP**)

- result: Tamm-Dancoff and PT increase in accuracy along the RGPEP evolution

Model computation exhibits the RGPEP method without clutter.

stglazek@fuw.edu.pl

Derivation of the model

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi + \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\mu^2\phi^2 - g\bar{\psi}\phi\psi$$

$$\mathcal{L} \to \mathcal{T}^{\mu\nu} \to \mathcal{H} = \mathcal{T}^{+-} \to H = \int_F \mathcal{H} \quad \to \quad \hat{H} = \int_F :\mathcal{H}(\hat{\psi}, \hat{\phi}):$$

$$m \gg \mu \gg \Delta$$

stglazek@fuw.edu.pl

Canonical Yukawa theory Hamiltonian

$$H_{FF} = H_f + H_b + H_{fb} + [\text{other terms}]$$

$$p^\pm = p^0 \pm p^3 \qquad p^2 = p^+ p^- - p^{\perp 2} = m^2 \qquad p^- = (m^2 + p^{\perp 2})/p^+$$

$$H_f = \sum_\sigma \int [p] \, \frac{m^2 + p^{\perp\, 2}}{p^+} \, b^\dagger_{p\sigma} b_{p\sigma} \qquad H_b = \int [p] \, \frac{\mu^2 + p^{\perp\, 2}}{p^+} \, a^\dagger_p a_p$$

$$H_{fb} = g \sum_{\sigma_1, \sigma_2} \int [p_1 p_2 p_3] \, \delta_{c.a} \, \bar u_1 \Gamma u_2 \; b^\dagger_{p_1 \sigma_1} (a^\dagger_{p_3} + a_{p_3}) b_{p_2 \sigma_2} + \ldots$$

$$\int [p] = \int d^4 p \, \delta(p^2 - m^2) \, \theta(p^+) \quad \rightarrow \quad \int [p] \; \theta(\Delta - |\vec p\,|)$$

stglazek@fuw.edu.pl

Model simplifications

fermion mass $m \gg \Delta$ $\quad\rightarrow\quad$ one fermion and many bosons

boson mass $\mu \gg \Delta$ $\quad\rightarrow\quad$ boson-fermion relative motion $|\vec{k}\,| \ll \Delta \ll \mu$

small-$\vec{k}$ boson states replaced by one, static mode with $E_b = \mu$

$\bar{u}_1 u_2 \rightarrow 2m$ etc.

Model Hamiltonian $\quad H_{FF} = H_f + H_b + H_{fb} + \text{[other terms]}$

$$\rightarrow \quad H_{\rm model} = E_f \, b^\dagger b + E_b \, a^\dagger a + g E_I \, b^\dagger (a^\dagger + a) b$$

$\{b, b^\dagger\} = 1, \quad [a, a^\dagger] = 1$ etc. (many ways to solve)

stglazek@fuw.edu.pl

The RGPEP equation

introduce $t = s^2$, a scale parameter, $s \sim$ size of effective quanta, $[s] \sim 1/E$

$$b_t = \mathcal{U}_t \, b \, \mathcal{U}_t^\dagger \qquad a_t = \mathcal{U}_t \, a \, \mathcal{U}_t^\dagger \qquad \text{etc.}$$

$b = b_0$, $a = a_0$ etc. $\quad 0 = t = s^2 \leftrightarrow$ point-like quanta of canonical Yukawa theory

b_t, a_t etc. $\quad 0 < t = s^2 \leftrightarrow$ the effective quanta

$$H_{\text{model}} = E_f \, b^\dagger b + E_b \, a^\dagger a + g E_I \, b^\dagger (a^\dagger + a) b \; = \; H_t \; = \sum_{k,l,m,n=0}^{\infty} C_t^{klmn} \, b_t^{\dagger k} \, b_t^l \, a_t^{\dagger m} \, a_t^n$$

$$\frac{d}{dt} \mathcal{H}_t = \left[\left[H_f + H_b, \mathcal{H}_t \right], \mathcal{H}_t \right] \quad \text{and} \quad C_0^{1100} = E_f, \; C_0^{0011} = E_b, \; C_0^{1110} = C_0^{1101} = g E_I$$

stglazek@fuw.edu.pl

Solution of the RGPEP equation in the model

$$H_{\rm model} = \quad E_f\, b^\dagger b \qquad\qquad + E_b\, a^\dagger a \quad + g E_I\, b^\dagger (a^\dagger + a) b$$

