LIGHT-FRONT QUANTIZATION FROM THEN UNTIL NOW

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P. D. Mannheim, P. Lowdon, S. J. Brodsky: Structure of light front vacuum sector diagrams, Phys. Lett. B 797, 134916 (2019). (arXiv:1904.05253)

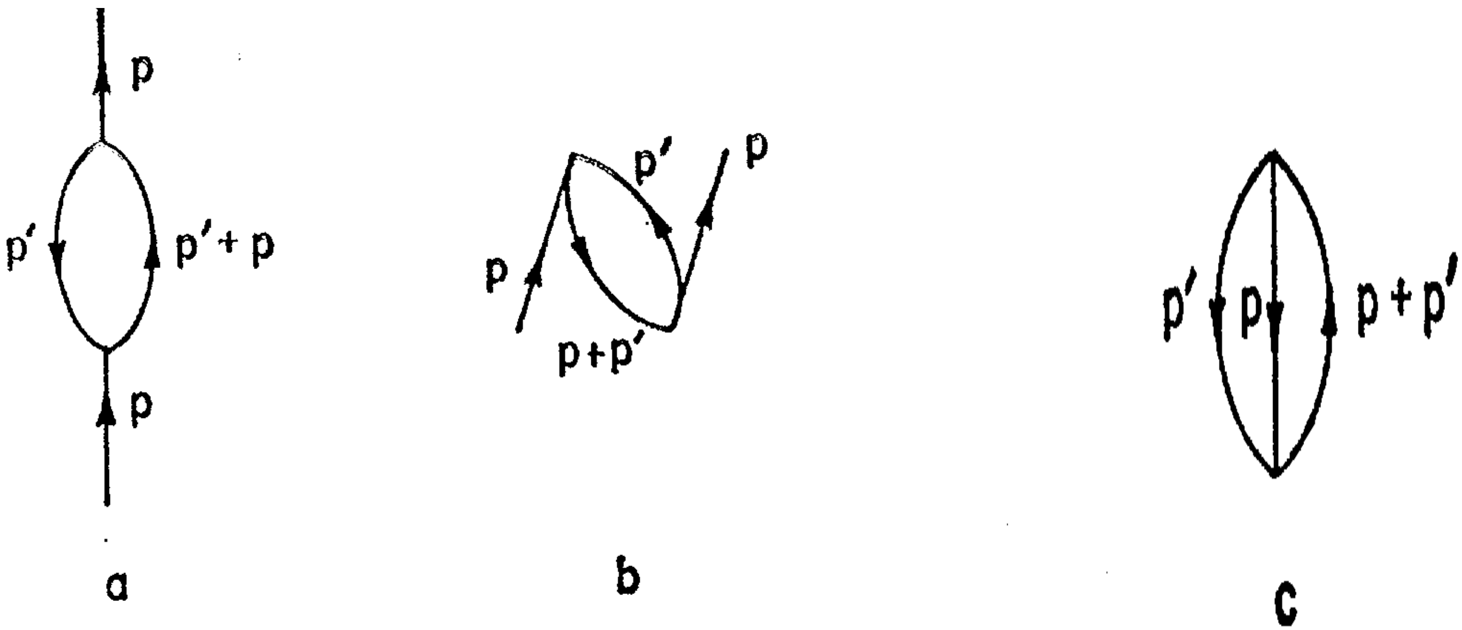
P. D. Mannheim: Equivalence of light-front quantization and instant-time quantization, Phys. Rev. D 102, 025020 (2020). (arXiv:1909.03548)

P. D. Mannheim: Light-front quantization is the same as instant-time quantization, Proc. Sci. (LC2019) 062 (2019). (arXiv:2001.04603)

P. D. Mannheim, P. Lowdon, S. J. Brodsky: Comparing light-front quantization with instant-time quantization, Phys. Rep. 891, 1 (2021). (arXiv:2005.00109)

1 INFINITE MOMENTUM FRAME

In 1966 Weinberg (Phys. Rev. 150, 1313 (1966)) showed that instant-time quantization perturbation theory would be simplified in the frame in which an observer moved with an infinite three-momentum with respect to the center of mass system of a scattering process, i.e., $p^3 = \alpha P$ where P is large and α is a constant and $p^0 = [(p^3)^2 + (p^1)^2 + (p^2)^2 + m^2]^{1/2} \to \alpha P + [(p^1)^2 + (p^2)^2 + m^2]/2\alpha P$.



Specifically, Graph (b) would be suppressed with respect to Graph (a). Graph (c) was not discussed. In Weinberg's case the x^0 time axis runs up the diagram and the analysis was made using old-fashioned perturbation theory. Old-fashioned (i.e. pre-Feynman) perturbation theory is off the energy shell but on the mass shell. (The Feynman approach is off the mass shell).

2 INSTANT-TIME FEYNMAN GRAPHS AND OLD-FASHIONED PERTURBATION THEORY

In the instant-time case one can take an instant-time forward in time Green's function such as $D(x^0 > 0, \mathrm{instant}) = -i\langle\Omega_I|\theta(x^0)\phi(x^0,x^1,x^2,x^3)\phi(0)|\Omega_I\rangle$ as evaluated in the instant-time vacuum $|\Omega_I\rangle$, and expand the field in terms of instant-time creation and annihilation operators that create and annihilate particles out of that vacuum as

$$\phi(x^0,\vec{x}) = \int \frac{d^3p}{(2\pi)^{3/2}(2E_p)^{1/2}}[a(\vec{p})\exp(-iE_pt+i\vec{p}\cdot\vec{x}) + a^\dagger(\vec{p})\exp(+iE_pt-i\vec{p}\cdot\vec{x})], \tag{2.1}$$

where $E_p = (\vec{p}^2+m^2)^{1/2}$ and $[a(\vec{p}), a^\dagger(\vec{p}\,')] = \delta^3(\vec{p}-\vec{p}\,')$. The insertion of $\phi(\vec{x},x^0)$ into $D(x^0 > 0, \mathrm{instant})$ immediately leads to the on-shell three-dimensional integral

$$D(x^0>0,\mathrm{instant},\mathrm{Fock}) = -\frac{i\theta(x^0)}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d^3p}{2E_p}e^{-iE_px^0+i\vec{p}\cdot\vec{x}}. \tag{2.2}$$

Alternatively, one can look for solutions to $(\partial_\alpha\partial^\alpha + m^2)D(x^\mu,\mathrm{instant}) = -\delta^4(x)$, and obtain the off-shell four-dimensional integral

$$D(x^\mu,\mathrm{instant}) = \frac{1}{(2\pi)^4}\int d^4p\frac{e^{-ip\cdot x}}{p^2-m^2+i\epsilon} = \frac{1}{(2\pi)^4}\int\frac{d^4p}{2E_p}e^{-ip\cdot x}\left[\frac{1}{p_0-E_p+i\epsilon}-\frac{1}{p_0+E_p-i\epsilon}\right], \tag{2.3}$$

with the p_0 integration being along a contour integral in the complex p_0 plane. One can then proceed from (2.3) to (2.2) by closing the Feynman contour below the real p_0 axis, to yield a contour integral in which the lower-half p_0 plane circle at infinity makes no contribution when the instant-time x^0 is positive, while the pole term yields (2.2).

Similarly, one can proceed from (2.2) to (2.3) by writing the theta function as a contour integral in the complex ω plane:

$$\theta(x^0) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega x^0}}{\omega + i\epsilon}, \tag{2.4}$$

so that the pole contribution yields $\theta(x^0) = 1$ when $x^0 > 0$ and yields $\theta(x^0) = 0$ when $x^0 < 0$. With this representation of the theta function (2.2) takes the form

$$D(x^0 > 0, \text{instant}) = \frac{1}{(2\pi)^4}\int \frac{d^3p}{2E_p}\int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega x^0}}{\omega + i\epsilon} e^{-iE_p x^0 + i\vec{p}\cdot\vec{x}}. \tag{2.5}$$

On setting $p_0 = \omega + E_p$, we can rewrite (2.5) as

$$D(x^0 > 0, \text{instant}) = \frac{1}{(2\pi)^4}\int \frac{d^4p}{2E_p}\frac{e^{-ip_0 x^0 + i\vec{p}\cdot\vec{x}}}{(p_0 - E_p + i\epsilon)}. \tag{2.6}$$

We recognize (2.6) as the forward in time, positive frequency component of (2.3), and thus establish the equivalence of the instant-time off-shell four-dimensional Feynman and on-shell three-dimensional Hamiltonian (Fock space) formalisms, and see that the equivalence occurs because the four-dimensional Feynman contour is given by **on-shell poles alone**. Pole dominance thus leads to old-fashioned perturbation theory.

When $x^0 = 0$ we obtain

$$\theta(0) = -\frac{1}{2\pi i}\oint_{-\infty}^{\infty} d\omega \frac{1}{\omega + i\epsilon} = -\frac{1}{2\pi i}[-2\pi i + \pi i] = \frac{1}{2}, \tag{2.7}$$

with there being a circle at infinity contribution and not just a pole term. For the instant-time case the circle contribution is suppressed because there are two powers of p_0 in the denominator of $D(x^0 = 0, \text{instant})$. However, in the light front case there is only one power of p_+ and the circle does contribute to $D(x^+ = 0, \text{front})$.