$$H_t = \left(E_f + g_t^2 \Delta_t\right)\, b_t^\dagger b_t + E_b\, a_t^\dagger a_t + g_t\; E_I\, b_t^\dagger (a_t^\dagger + a_t) b_t$$

$$a_t = a + c_t\, b^\dagger b \qquad b_t \;=\; e^{c_t\,(a^\dagger - a)} b \qquad c_t \;=\; (g - g_t) E_I / E_b$$

$$g_t = g e^{-E_b^2 t} \qquad \Delta_t \;=\; (1 - e^{2 E_b^2 t}) E_I^2 / E_b$$

stglazek@fuw.edu.pl

spectral decomposition at $t \to \infty$

$$H_\infty \;=\; E_{\rm fermion}\, b^\dagger_\infty b_\infty + E_b\, a^\dagger_\infty a_\infty$$

$$E_{\rm fermion} \;=\; \lim_{t\to\infty} \left(E_f + g_t^2 \Delta_t\right)$$

$$E_{n\,{\rm bosons}} \;=\; nE_b$$

$$E_{\rm fermion} = E_f - g^2 E_I^2/E_b$$

$$E_{{\rm fermion}+n\,{\rm bosons}} \;=\; E_{\rm fermion} + nE_b$$

$$g_\infty = 0$$

$$|{\rm fermion}\rangle = b^\dagger_\infty |0\rangle$$

$$|n\,{\rm bosons}\rangle = \frac{1}{\sqrt{n!}} a^{\dagger n}_\infty |0\rangle$$

$$|{\rm fermion} + n\,{\rm bosons}\rangle = \frac{1}{\sqrt{n!}} a^{\dagger n}_\infty b^\dagger_\infty |0\rangle$$

$$b^\dagger_\infty \;=\; e^{-g(E_I/E_b)\,(a^\dagger - a)} b^\dagger$$

$$a^\dagger_\infty \;=\; a^\dagger + g(E_I/E_b)\, b^\dagger b$$

stglazek@fuw.edu.pl

Weak-coupling expansion power series in g_t

$$g_t = g e^{-E_b^2 t}$$

example of expansion

$$|\text{fermion}\rangle = N \left[b_t^\dagger |0\rangle - \frac{1}{E_b} g_t E_I \, a_t^\dagger b_t^\dagger |0\rangle + \frac{1}{2} \left(\frac{1}{E_b} g_t E_I \right)^2 a_t^\dagger a_t^\dagger b_t^\dagger |0\rangle \right] + O(g_t^3)$$

$$\lim_{t \to \infty} g_t = 0$$

Accelerated convergence for growing size of effective particles

$$g_t = g e^{-t(E_{f+b} - E_f)^2} \to g e^{-t^2(\mathcal{M}_{f+b}^2 - m_f^2)^2}$$

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Tamm-Dancoff approximation limit the number of Fock components

$$|\text{fermion}_{\text{TD}}\rangle = x_{t0}\, b_t^\dagger|0\rangle + x_{t1}\, a_t^\dagger b_t^\dagger|0\rangle + \frac{1}{\sqrt{2}} x_{t2}\, a_t^{\dagger\,2} b_t^\dagger|0\rangle$$

$$H_{TD}|\text{fermion}_{\text{TD}}\rangle = E_{TD}|\text{fermion}_{\text{TD}}\rangle$$

How many effective bosons are needed?