3 LIGHT-FRONT VARIABLES

In 1969 Chang and Ma (Phys. Rev. 180, 1506 (1969)) recovered Weinberg's infinite momentum frame result by working with the light-front variables $p^+ = p^0 + p^3$, $p^- = p^0 - p^3$. Under a Lorentz boost in the z direction with velocity u these variables transform as

$$p^0 + p^3 \to (p^0 + p^3)\left(\frac{1+u}{1-u}\right)^{1/2}, \quad p^0 - p^3 \to (p^0 - p^3)\left(\frac{1-u}{1+u}\right)^{1/2} \tag{3.1}$$

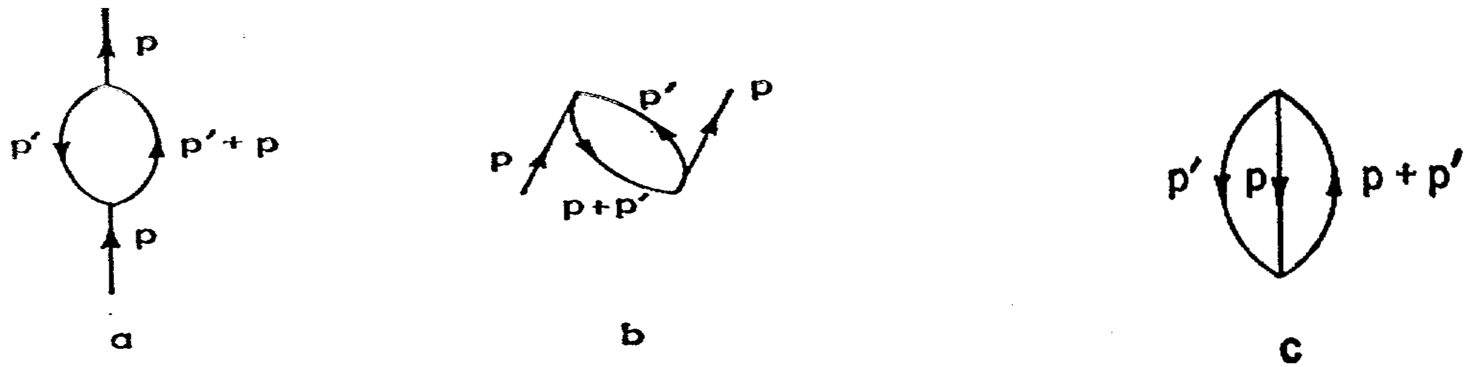
Setting $(1+u)^{1/2}/(1-u)^{1/2} = 1/2P$, $p^3 = \alpha P$, for large P and $(p^0)^2 - (p^3)^2 = p^+p^- = m^2 + (p^1)^2 + (p^2)^2$ we obtain

$$p^0 + p^3 \to \frac{2\alpha P}{2P} = \alpha, \quad p^0 - p^3 \to \frac{[m^2 + (p^1)^2 + (p^2)^2]}{2\alpha P}2P = \frac{[m^2 + (p^1)^2 + (p^2)^2]}{2\alpha}, \tag{3.2}$$

i.e., we recover the momenta used by Weinberg. With this choice a Green's function as evaluated with a complex plane p_+ contour becomes equal to Graph (a) when Graph (a) is evaluated with a complex plane p_0 contour at large p^3.

There is a caveat. In the infinite momentum frame case the flow of time is forward in x^0, while the flow of time in the light-front case is forward in $x^+ = x^0 + x^3$. But for timelike or lightlike events $(x^0)^2 - (x^3)^2 = x^+x^- \geq (x^1)^2 + (x^2)^2$ is positive, where $x^- = x^0 - x^3$. Thus x^+x^- is positive. Consequently, x^+ and x^- have the same sign. And thus for $x^0 > 0$ (a Lorentz invariant for timelike or lightlike events) it follows that x^+ is positive too. Thus for timelike or lightlike events, forward in x^+ is the same as forward in x^0.

4 THE TAKEAWAY



In their work Chang and Ma showed that

for Graph (a) x^+ is positive and all the p^- poles have both p^- and p^+ positive,

for Graph (b) x^+ is negative and all the p^- poles have both p^- and p^+ negative,

for Graph (c) x^+ is zero and so is p^+. But if p^+ is zero then p^- is infinite. Thus $p_+ = p^-/2$ is infinite too, just as it should be since it is the conjugate of x^+. $(\Delta x^+ \Delta p_+ > \hbar)$.

However, and this is the key point, **all of these statements are true without going to the infinite momentum frame.** They thus can define a strategy for evaluating diagrams as diagrams are segregated by the sign of the time variable x^+. And since x^+ is positive for scattering processes they only involve positive p^- and p^+, with the p^- pole contributions then corresponding to old-fashioned perturbation theory diagrams. Only needing positive p^- and p^+ provides enormous computational benefits.

The vacuum Graph (c) is expressly non-zero, something known as early as 1969. However it involves $p^+ = 0$ zero modes, whose evaluation is tricky. Resolved in Mannheim, Lowdon and Brodsky 2019.

But what about the instant-time graphs that are not at infinite momentum. Are they different from or the same as the light-front graphs. And if they are different, then which ones describe the real world. In Mannheim, Lowdon and Brodsky (2019) they were shown to be the same, though developments since 1969 would suggest that this would be far from the case.

5 LIGHT-FRONT QUANTUM FIELD THEORY

Instead of replacing instant-time momenta by light-front momenta in Feynman diagrams, we can obtain a fully-fledged light-front quantum field theory by constructing equal x^+ commutators rather than equal x^0 commutators. For a scalar field [Neville and Rohrlich, Nuovo Cimento A 1, 625 (1971)]

Scalar field light-front commutators at equal x^+

$$[\phi(x^+,x^1,x^2,x^-),\phi(x^+,y^1,y^2,y^-)] = -\frac{i}{4}\epsilon(x^- - y^-)\delta(x^1-y^1)\delta(x^2-y^2),$$
$$[\phi(x^+,x^1,x^2,x^-),2\partial_-\phi(x^+,y^1,y^2,y^-)] = i\delta(x^1-y^1)\delta(x^2-y^2)\delta(x^- - y^-). \tag{5.1}$$

Scalar field instant-time commutators at equal x^0

$$[\phi(x^0,x^1,x^2,x^3),\partial_0\phi(x^0,y^1,y^2,y^3)] = i\delta(x^1-y^1)\delta(x^2-y^2)\delta(x^3-y^3),$$
$$[\phi(x^0,x^1,x^2,x^3),\phi(x^0,y^1,y^2,y^3)] = 0. \tag{5.2}$$

Gauge field instant-time commutators at equal x^0

$$[A_\nu(x^0,x^1,x^2,x^3),\partial_0 A_\mu(x^0,y^1,y^2,y^3)] = -ig_{\mu\nu}\delta(x^1-y^1)\delta(x^2-y^2)\delta(x^3-y^3),$$
$$[A_\nu(x^0,x^1,x^2,x^3),A_\mu(x^0,y^1,y^2,y^3)] = 0. \tag{5.3}$$

Using gauge fixing, for light-front gauge fields we obtain (Mannheim, Lowdon and Brodsky 2021)

Gauge field light-front commutators at equal x^+

$$[A_\nu(x^+,x^1,x^2,x^-),2\partial_- A_\mu(x^+,y^1,y^2,y^-)] = -ig_{\mu\nu}\delta(x^1-y^1)\delta(x^2-y^2)\delta(x^- - y^-),$$
$$[A_\nu(x^+,x^1,x^2,x^-),A_\mu(x^+,y^1,y^2,y^-)] = \frac{i}{4}g_{\mu\nu}\epsilon(x^- - y^-)\delta(x^1-y^1)\delta(x^2-y^2). \tag{5.4}$$

Analogous results in the non-Abelian case.

The instant-time and light-front commutators are completely different.

6 INSTANT-TIME AND LIGHT-FRONT ANTICOMMUTATORS

Fermion instant-time anticommutators at equal x^0

$$\left\{\psi_\alpha(x^0,x^1,x^2,x^3),\psi^\dagger_\beta(x^0,y^1,y^2,y^3)\right\}=\delta_{\alpha\beta}\delta(x^1-y^1)\delta(x^2-y^2)\delta(x^3-y^3). \tag{6.1}$$

Fermion light-front anticommutators at equal x^+

$$\left\{[\psi_{(+)}]_\alpha(x^+,x^1,x^2,x^-),[\psi^\dagger_{(+)}]_\beta(x^+,y^1,y^2,y^-)\right\}=\Lambda^+_{\alpha\beta}\delta(x^--y^-)\delta(x^1-y^1)\delta(x^2-y^2). \tag{6.2}$$

[Chang, Root and Yan, Phys. Rev. D 7, 1133 (1973).]

Non-Invertible Projectors

$$\Lambda^\pm=\tfrac{1}{2}(1\pm\gamma^0\gamma^3),\quad \Lambda^++\Lambda^-=I,\quad (\Lambda^+)^2=\Lambda^+,\quad (\Lambda^-)^2=\Lambda^-,\quad \Lambda^+\Lambda^-=0,\quad \gamma^\pm=\gamma^0\pm\gamma^3,\quad (\gamma^\pm)^2=0,$$

$\psi_{(\pm)}=\Lambda_\pm\psi$, $\psi_{(-)}$ **is a constrained variable:** (6.3)

$$\psi_{(-)}(x^+,x^1,x^2,x^-)=-\frac{i}{4}\int du^-\epsilon(x^--u^-)[-i\gamma^0(\gamma^1\partial_1+\gamma^2\partial_2)+m\gamma^0]\psi_{(+)}(x^+,x^1,x^2,u^-). \tag{6.4}$$

$$\left\{[\psi_{(+)}]_\nu(x),[\psi^\dagger_{(-)}]_\sigma(y)\right\}=\tfrac{i}{8}\epsilon(x^--y^-)[i(\gamma^-\gamma^1\partial^x_1+\gamma^-\gamma^2\partial^x_2)-m\gamma^-]_{\nu\sigma}\delta(x^1-y^1)\delta(x^2-y^2), \tag{6.5}$$

$$\begin{aligned}&\left\{\psi^{(-)}_\mu(x^+,x^1,x^2,x^-),[\psi^\dagger_{(-)}]_\nu(x^+,y^1,y^2,y^-)\right\}\\&=\frac{1}{16}\Lambda^-_{\mu\nu}\left[-\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^1}-\frac{\partial}{\partial x^2}\frac{\partial}{\partial x^2}+m^2\right]\int du^-\epsilon(x^--u^-)\epsilon(y^--u^-)\delta(x^1-y^1)\delta(x^2-y^2).\end{aligned} \tag{6.6}$$

The instant-time and light-front anticommutators are completely different and even not invertible.

7 PROPAGATORS AND TIME-ORDERED PRODUCTS

Things get even worse. The x^+-ordered product does not always satisfy the field wave equation with a delta function source (the propagator equation). This is not a problem for scalar fields, but for fermions we obtain [Yan, Phys. Rev. D 7, 1780 (1973)]

$$
\begin{aligned}
&-i\langle\Omega|[\theta(x^+)\psi_\beta(x^\mu)\bar{\psi}_\alpha(0)-\theta(-x^+)\bar{\psi}_\alpha(0)\psi_\beta(x^\mu)]|\Omega\rangle=\tfrac{i}{4}\gamma^+_{\beta\alpha}\delta(x^+)\epsilon(x^-)\delta(x^1)\delta(x^2)\\
&+\frac{2}{(2\pi)^4}\int_{-\infty}^{\infty}dp_+dp_1dp_2dp_-\Big[\frac{e^{-i(p_+x^++p_1x^1+p_2x^2+p_-x^-)}}{\gamma^+p_++\gamma^-p_-+\gamma^1p_1+\gamma^2p_2-m+i\epsilon}\Big]_{\beta\alpha},
\end{aligned}
\tag{7.1}
$$

i.e., a propagator plus a delta function term. This delta function term only contributes at $x^+=0$, and thus can only contribute in vacuum graphs.

For gauge fields quantized in the $A^+=0$ axial gauge we have [Harindranath, arXiv:hep-ph/9612244]

$$
\begin{aligned}
&-i\langle\Omega|[\theta(x^+)A^\mu(x)A^\nu(0)+\theta(-x^+)A^\nu(0)A^\mu(x)]|\Omega\rangle\\
&=2\int\frac{dp_+dp_-dp_1dp_2}{(2\pi)^4}\frac{e^{-ip\cdot x}}{p^2+i\epsilon}\left(g^{\mu\nu}-\frac{n^\mu p^\nu+n^\nu p^\mu}{n\cdot p}+\frac{p^2}{(n\cdot p)^2}n^\mu n^\nu\right),
\end{aligned}
\tag{7.2}
$$

i.e., a propagator plus an n^μ-dependent term with only non-zero element $n_+=1$. The n^μ-dependent terms are absent in the instant-time case and lead to a zero mode problem at $p^+=0$.

Fortunately, both of the fermion and gauge field problems are readily fixable. The gauge field n^{μ}-dependent term does not appear at all if we use gauge fixing. Rather, if one takes the action to be of the form

$$I_G = \int d^4x \left[-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}(\partial_\mu A^\mu)^2\right] = \int d^4x \left[-\tfrac{1}{2}\partial_\nu A_\mu \partial^\nu A^\mu\right], \tag{7.3}$$

the x^+-ordered product is then nicely given by $D^{\mu\nu}(p) = g^{\mu\nu}/(p^2 + i\epsilon)$ [Mannheim, Lowdon and Brodsky 2021], just as in the instant-time x^0-ordered case. So no zero mode problem.

For fermions we note that because of Lorentz invariance the vacuum graphs have no external indices, and so the α and β indices in (7.1) must be contracted with $\delta_{\alpha\beta}$. But γ^+ is traceless, and so the delta function term in (7.1) decouples [Mannheim, Lowdon and Brodsky 2021].

We thus see that the instant-time and light-front propagators (and thus Dyson-Wick expansions) are identical in form, and only differ from each other by a change of integration variable from p^0, p^3 to p^+, p^- in expressions that are Poincare invariant. Thus unlike in the infinite momentum frame study, now we can identify the two sets of propagators and Feynman diagrams at all momenta. The two theories are thus equivalent.

But what about the commutators and anticommutators?

8 UNEQUAL TIME COMMUTATORS AND ANTICOMMUTATORS

Following Mannheim (2020):

UNEQUAL TIME Scalar instant-time commutator

$$\begin{aligned}
i\Delta(x-y) &= [\phi(x^0,x^1,x^2,x^3),\phi(y^0,y^1,y^2,y^3)] \\
&= \int \frac{d^3p d^3q}{(2\pi)^3(2p)^{1/2}(2q)^{1/2}}\Big([a(\vec{p}),a^\dagger(\vec{q})]e^{-ip\cdot x+iq\cdot y}+[a^\dagger(\vec{p}),a(\vec{q})]e^{ip\cdot x-iq\cdot y}\Big) \\
&= \int \frac{d^3p}{(2\pi)^3 2p}\big(e^{-ip\cdot(x-y)}-e^{ip\cdot(x-y)}\big) \\
&= -\frac{i}{2\pi}\frac{\delta(x^0-y^0-|\vec{x}-\vec{y}|)-\delta(x^0-y^0+|\vec{x}-\vec{y}|)}{2|\vec{x}-\vec{y}|} \\
&= -\frac{i}{2\pi}\epsilon(x^0-y^0)\delta[(x^0-y^0)^2-(x^1-y^1)^2-(x^2-y^2)^2-(x^3-y^3)^2]. \qquad (8.1)
\end{aligned}$$

Since it holds at ALL times, it also holds at EQUAL light front time.

Substitute $x^0=(x^++x^-)/2$**,** $x^3=(x^+-x^-)/2$**,** $y^0=(y^++y^-)/2$**,** $y^3=(y^+-y^-)/2$**:**

$$i\Delta(x-y)=-\frac{i}{2\pi}\epsilon[\tfrac{1}{2}(x^++x^--y^+-y^-)]\delta[(x^+-y^+)(x^--y^-)-(x^1-y^1)^2-(x^2-y^2)^2]. \qquad (8.2)$$

$$i\Delta(x-y)\big|_{x^+=y^+} = [\phi(x^+,x^1,x^2,x^-),\phi(x^+,y^1,y^2,y^-)]=-\frac{i}{4}\epsilon(x^--y^-)\delta(x^1-y^1)\delta(x^2-y^2). \qquad (8.3)$$

At $x^+=y^+$ **UNEQUAL instant-time commutator is EQUAL light-front time commutator**

Light-front quantization is instant-time quantization, and does not need to be independently postulated.

UNEQUAL TIME Abelian gauge field instant-time commutator

$$
\begin{aligned}
&[A_\nu(x^0,x^1,x^2,x^3),A_\mu(y^0,y^1,y^2,y^3)]=ig_{\mu\nu}\Delta(x-y)\\
&=-\frac{i}{2\pi}g_{\mu\nu}\epsilon(x^0-y^0)\delta[(x^0)^2-(x^1)^2-(x^2)^2-(x^3)^2].
\end{aligned}
\tag{8.4}
$$

Leads to

$$
[A_\nu(x^+,x^1,x^2,x^-),A_\mu(x^+,y^1,y^2,y^-)]=\frac{i}{4}g_{\mu\nu}\epsilon(x^--y^-)\delta(x^1-y^1)\delta(x^2-y^2).
\tag{8.5}
$$

At $x^+=y^+$ UNEQUAL instant-time commutator is EQUAL light-front time commutator Similar result holds for non-Abelian gauge field.

9 FERMION UNEQUAL INSTANT-TIME ANTICOMMUTATOR

$$
\left\{\psi_\alpha(x^0,x^1,x^2,x^3),\psi^\dagger_\beta(y^0,y^1,y^2,y^3)\right\}=\left[(i\gamma^\mu\gamma^0\partial_\mu\right]_{\alpha\beta}i\Delta(x-y).
\tag{9.1}
$$

Apply projector and set $x^+=y^+$

$$
\begin{aligned}
&\Lambda^+_{\alpha\gamma}\left\{\psi_\gamma(x^+,x^1,x^2,x^-),\psi_\delta(x^+,y^1,y^2,y^-)\right\}\Lambda^+_{\delta\beta}\\
&=\left\{[\psi_{(+)}(x^+,x^1,x^2,x^-)]_\alpha,[\psi^\dagger_{(+)}]_\beta(x^+,y^1,y^2,y^-)\right\}=\Lambda^+_{\alpha\beta}\delta(x^--y^-)\delta(x^1-y^1)\delta(x^2-y^2).
\end{aligned}
\tag{9.2}
$$

At $x^+=y^+$ UNEQUAL instant-time anticommutator is EQUAL light-front time anticommutator. Can also derive anticommutators involving bad fermions in the same way.

All cases discussed in Mannheim (2020).

Light-front quantization is instant-time quantization, and does not need to be independently postulated. The seemingly different structure between EQUAL instant-time and EQUAL light-front time commutators is actually a consequence of the structure of UNEQUAL instant-time time commutators and anticommutators as restricted to equal x^0 or equal x^+.

Now the transformation $x^+ = x^0 + x^3$, $x^- = x^0 - x^3$ is not a Lorentz transformation but a translation, i.e., a general coordinate transformation. But for theories that are Poincare invariant this is a symmetry. Thus:

GENERAL RULE: ANY TWO DIRECTIONS OF QUANTIZATION THAT CAN BE CONNECTED BY A GENERAL COORDINATE TRANSFORMATION DESCRIBE THE SAME THEORY.

BUT IN THE QUANTUM THEORY TRANSLATIONS ARE UNITARY TRANSFORMATIONS. THUS INSTANT-TIME AND LIGHT-FRONT THEORIES ARE UNITARILY EQUIVALENT, AND ARE THUS ONE AND THE SAME THEORY.