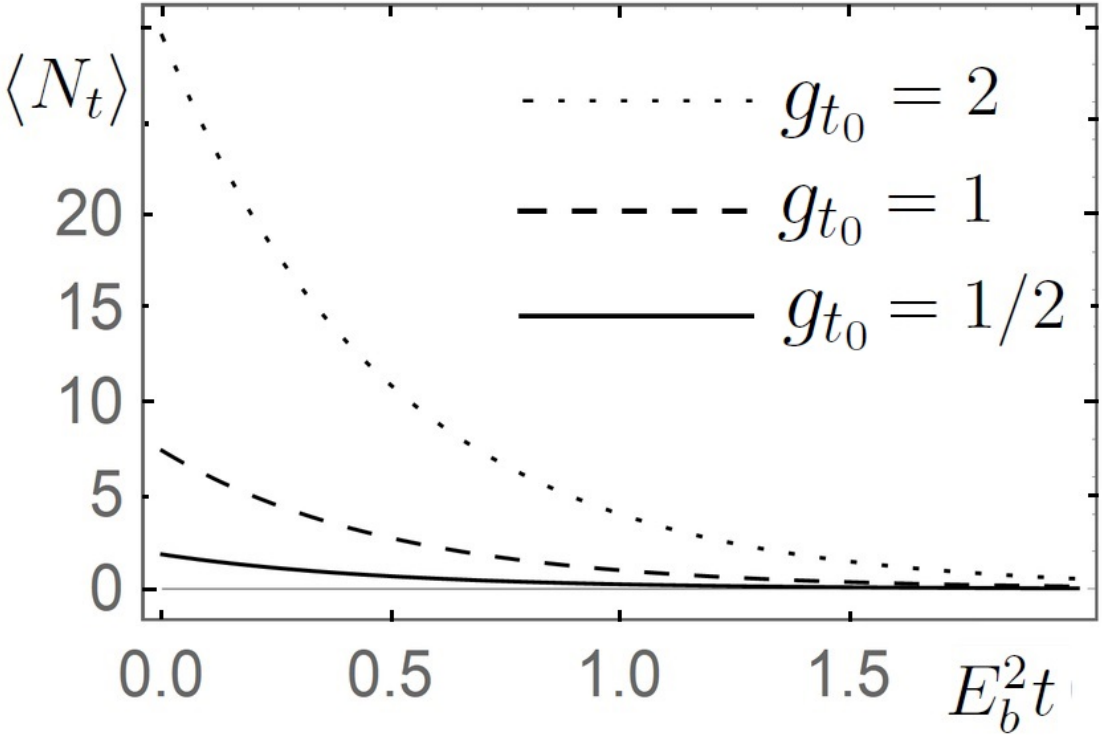
$$\langle N_t\rangle = \langle \text{fermion}|a_t^\dagger a_t|\text{fermion}\rangle \;=\; g_t^2 \;=\; g_{t_0}^2\, e^{2E_b^2(t_0-t)}$$

g_{t_0} corresponds to effective bosons of size $s_0 = \sqrt{t_0} = 1/E_b$, examples with $E_I = E_b$

$g_{t_0} = 1/2$	$g_{t_0}^2 = 1/4$	$\alpha_\Upsilon \sim 1/4$
$g_{t_0} = 1$	$g_{t_0}^2 = 1$	convergence radius for geometric series in g_0
$g_{t_0} = 2$	$g_{t_0}^2 = 4$	potentially non-perturbative dynamics

stglazek@fuw.edu.pl

Accuracy of the weak-coupling and Tamm-Dancoff approximations



stglazek@fuw.edu.pl

Concluding remarks concerning QFT

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1. Some terms in H_{eff} order g^2 and g^3 are known in QED and QCD — AF
2. 4th-order terms are only known in models — two loops
3. Phenomenology of heavy-quarkonia (QCD + gluon mass) — oscillator potential
4. Regularization and the vacuum problem (small x) — $p^+ > \epsilon^+$
5. Poincaré symmetry, quantum mechanics and QFT

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$$\mathcal{U}_t^\dagger = 1 + \left[e^{c_t\,(a^\dagger - a)} - 1 \right] \; b^\dagger b$$

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Tamm-Dancoff approximation limit the number of Fock components

$$|\mathrm{fermion_{TD}}\rangle = x_{t0}\, b_t^\dagger|0\rangle + x_{t1}\, a_t^\dagger b_t^\dagger|0\rangle + \frac{1}{\sqrt{2}} x_{t2}\, a_t^{\dagger\, 2} b_t^\dagger|0\rangle$$

$$H_{TD}|\mathrm{fermion_{TD}}\rangle = E_{TD}|\mathrm{fermion_{TD}}\rangle$$

$$\begin{bmatrix} h_{2,2} & h_{2,1} & 0 \\ h_{1,2} & h_{1,1} & h_{1,0} \\ 0 & h_{0,1} & h_{0,0} \end{bmatrix} \begin{bmatrix} x_{t2} \\ x_{t1} \\ x_{t0} \end{bmatrix} = E_{\mathrm{TD}} \begin{bmatrix} x_{t2} \\ x_{t1} \\ x_{t0} \end{bmatrix}$$

$$h_{m,n} = \frac{1}{\sqrt{m!n!}} \langle 0|b_t a_t^m\, H_t a_t^{\dagger n} b_t^\dagger|0\rangle \qquad m, n \le 2$$

$$E_{TD} = E_f + g_t^2 \Delta_t + x E_b \quad \Rightarrow \quad (2-x)(1-x)x + \alpha_t(2-3x) = 0$$

$$x = n + \alpha_t y \qquad E_{TDn} = E_{\mathrm{fermion}} + n E_B \qquad \alpha_t = (g_t E_I / E_b)^2$$

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