11 UNITARY EQUIVALENCE VIA TRANSLATION INVARIANCE

So far the discussion has only dealt with free theory commutators, and they just happen to be c-numbers. However, for interacting theories we can only discuss matrix elements. With

$$[\hat{P}_\mu, \phi] = -i\partial_\mu \phi, \quad [\hat{P}_\mu, \hat{P}_\nu] = 0 \tag{11.1}$$

to all orders in perturbation theory because of Poincare invariance, we introduce

$$U(\hat{P}_0, \hat{P}_3) = \exp(ix^3 \hat{P}_0)\exp(ix^0 \hat{P}_3). \tag{11.2}$$

It effects

$$U\phi(IT; x^0, x^1, x^2, -x^3)U^{-1} = \phi(IT; x^0 + x^3, x^1, x^2, x^0 - x^3) = \phi(LF; x^+, x^1, x^2, x^-) \tag{11.3}$$

Then with a light-front vacuum of the form $|\Omega_F\rangle = U|\Omega_I\rangle$ we obtain

$$\begin{aligned} -i\langle\Omega_I|[\phi(IT; x^0, x^1, x^2, -x^3), \phi(0)]|\Omega_I\rangle &= -i\langle\Omega_I|U^\dagger U[\phi(IT; x^0, x^1, x^2, -x^3), \phi(0)]U^\dagger U|\Omega_I\rangle \\ &= -i\langle\Omega_F|[\phi(LF; x^+, x^1, x^2, x^-), \phi(0)]|\Omega_F\rangle, \end{aligned} \tag{11.4}$$

to all orders in perturbation theory. We thus establish the unitary equivalence of matrix elements of instant-time and light-front commutators to all orders.

The same equivalence holds for the all-order Lehmann representations. For the instant-time case we have

$$\langle\Omega|[\phi(IT;x),\phi(IT;y)]|\Omega\rangle = \frac{1}{(2\pi)^3}\int_0^\infty d\sigma^2\rho(\sigma^2,IT)\int d^4q\epsilon(q_0)\delta(q^2-\sigma^2)e^{-iq\cdot(x-y)}$$
$$= \int_0^\infty d\sigma^2\rho(\sigma^2,IT)i\Delta(IT,FREE;x-y,\sigma^2), \tag{11.5}$$

where

$$\rho(q^2,IT)\theta(q_0) = (2\pi)^3\sum_n \delta^4(p_\mu^n-q_\mu)|\langle\Omega|\phi(0)|p_\mu^n\rangle|^2, \quad \hat{P}_\mu|p_\mu^n\rangle = p_\mu^n|p_\mu^n\rangle, \tag{11.6}$$

as written in instant-time momentum eigenstates.

For the light-front case we have

$$\langle\Omega|[\phi(LF;x),\phi(LF;y)]|\Omega\rangle = \frac{2}{(2\pi)^3}\int_0^\infty d\sigma^2\rho(\sigma^2,LF)\int d^4q\epsilon(q_+)\delta(q^2-\sigma^2)e^{-iq\cdot(x-y)}.$$
$$= \int_0^\infty d\sigma^2\rho(\sigma^2,LF)i\Delta(LF,FREE;x-y,\sigma^2), \tag{11.7}$$

where

$$\rho(q_\mu,LF) = \frac{(2\pi)^3}{2}\sum_n \delta^4(p_\mu^n-q_\mu)|\langle\Omega|\phi(0)|p_\mu^n\rangle|^2 = \rho(q^2,LF)\theta(q_+), \tag{11.8}$$

as written in light-front momentum eigenstates. Then with

$$U|p_0^n\rangle = |p_+^n\rangle, \quad U|p_3^n\rangle = |p_-^n\rangle, \quad U|p_1^n\rangle = |p_1^n\rangle, \quad U|p_2^n\rangle = |p_2^n\rangle \tag{11.9}$$

we obtain the all-order

$$\langle\Omega|[\phi(IT;x),\phi(IT;y)]|\Omega\rangle = \langle\Omega|[\phi(LF;x),\phi(LF;y)]|\Omega\rangle. \tag{11.10}$$

With the all-order momentum operators having real and complete eigenspectra we have the all-order

$$\hat{P}_\mu(IT) = \sum |p^n(IT)\rangle p^n_\mu(IT)\langle p^n(IT)|, \quad \hat{P}_\mu(LF) = \sum |p^n(LF)\rangle p^n_\mu(LF)\langle p^n(LF)|. \tag{11.11}$$

With eigenvalues not changing under a unitary transformation, we obtain

$$\begin{aligned}\hat{P}_0(IT) = U\hat{P}_0(IT)U^{-1} = U\sum |p^n(IT)\rangle p^n_0\langle p^n(IT)|U^\dagger \\ = \sum |p^n(LF)\rangle(p^n_+ + p^n_-)\langle p^n(LF)| = \hat{P}_+(LF) + \hat{P}_-(LF).\end{aligned} \tag{11.12}$$

Given (11.11) and (11.12), there initially appears to be a mismatch between the eigenstates of $\hat{P}_0(IT)$ and $\hat{P}_+(LF)$. However, for any timelike set of instant-time momentum eigenvalues we can Lorentz boost p_1, p_2 and p_3 to zero, to yield

$$p_1 = 0, \quad p_2 = 0, \quad p_3 = 0, \quad p_0 = m. \tag{11.13}$$

If we impose this same $p_1 = 0$, $p_2 = 0$, $p_3 = 0$ condition on the light-front momentum eigenvalues we would set $p_+ = p_-$, $p^2 = 4p_+^2 = m^2$, and thus obtain

$$p_1 = 0, \quad p_2 = 0, \quad p_+ = p_-, \quad p_0 = 2p_+ = m \tag{11.14}$$

When written in terms of contravariant vectors with $p^\mu = g^{\mu\nu}p_\nu$ this condition takes the form

$$p^0 = p^- = m. \tag{11.15}$$

Thus in **the instant-time rest frame** the eigenvalues of the contra variant $\hat{P}^0(IT)$ and $\hat{P}^-(LF)$ coincide. In this sense then instant-time and light-front Hamiltonians are equivalent. And non-relativistic in the light-front case still means $p_3 = 0$, i.e., $p_+ = p_-$, and not $p_- = p^+/2 = 0$.

Having now established the equivalence of commutators and the equivalence of Hamiltonian operators, we now proceed to establish the same equivalence for both free and interacting instant-time and light-front Green's functions.

12 INSTANT-TIME AND LIGHT-FRONT FOCK SPACE EXPANSIONS

Instant-Time Scalar Field Fock Space Expansion with $E_p^2 = p_1^2 + p_2^2 + p_3^2 + m^2$

$$\phi(x^0, x^1, x^2, x^3) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{(2E_p)^{1/2}} [a(\vec{p})e^{-iE_pt+i\vec{p}\cdot\vec{x}} + a^\dagger(\vec{p})e^{+iE_pt-i\vec{p}\cdot\vec{x}}]. \tag{12.1}$$

Contains $-\infty \leq p_3 \leq \infty$, well-behaved at $p_3 = 0$.

Light-Front Scalar Field Fock Space Expansion with $F_p^2 = (p_1)^2 + (p_2)^2 + m^2$

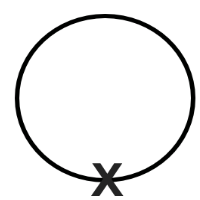
$$\phi(x^+, x^1, x^2, x^-) = \frac{2}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_0^{\infty} \frac{dp_-}{(4p_-)^{1/2}}$$
$$\times \left[e^{-i(F_p^2x^+/4p_-+p_-x^-+p_1x^1+p_2x^2)}a(p_1, p_2, p_-) + e^{i(F_p^2x^+/4p_-+p_-x^-+p_1x^1+p_2x^2)}a_p^\dagger(p_1, p_2, p_-)\right]. \tag{12.2}$$

Singular at $p_- = 0$, undefined at $x^+ = 0,\ p_- = 0$. ($p_- = p^+/2$, $p_+ = p^-/2$).

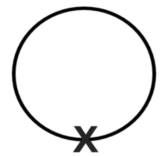
Contains $0 \leq p_- \leq \infty$ only, Light-Front Hamiltonian approach restricts to $p_- > 0$, $p_+ < \infty$.

Thus go beyond Light-Front Hamiltonian if have processes with $p_- = 0$.

This happens in vacuum sector where tadpole is $-i\langle\Omega|\phi(0)\phi(0)|\Omega\rangle$ with $x^+ = 0$.
If bring zero four-momentum into cross in vacuum tadpole then only allowed momentum in loop has $p_- = 0$. If exclude $p_- = 0$ then tadpole is zero. Potential solution to cosmological constant problem. Fails since have to deal with indeterminacy of x^+/p_- at $x^+ = 0$, $p_- = 0$.



13 EQUIVALENCE OF INSTANT-TIME AND LIGHT-FRONT PROPAGATORS AND TADPOLES



Construct tadpole as $x^\mu \to 0$ limit of propagator (not two-point function), i.e., use x^μ as a regulator.

$$D(x^\mu) = -i\langle\Omega|[\theta(\sigma)\phi(x)\phi(0) + \theta(-\sigma)\phi(0)\phi(x)]|\Omega\rangle = \frac{1}{(2\pi)^4}\int d^4p \frac{e^{-ip\cdot x}}{p^2 - m^2 + i\epsilon}, \qquad \sigma = \mathrm{x}^0 \text{ or } \sigma = \mathrm{x}^+ \tag{13.1}$$

$$D(x^\mu = 0) = -i\langle\Omega|\phi(0)\phi(0)|\Omega\rangle = \frac{1}{(2\pi)^4}\int d^4p \frac{1}{p^2 - m^2 + i\epsilon}. \tag{13.2}$$

$$\begin{aligned}
D(x^\mu, \text{instant}) &= \frac{1}{(2\pi)^4}\int dp_0 dp_1 dp_2 dp_3 \frac{e^{-i(p_0x^0 + p_1x^1 + p_2x^2 + p_3x^3)}}{(p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2 - m^2 + i\epsilon}, \\
D(x^\mu, \text{front}) &= \frac{2}{(2\pi)^4}\int dp_+ dp_1 dp_2 dp_- \frac{e^{-i(p_+x^+ + p_1x^1 + p_2x^2 + p_-x^-)}}{4p_+p_- - (p_1)^2 - (p_2)^2 - m^2 + i\epsilon}, \\
D(x^\mu = 0, \text{instant}) &= \frac{1}{(2\pi)^4}\int dp_0 dp_1 dp_2 dp_3 \frac{1}{(p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2 - m^2 + i\epsilon}, \\
D(x^\mu = 0, \text{front}) &= \frac{2}{(2\pi)^4}\int dp_+ dp_1 dp_2 dp_- \frac{1}{4p_+p_- - (p_1)^2 - (p_2)^2 - m^2 + i\epsilon}.
\end{aligned} \tag{13.3}$$

For all of these Feynman contours there are only poles, except $D(x^\mu = 0, \text{front})$, for which the circle at infinity in the complex p_+ plane is not suppressed.

14 THE NON-VACUUM INSTANT-TIME CASE

In the instant-time case the Feynman integral is readily performed since it is just pole terms and for the **forward** $D(x^0 > 0, \text{instant}) = -i\langle\Omega_I|\theta(x^0)\phi(x^0, x^1, x^2, x^3)\phi(0)|\Omega_I\rangle$ we obtain

$$\begin{aligned} D(x^0 > 0, \text{instant}) &= D(x^0 > 0, \text{instant}, \text{pole}) \\ &= -\frac{i}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d^3p}{2E_p}e^{-iE_px^0+i\vec{p}\cdot\vec{x}} = \frac{1}{8\pi}\left(\frac{m^2}{x^2}\right)^{1/2} H_1^{(2)}(m(x^2)^{1/2}). \end{aligned} \tag{14.1}$$

Insertion of the Fock space expansion for $\phi(x^0, x^1, x^2, x^3)$ yields

$$D(x^0 > 0, \text{instant},\ \text{Fock}) = -\frac{i}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d^3p}{2E_p}e^{-iE_px^0+i\vec{p}\cdot\vec{x}}. \tag{14.2}$$

We recognize (14.2) as (14.1), to thus establish the equivalence of the instant-time Feynman and Fock space prescriptions.

15 THE NON-VACUUM LIGHT-FRONT CASE

In the light-front case poles in the complex p_+ plane occur at

$$p_+ = E'_p - \frac{i\epsilon}{4p_-}, \quad E'_p = \frac{(p_1)^2 + (p_2)^2 + m^2}{4p_-}. \tag{15.1}$$

Poles with $p_- \geq 0^+$ thus all lie below the real p_+ axis and have positive E'_p, while poles with $p_- \leq 0^-$ all lie above the real p_+ axis and have negative E'_p. For $x^+ > 0$, closing the p_+ contour below the real axis (which for $x^+ > 0$ suppresses the circle at infinity contribution) then restricts to poles with $E'_p > 0$, $p_- \geq 0^+$. However, in order to evaluate the pole terms one has to deal with the fact that the pole at $p_- = 0^+$ has $E'_p = \infty$. Momentarily exclude the region around $p_- = 0$, and thus only consider poles below the real p_+ axis that have $p_- \geq \delta$. Evaluating the contour integral in the lower half of the complex p_+ plane thus gives

$$\begin{aligned} D(x^+ > 0, \text{front}, \text{pole}) &= -\frac{2i}{(2\pi)^3} \int_\delta^\infty \frac{dp_-}{4p_-} \int_{-\infty}^\infty dp_1 \int_{-\infty}^\infty dp_2 e^{-i(E'_p x^+ + p_- x^- + p_1 x^1 + p_2 x^2) - \epsilon x^+/4p_-} \\ &= -\frac{1}{4\pi^2 x^+} \int_\delta^\infty dp_- e^{-ip_- x^- + i[(x^1)^2 + (x^2)^2]p_-/x^+ - im^2 x^+/4p_- - \epsilon x^+/4p_-} \\ &= -\frac{1}{4\pi^2 x^+} \int_\delta^\infty dp_- e^{-ip_- x^2/x^+ - im^2 x^+/4p_- - \epsilon x^+/4p_-}. \end{aligned} \tag{15.2}$$

If we now set $\alpha = x^+/4p_-$, we obtain

$$D(x^+ > 0, \text{front}, \text{pole}) = -\frac{1}{16\pi^2} \int_0^{x^+/4\delta} \frac{d\alpha}{\alpha^2} e^{-ix^2/4\alpha - i\alpha m^2 - \alpha\epsilon}. \tag{15.3}$$

In (15.3) we can now take the limit $\delta \to 0$, $x^+/4\delta \to \infty$ without encountering any ambiguity **AS LONG AS x^+ IS NONZERO**, and with $x^+ > 0$ thus obtain

$$D(x^+ > 0, \text{front}, \text{pole}) = -\frac{1}{16\pi^2} \int_0^\infty \frac{d\alpha}{\alpha^2} e^{-ix^2/4\alpha - i\alpha m^2 - \alpha\epsilon} = \frac{1}{8\pi} \left(\frac{m^2}{x^2}\right)^{1/2} H_1^{(2)}(m(x^2)^{1/2}). \tag{15.4}$$

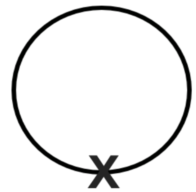
Comparing with (14.1) we see that $D(x^+ > 0, \text{instant})$ and $D(x^+ > 0, \text{front})$ are equal.

Inserting the Fock space expansion for $\phi(x^+, x^1, x^2, x^-)$ gives precisely the same result, and thus we obtain

$$\begin{aligned} &D(x^0 > 0, \text{instant}) = D(x^0 > 0, \text{instant}, \text{pole}) = D(x^0 > 0, \text{instant}, \text{Fock}) \\ &= D(x^+ > 0, \text{front}) = D(x^+ > 0, \text{front}, \text{pole}) = D(x^+ > 0, \text{front}, \text{Fock}). \end{aligned} \tag{15.5}$$

General rule: the Feynman and Fock space prescriptions will coincide whenever the only contribution to Feynman contours is poles. Thus for $x^+ > 0$ the Feynman and Light-Front Hamiltonian approaches coincide. But what about $x^+ = 0$?

16 THE INSTANT-TIME VACUUM CASE



In the instant-time case one can readily set x^μ to zero, and obtain

$$
\begin{aligned}
D(x^\mu = 0, \text{instant}) &= \frac{1}{(2\pi)^4} \int dp_0 dp_1 dp_2 dp_3 \frac{1}{(p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2 - m^2 + i\epsilon} \\
&= D(x^\mu = 0, \text{instant}, \text{pole}) = D(x^\mu = 0, \text{instant}, \text{Fock}) \\
&= -\frac{i}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d^3p}{2E_p} = -\frac{1}{16\pi^2} \int_0^\infty \frac{d\alpha}{\alpha^2} e^{-i\alpha m^2 - \alpha\epsilon}.
\end{aligned}
\tag{16.1}
$$

17 THE LIGHT-FRONT VACUUM CASE - POLE AND FOCK SPACE CONTRIBUTIONS

In the light-front case we set x^μ to zero and evaluate

$$D(x^\mu = 0, \text{front}) = \frac{2}{(2\pi)^4}\int dp_+ dp_1 dp_2 dp_- \frac{1}{4p_+p_- - (p_1)^2 - (p_2)^2 - m^2 + i\epsilon}. \tag{17.1}$$

Again we need to take care of the $p_- = 0$ region, so we again introduce the δ cutoff at small p_-. On closing below the real p_+ axis the only poles are those with $p_- > 0$, and for them we obtain a pole contribution of the form

$$D(x^\mu = 0, \text{front}, \text{pole}) = -\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_{\delta}^{\infty} \frac{dp_-}{4p_-}. \tag{17.2}$$

Then on setting $p_- = 1/\alpha$, we are able to let p_- go to zero, to obtain

$$D(x^\mu = 0, \text{front}, \text{pole}) = -\frac{i}{16\pi^3}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_{0}^{1/\delta} \frac{d\alpha}{\alpha} = -\frac{i}{16\pi^3}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_{0}^{\infty} \frac{d\alpha}{\alpha}. \tag{17.3}$$

For the Fock space prescription we set $x^\mu = 0$ in (12.2), viz.

$$\phi(0) = \frac{2}{(2\pi)^{3/2}}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_{0}^{\infty} \frac{dp_-}{(4p_-)^{1/2}}[a_p + a_p^\dagger], \tag{17.4}$$

and on inserting $\phi(0)$ into $-i\langle\Omega|\phi(0)\phi(0)]|\Omega\rangle$ obtain

$$D(x^\mu = 0, \text{front}, \text{Fock}) = -\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_{0}^{\infty} \frac{dp_-}{4p_-} = D(x^\mu = 0, \text{front}, \text{pole}). \tag{17.5}$$

Comparing with (17.2) we again see the equivalence of the pole and Fock space prescriptions.

However, something is wrong. We are evaluating the m-dependent $D(x^\mu = 0, \text{front})$ as given in (17.1), and yet we obtain an answer that does not depend on m at all. **What went wrong is that we left out the circle at infinity.**

18 THE LIGHT-FRONT VACUUM CASE - CIRCLE AT INFINITY CONTRIBUTION

To evaluate the circle at infinity contribution we introduce the regulator

$$\frac{1}{(A+i\epsilon)} = -i\int_0^\infty d\alpha e^{i\alpha(A+i\epsilon)}. \tag{18.1}$$

For $p_- > 0$ the regulator converges on the **UPPER** half circle, and there are no poles at all. We obtain

$$\begin{aligned}
&D(x^\mu = 0, p_- > 0, \text{front}, \text{upper circle}) \\
&= \frac{2i}{(2\pi)^4}\int_0^\infty dp_- \int_{-\infty}^\infty dp_1 \int_{-\infty}^\infty dp_2 \int_0^\pi iRe^{i\theta}d\theta \int_0^\infty d\alpha e^{i\alpha(4p_-Re^{i\theta}-(p_1)^2-(p_2)^2-m^2+i\epsilon)} \\
&= \frac{1}{8\pi^3}\int_0^\infty dp_- \int_0^\infty \frac{d\alpha}{\alpha}e^{-i\alpha m^2-\alpha\epsilon}\int_0^\pi iRe^{i\theta}d\theta e^{4i\alpha p_-Re^{i\theta}} \\
&= \frac{1}{8\pi^3}\int_0^\infty dp_- \int_0^\infty \frac{d\alpha}{\alpha}e^{-i\alpha m^2-\alpha\epsilon}\frac{(e^{4i\alpha p_-Re^{i\theta}} - e^{-4i\alpha p_-Re^{i\theta}})}{4i\alpha p_-}\Bigg|_0^\pi \\
&= \frac{1}{8\pi^3}\int_0^\infty dp_- \int_0^\infty \frac{d\alpha}{\alpha}e^{-i\alpha m^2-\alpha\epsilon}\frac{(e^{-4i\alpha p_-R} - e^{4i\alpha p_-R})}{4i\alpha p_-} \\
&= -\frac{1}{4\pi^3}\int_0^\infty dp_- \int_0^\infty \frac{d\alpha}{\alpha}e^{-i\alpha m^2-\alpha\epsilon}\frac{\sin(4\alpha p_-R)}{4\alpha p_-}.
\end{aligned} \tag{18.2}$$

Then, on letting R go to infinity we obtain

$$\begin{aligned}
&D(x^\mu = 0, p_- > 0, \text{front}, \text{upper circle}) = -\frac{1}{4\pi^2}\int_0^\infty dp_- \int_0^\infty \frac{d\alpha}{\alpha}e^{-i\alpha m^2-\alpha\epsilon}\delta(4\alpha p_-) \\
&= -\frac{1}{8\pi^2}\int_{-\infty}^\infty dp_- \int_0^\infty \frac{d\alpha}{\alpha}e^{-i\alpha m^2-\alpha\epsilon}\delta(4\alpha p_-) = -\frac{1}{32\pi^2}\int_0^\infty \frac{d\alpha}{\alpha^2}e^{-i\alpha m^2-\alpha\epsilon}.
\end{aligned} \tag{18.3}$$

We thus establish the centrality of $p_- = 0$ modes.

Similarly, for $p_- < 0$ close on the **LOWER** half circle, and again there are no poles. We obtain

$$D(x^\mu = 0, p_- > 0, \text{front}, \text{upper circle}) = D(x^\mu = 0, p_- < 0, \text{front}, \text{lower circle}), \tag{18.4}$$

and thus

$$D(x^\mu = 0, \mathrm{front}) = D(x^\mu = 0, p_- > 0, \mathrm{front}, \mathrm{upper\ circle}) + D(x^\mu = 0, p_- < 0, \mathrm{front}, \mathrm{lower\ circle})$$
$$= -\frac{1}{16\pi^2}\int_0^\infty \frac{d\alpha}{\alpha^2} e^{-i\alpha m^2 - \alpha\epsilon}. \tag{18.5}$$

Now not only is there now an m dependence, we obtain

$$D(x^\mu = 0, \mathrm{front}) = D(x^\mu = 0, \mathrm{instant}). \tag{18.6}$$

So again, light-front quantization is instant-time quantization. And even though there is only a circle at infinity contribution in the light front case, it is this circle at infinity that enables the light-front and instant-time vacuum graphs to be the same.

19 RECONCILING THE FOCK SPACE AND FEYNMAN CALCULATIONS

To avoid $p_-=0$ difficulties we use the regulator on the real p_+ axis, and set

$$
\begin{aligned}
&D(x^\mu,\mathrm{front},\mathrm{regulator})\\
&=-\frac{2i}{(2\pi)^4}\int_{-\infty}^{\infty}dp_+\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_{-\infty}^{\infty}dp_-e^{-i(p_+x^++p_-x^-+p_1x^1+p_2x^2)}\int_0^\infty d\alpha e^{i\alpha(4p_+p_--(p_1)^2-(p_2)^2-m^2+i\epsilon)}\\
&=-\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_0^{\infty}dp_-e^{-i(p_-x^-+p_1x^1+p_2x^2)}\int_0^\infty d\alpha e^{i\alpha(-(p_1)^2-(p_2)^2-m^2+i\epsilon)}\delta(4\alpha p_--x^+)\\
&-\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_{-\infty}^{0}dp_-e^{-i(p_-x^-+p_1x^1+p_2x^2)}\int_0^\infty d\alpha e^{i\alpha(-(p_1)^2-(p_2)^2-m^2+i\epsilon)}\delta(4\alpha p_--x^+).
\end{aligned}
\tag{19.1}
$$

On changing the signs of p_-, p_1 and p_2 in the last integral and setting F_p^2 equal to the positive $(p_1)^2+(p_2)^2+m^2$ we obtain

$$
\begin{aligned}
&D(x^\mu,\mathrm{front},\mathrm{regulator})\\
&=-\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_0^{\infty}\frac{dp_-}{4p_-}e^{-i(p_-x^-+p_1x^1+p_2x^2)}\int_0^\infty d\alpha e^{ix^+(-F_p^2+i\epsilon)/4p_-}\delta(\alpha-x^+/4p_-)\\
&-\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_0^{\infty}\frac{dp_-}{4p_-}e^{i(p_-x^-+p_1x^1+p_2x^2)}\int_0^\infty d\alpha e^{ix^+(F_p^2-i\epsilon)/4p_-}\delta(\alpha+x^+/4p_-)\\
&=-\frac{2i\theta(x^+)}{(2\pi)^3}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_0^{\infty}\frac{dp_-}{4p_-}e^{-i(F_p^2x^+/4p_-+p_-x^-+p_1x^1+p_2x^2+ix^+\epsilon/4p_-)}\\
&-\frac{2i\theta(-x^+)}{(2\pi)^3}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_0^{\infty}\frac{dp_-}{4p_-}e^{i(F_p^2x^+/4p_-+p_-x^-+p_1x^1+p_2x^2-ix^+\epsilon/4p_-)},
\end{aligned}
\tag{19.2}
$$

and note that the structure of (19.2) is such that for $x^+>0$ (forward in time) one only has positive energy propagation, while for $x^+<0$ (backward in time) one only has negative energy propagation. With the insertion into $D(x^\mu)=-i\langle\Omega|[\theta(x^+)\phi(x)\phi(0)+\theta(-x^+)\phi(0)\phi(x)]|\Omega\rangle$ of the Fock space expansion for $\phi(x^\mu)$ given in (12.2) precisely leading to (19.2), we recognize (19.2) as the $x^\mu\neq 0$ $D(x^\mu,\mathrm{front},\mathrm{Fock})$.

Now if we set $x^\mu = 0$ in (19.2) we would appear to obtain the m-independent $D(x^\mu = 0, \mathrm{front}, \mathrm{Fock})$ given in (17.5). However, we cannot take the $x^+ \to 0$ limit since the quantity $x^+/4p_-$ is undefined if p_- is zero, and $p_- = 0$ is included in the integration range. Hence, just as discussed in regard to (15.3), the limit is singular.

To obtain a limit that is not singular we note that we can set x^μ to zero in (19.1) as there the limit is well-defined, and this leads to

$$\begin{aligned}
&D(x^\mu = 0, \mathrm{front}, \mathrm{regulator})\\
&= -\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_{0}^{\infty} dp_- \int_{0}^{\infty} d\alpha e^{i\alpha(-(p_1)^2-(p_2)^2-m^2+i\epsilon)}\delta(4\alpha p_-)\\
&-\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_{-\infty}^{0} dp_- \int_{0}^{\infty} d\alpha e^{i\alpha(-(p_1)^2-(p_2)^2-m^2+i\epsilon)}\delta(4\alpha p_-)\\
&= -\frac{2i}{(2\pi)^3}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_{-\infty}^{\infty} dp_- \int_{0}^{\infty} \frac{d\alpha}{4\alpha} e^{i\alpha(-(p_1)^2-(p_2)^2-m^2+i\epsilon)}\delta(p_-),
\end{aligned} \tag{19.3}$$

and again see the **centrality of $p_- = 0$ modes**. If we do the momentum integrations we obtain the m-dependent

$$D(x^\mu = 0, \mathrm{front}, \mathrm{regulator}) = -\frac{1}{16\pi^2}\int_0^\infty \frac{d\alpha}{\alpha^2} e^{-i\alpha m^2 - \alpha\epsilon}. \tag{19.4}$$

We recognize (19.4) as being of the same form as the m-dependent $D(x^\mu = 0, \mathrm{front})$ given in (18.5). We thus have to conclude that the limit $x^\mu \to 0$ of (19.2) is not (17.5) but is (19.4) instead, and that

$$D(x^\mu = 0, \mathrm{front}) = D(x^\mu = 0, \mathrm{instant}) = -\frac{1}{16\pi^2}\int_0^\infty \frac{d\alpha}{\alpha^2} e^{-i\alpha m^2 - \alpha\epsilon}. \tag{19.5}$$

Setting $p_- = 0$ and then $x^+ = 0$ is not the same as setting $x^+ = 0$ and then $p_- = 0$.

Thus because of singularities we first have to point split, and when we do so we find that it is the m-dependent (19.4) that is the correct value for the light-front vacuum graph. And it is equal to the instant-time vacuum graph.

20 RELATIVISTIC EIKONALIZATION AND THE LIGHT FRONT

For eikonalization of a light wave one defines $A_\mu = \epsilon_\mu e^{iT}$ and takes the eikonal phase to obey

$$\partial_\mu T = \frac{dx_\mu}{dq} = k_\mu, \quad k_\mu k^\mu = 0, \tag{20.1}$$

where q is an affine parameter that measures distance along the light ray (the normal to the propagating wavefront). But if we set $T = \int^x k_\mu dx^\mu$, we would have $T = 0$. If momentarily we nonetheless do set $T = \int^x k_\mu dx^\mu$, then for $k_\mu = (k, 0, 0, k)$ we would have

$$(\partial_0 + \partial_3)T = 0, \tag{20.2}$$

which we recognize as a light-front constraint. Now in light-front coordinates we have

$$k_\mu k^\mu = 4k_+ k_- - k_1^2 - k_2^2, \quad \partial_+ = \frac{1}{2}(\partial_0 + \partial_3), \qquad \partial_- = \frac{1}{2}(\partial_0 - \partial_3) \tag{20.3}$$

Now we can be on the light cone if $k_+ = k_1 = k_2 = 0$, with k_- unconstrained. Thus we can now set

$$T = \int^x k_- dx^-, \tag{20.4}$$

a quantity that is non-zero on the light cone. Since T does not depend on x^+ it still obeys $\partial_+ T = 0$.

The eikonalized ray thus travels on a light-front trajectory and not on an instant-time one. (Mannheim arXiv:2105.08556 [gr-qc].)

21 THE MORAL OF THE STORY

When we let $p_- \to 0$ we are letting $p_+ = [(p_1)^2 + (p_2)^2 + m^2]/4p_- \to \infty$.

However x^+ is the conjugate of p_+, and thus as $p_+ \to \infty$, $x^+ \to 0$.

The $p_- \to 0$ and the $x^+ \to 0$ limits are thus intertwined.

If we stay away from $x^+ = 0$ and restrict to $x^+ > 0$ and thus $p_- > 0$ as in the Light-Front Hamiltonian approach, there is no difficulty as there are only poles and nothing is singular, with the forward scattering on-shell Light-Front Hamiltonian approach thus being validated.

However this does become a concern for tadpole graphs as they have $x^+ = 0$, since we need both $\theta(x^+)$ and $\theta(-x^+)$ time orderings in the limit, with $\langle\Omega|[\theta(x^+)\phi(x)\phi(0) + \theta(-x^+)\phi(0)\phi(x)]|\Omega\rangle \to \langle\Omega|[\theta(0^+)\phi(0)\phi(0) + \theta(0^-)\phi(0)\phi(0)]|\Omega\rangle = \langle\Omega|\phi(0)\phi(0)|\Omega\rangle$.

If we compare

$$\begin{aligned}
D(x^\mu, \text{instant}) &= \frac{1}{(2\pi)^4}\int dp_0 dp_1 dp_2 dp_3 \frac{e^{-i(p_0x^0+p_1x^1+p_2x^2+p_3x^3)}}{(p_0)^2-(p_1)^2-(p_2)^2-(p_3)^2-m^2+i\epsilon},\\
D(x^\mu, \text{front}) &= \frac{2}{(2\pi)^4}\int dp_+ dp_1 dp_2 dp_- \frac{e^{-i(p_+x^++p_1x^1+p_2x^2+p_-x^-)}}{4p_+p_- -(p_1)^2-(p_2)^2-m^2+i\epsilon},
\end{aligned} \tag{21.1}$$

$$\begin{aligned}
D(x^\mu = 0, \text{instant}) &= \frac{1}{(2\pi)^4}\int dp_0 dp_1 dp_2 dp_3 \frac{1}{(p_0)^2-(p_1)^2-(p_2)^2-(p_3)^2-m^2+i\epsilon},\\
D(x^\mu = 0, \text{front}) &= \frac{2}{(2\pi)^4}\int dp_+ dp_1 dp_2 dp_- \frac{1}{4p_+p_- -(p_1)^2-(p_2)^2-m^2+i\epsilon},
\end{aligned} \tag{21.2}$$

we can transform each instant-time graph into each corresponding light-front graph by a change of variable. Thus they must be equal. However, that does not mean that pole equals pole or that circle equals circle, only that pole plus circle equals pole plus circle, as it is only on the full closed contour that the integrals are equal.

The transformation $x^0 \to x^0 + x^3$, $x^3 \to x^0 - x^3$ is a spacetime-dependent general coordinate transformation (not a Lorentz transformation), and thus by the general coordinate invariance of the fundamental interactions it must be the case that
LIGHT-FRONT QUANTIZATION IS INSTANT-TIME QUANTIZATION, JUST ONE THEORY.

22 INFINITE MOMENTUM FRAME CONSIDERATIONS

Under a Lorentz boost with velocity u in the 3-direction the contravariant and covariant components of a general four-vector A^μ transform as

$$A^0 \to \frac{A^0 + uA^3}{(1-u^2)^{1/2}}, \quad A^3 \to \frac{A^3 + uA^0}{(1-u^2)^{1/2}}, \quad A_0 \to \frac{A_0 - uA_3}{(1-u^2)^{1/2}}, \quad A_3 \to \frac{A_3 - uA_0}{(1-u^2)^{1/2}}. \tag{22.1}$$

If we set $(1-u) = \epsilon^2/2$, then with ϵ small, to leading order we obtain

$$\begin{aligned} &A^0 \to \frac{A^0 + A^3}{\epsilon} + O(\epsilon), \quad A^3 \to \frac{A^3 + A^0}{\epsilon} + O(\epsilon), \quad A_0 \to \frac{A_0 - A_3}{\epsilon} + O(\epsilon), \quad A_3 \to \frac{A_3 - A_0}{\epsilon} + O(\epsilon), \\ &(A^0)^2 - (A^3)^2 = A^+A^- \to A^+A^-, \end{aligned} \tag{22.2}$$

where $A^\pm = A^0 \pm A^3$. This leads to

$$p^3 \to \frac{p^+}{\epsilon} = \frac{2p_-}{\epsilon}, \quad E_p \to \frac{2p_-}{\epsilon}, \quad \frac{dp^3}{E_p} \to \frac{dp_-}{p_-}, \tag{22.3}$$

where $E_p = [(p_3)^2 + (p_1)^2 + (p_2)^2 + m^]^{1/2}$.

On transforming to the infinite momentum frame we obtain

$$\begin{aligned} &D(x^\mu = 0, \text{instant}, \text{Fock}) = D(x^\mu = 0, \text{instant}, \text{pole}) = -\frac{i}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d^3p}{2E_p} \\ &\to -\frac{i}{(2\pi)^3}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_0^{\infty}\frac{dp_-}{2p_-} = D(x^\mu = 0, \text{front}, \text{Fock}) = D(x^\mu = 0, \text{front}, \text{pole}). \end{aligned} \tag{22.4}$$

$$
\begin{aligned}
D(x^\mu = 0, \text{instant}, \text{pole}) \quad &= -\frac{i}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d^3p}{2E_p} \\
&\to -\frac{i}{(2\pi)^3}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_{0}^{\infty}\frac{dp_-}{2p_-} = D(x^\mu = 0, \text{front}, \text{pole}) \qquad (22.5)
\end{aligned}
$$

and as such, the infinite momentum frame is doing what it is supposed to do, namely it is transforming an instant-time on-shell graph into a light-front on-shell graph. However, this is not the correct answer as it does not depend on m. As we showed in (19.5) the correct answer is the m-dependent

$$
D(x^\mu = 0, \text{front}) = D(x^\mu = 0, \text{instant}) = -\frac{1}{16\pi^2}\int_0^{\infty}\frac{d\alpha}{\alpha^2}e^{-i\alpha m^2 - \alpha\epsilon}. \qquad (22.6)
$$

Thus in this respect not only is the on-shell prescription failing for light-front vacuum graphs, so is the infinite momentum frame prescription.

We thus have two puzzles: How could the limit in (22.5) lose its m dependence to begin with if it is a Lorentz transformation. And second how do we recover the m dependence anyway.

For the first puzzle we note that since the mass-dependent quantity $dp_3/2E_p$ is Lorentz invariant, under a Lorentz transformation with a velocity less than the velocity of light it must transform into itself and thus must remain mass dependent. However, in the infinite momentum frame it transforms into a quantity $dp_-/2p_-$ that is mass independent. This is because velocity less than the velocity of light and velocity equal to the velocity of light are inequivalent, since an observer that is able to travel at less than the velocity of light is not able to travel at the velocity of light. Lorentz transformations at the velocity of light are different than those at less than the velocity of light, and at the velocity of light observers (viz. observers on the light cone) can lose any trace of mass.

The resolution to the second puzzle lies in the contribution of the circle at infinity to the Feynman contour. In the instant-time case the integral

$$\int \frac{dp_0 dp_3}{(p_0)^2-(p_3)^2-(p_1)^2-(p_2)^2-m^2+i\epsilon} \tag{22.7}$$

is suppressed on the circle at infinity in the complex p_0 plane (p_3 being finite), and only poles contribute. However, when one goes to the infinite momentum frame in the instant-time case dp_3 also becomes infinite ($p^3=mv/(1-v^2)^{1/2}$) and the circle contribution is no longer suppressed. Specifically, on the instant-time circle at infinity, the term that is of relevance behaves as

$$\int \frac{Rie^{i\theta}d\theta dp_3}{R^2e^{2i\theta}-(p_3)^2}, \tag{22.8}$$

and on setting $\epsilon=1/R$ in the infinite momentum frame limit, as per (22.3) the circle term behaves as the unsuppressed

$$\int \frac{Rie^{i\theta}d\theta R dp_-}{R^2e^{2i\theta}-R^2p_-^2}=\int \frac{ie^{i\theta}d\theta dp_-}{e^{2i\theta}-p_-^2}. \tag{22.9}$$

Thus in the instant-time case one cannot ignore the circle at infinity in the infinite momentum frame even though one can ignore it for observers moving with finite momentum. Consequently, the initial reduction from the instant-time Feynman diagram to the on-shell instant-time Hamiltonian prescription is not valid in the infinite momentum frame, and one has to do the full four-dimensional Feynman contour integral instead.

23 INTERACTIONS

Two c-number approaches: path integrals and Feynman diagrams. Path integrals involve integrals of classical variables in coordinate space. Feynman diagrams involve integrals of classical variables in momentum space. For both we can transform from instant-time to light-front coordinate and momentum variables using general coordinate transformations. Thus if underlying theory and its renormalization procedure are general coordinate invariant the equivalence of instant-time and light-front Green's functions is established.

However, there is a caveat. For Feynman diagrams we need to start out with fully covariant four-dimensional contour integrals if we want to establish the equivalence. We can obscure the equivalence if we do the pole integrations in the complex frequency plane first, as then we would have on-shell three-dimensional integrals. Also we would then have a zero momentum mode problem. We can avoid this by not doing the frequency integrations until after we have introduced the exponential regulators.

That the zero mode problem must be avoidable is apparent from the path integral approach as it is purely in coordinate space and involves no zero momentum modes at all.

24 MASSIVE FIELDS – SCALAR INSTANT-TIME CASE

$$\begin{aligned} i\Delta(IT;x-y) &= [\phi(x^0,x^1,x^2,x^3),\phi(y^0,y^1,y^2,y^3)] \\ &= \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_{-\infty}^{\infty} dp_3 \frac{1}{(2\pi)^3 2E_p}\left(e^{-iE_p(x^0-y^0)+i\vec{p}\cdot(\vec{x}-\vec{y})} - e^{iE_p(x^0-y^0)-i\vec{p}\cdot(\vec{x}-\vec{y})}\right). \end{aligned} \tag{24.1}$$

Here p_3 ranges from $-\infty$ to ∞ and integrand is well-behaved at $p_3 = 0$.

$$\begin{aligned} i\Delta(IT;(x-y)^2>0) &= \frac{im}{4\pi}\epsilon(x^0-y^0)\frac{J_1(m[(x-y)^2]^{1/2})}{[(x-y)^2]^{1/2}}, \\ i\Delta(IT;(x-y)^2=0) &= -\frac{i}{2\pi}\epsilon(x^0-y^0)\delta[(x-y)^2], \\ i\Delta(IT;(x-y)^2<0) &= 0. \end{aligned} \tag{24.2}$$

Discontinuous at $m = 0$, go off shell and write a contour integral in p_0 since $\epsilon(t) = \theta(t) - \theta(-t)$ and $\delta(t)$ are distributions with

$$\theta(t) = -\frac{1}{2\pi i}\oint d\omega \frac{e^{-i\omega t}}{\omega + i\epsilon} = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{\omega + i\epsilon}, \tag{24.3}$$

with $t \neq 0$ suppressing circle at infinity. As we will see, in $t = 0$ vacuum case, no suppression. Get $\theta(0) = 1/2$.

$$\begin{aligned} i\Delta(IT;x-y) = -\frac{1}{2\pi i}\frac{1}{8\pi^3}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_{-\infty}^{\infty} dp_3 \oint dp_0 \\ \times\left[\frac{\theta(x^0-y^0)e^{-ip\cdot(x-y)} - \theta(-x^0+y^0)e^{ip\cdot(x-y)}}{(p_0)^2-(p_3)^2-(p_1)^2-(p_2)^2-m^2+i\epsilon} + \frac{\theta(x^0-y^0)e^{ip\cdot(x-y)} - \theta(-x^0+y^0)e^{-ip\cdot(x-y)}}{(p_0)^2-(p_3)^2-(p_1)^2-(p_2)^2-m^2-i\epsilon}\right]. \end{aligned} \tag{24.4}$$

Introduce exponential regulator, with the $i\epsilon$ term suppressing the $\alpha = \infty$ contribution when A is real

$$\int_0^{\infty} d\alpha \exp[i\alpha(A+i\epsilon)] = -\frac{1}{iA}, \tag{24.5}$$

Obtain

$$i\Delta(IT;x-y) = -\frac{i}{4\pi^2}\epsilon(x^0-y^0)\int_0^{\infty}\frac{d\alpha}{4\alpha^2}\left[e^{-i(x-y)^2/4\alpha - i\alpha m^2 - \alpha\epsilon} + e^{i(x-y)^2/4\alpha + i\alpha m^2 - \alpha\epsilon}\right]. \tag{24.6}$$

25 MASSIVE FIELDS – SCALAR LIGHT-FRONT CASE

$$
\begin{aligned}
i\Delta(LF;x-y) &= [\phi(x^+,x^1,x^2,x^-),\phi(y^+,y^1,y^2,y^-)] \\
&= \frac{1}{4\pi^3}\int_{-\infty}^{\infty} dp_1 dp_2 \int_0^{\infty}\frac{dp_-}{4p_-}\left[e^{-i[F_p^2(x^+-y^+)/4p_-+p_-(x^--y^-)+p_1(x^1-y^1)+p_2(x^2-y^2)]} - e^{i[F_p^2(x^+-y^+)/4p_-+p_-(x^--y^-)+p_1(x^1-y^1)+p_2(x^2-y^2)]}\right].
\end{aligned}
\tag{25.1}
$$

Here p_- only ranges from 0 to ∞ and integrand is singular at $p_-=0$. So put p_- into the exponential.

$$
\begin{aligned}
i\Delta(LF;x-y) &= -\frac{1}{2\pi i}\frac{1}{4\pi^3}\int_{-\infty}^{\infty}dp_1\int_{-\infty}^{\infty}dp_2\int_0^{\infty}dp_-\oint dp_+ \\
&\times\left[\frac{\theta(x^+-y^+)e^{-ip\cdot(x-y)}-\theta(-x^++y^+)e^{ip\cdot(x-y)}}{4p_+p_- -(p_1)^2-(p_2)^2-m^2+i\epsilon}+\frac{\theta(x^+-y^+)e^{ip\cdot(x-y)}-\theta(-x^++y^+)e^{-ip\cdot(x-y)}}{4p_+p_- -(p_1)^2-(p_2)^2-m^2-i\epsilon}\right] \\
&= -\frac{i}{4\pi^2}\epsilon(x^+-y^+)\int_0^{\infty}\frac{d\alpha}{4\alpha^2}\left[e^{-i(x-y)^2/4\alpha-i\alpha m^2-\alpha\epsilon}+e^{i(x-y)^2/4\alpha+i\alpha m^2-\alpha\epsilon}\right].
\end{aligned}
\tag{25.2}
$$

$$
\begin{aligned}
i\Delta(LF;(x-y)^2>0) &= \frac{im}{4\pi}\epsilon(x^+-y^+)\frac{J_1(m[(x-y)^2]^{1/2})}{[(x-y)^2]^{1/2}} = \frac{im}{4\pi}\epsilon(x^--y^-)\frac{J_1(m[(x-y)^2]^{1/2})}{[(x-y)^2]^{1/2}}, \\
i\Delta(LF;(x-y)^2=0) &= -\frac{i}{2\pi}\epsilon(x^+-y^+)\delta[(x-y)^2] = -\frac{i}{2\pi}\epsilon(x^--y^-)\delta[(x-y)^2], \\
i\Delta(LF;(x-y)^2<0) &= 0.
\end{aligned}
\tag{25.3}
$$

$$
\begin{aligned}
i\Delta(LF;x-y) &= -\frac{i}{4\pi^2}\epsilon(x^--y^-)\int_0^{\infty}\frac{d\alpha}{4\alpha^2}\left[e^{-i(x-y)^2/4\alpha-i\alpha m^2-\alpha\epsilon}+e^{i(x-y)^2/4\alpha+i\alpha m^2-\alpha\epsilon}\right], \\
i\Delta(IT;x-y) &= -\frac{i}{4\pi^2}\epsilon(x^0-y^0)\int_0^{\infty}\frac{d\alpha}{4\alpha^2}\left[e^{-i(x-y)^2/4\alpha-i\alpha m^2-\alpha\epsilon}+e^{i(x-y)^2/4\alpha+i\alpha m^2-\alpha\epsilon}\right].
\end{aligned}
\tag{25.4}
$$

Substitute $x^0=(x^++x^-)/2$, $x^3=(x^+-x^-)/2$, $y^0=(y^++y^-)/2$, $y^3=(y^+-y^-)/2$, so that $(x-y)^2=(x^0-y^0)^2-(x^3-y^3)^2-(x^1-y^1)^2-(x^2-y^2)^2\to(x^+-y^+)(x^--y^-)-(x^1-y^1)^2-(x^2-y^2)^2$ the instant-time $i\Delta(IT;x-y)$ transforms into the light-front $i\Delta(LF;x-y)$. We have thus achieved our main objective, showing that $i\Delta(IT;x-y)$ and $i\Delta(LF;x-y)$ are related by a coordinate transformation, and are thus **COMPLETELY EQUIVALENT**.

For instant-time case need **FOUR**-component fermion

$$\left\{\psi_\alpha(x^0,x^1,x^2,x^3),\psi_\beta^\dagger(y^0,y^1,y^2,y^3)\right\}$$
$$=\left[\left(i\gamma^0\partial_0^x+i\gamma^3\partial_3^x+i\gamma^1\partial_1^x+i\gamma^2\partial_2^x+m\right)\gamma^0\right]_{\alpha\beta}i\Delta(IT;x-y). \tag{26.1}$$

For light-front case again need **FOUR**-component fermion

$$\{\psi_\alpha(x^+,x^1,x^2,x^-),\psi_\beta^\dagger(y^+,y^1,y^2,y^-)\}$$
$$=\left[\left(i\gamma^+\partial_+^x+i\gamma^-\partial_-^x+i\gamma^1\partial_1^x+i\gamma^2\partial_2^x+m\right)\gamma^0\right]_{\alpha\beta}i\Delta(LF;x-y). \tag{26.2}$$

Thus can derive unequal light-front time anticommutators from unequal instant-time anticommutators. **PROVIDED INCLUDE GOOD AND BAD FERMIONS**

But what happened to projected fermion anticommutators. We now derive them by projecting (26.2).

$$\left\{[\psi_{(+)}]_\alpha(x^+,x^1,x^2,x^-),[\psi^\dagger_{(+)}]_\beta(y^+,y^1,y^2,y^-)\right\} = 2\Lambda^+_{\alpha\beta} i\frac{\partial}{\partial x^-} i\Delta(LF;x-y), \tag{26.3}$$

$$\left\{[\psi_{(-)}]_\alpha(x^+,x^1,x^2,x^-),[\psi^\dagger_{(-)}]_\beta(y^+,y^1,y^2,y^-)\right\} = 2\Lambda^-_{\alpha\beta} i\frac{\partial}{\partial x^+} i\Delta(LF;x-y). \tag{26.4}$$

$$\left\{[\psi_{(+)}]_\alpha(x^+,x^1,x^2,x^-),[\psi^\dagger_{(+)}]_\beta(x^+,y^1,y^2,y^-)\right\} = \Lambda^+_{\alpha\beta}\delta(x^- - y^-)\delta(x^1-y^1)\delta(x^2-y^2). \tag{26.5}$$

$$\begin{aligned}&\left\{\frac{\partial}{\partial x^-}\psi^{(-)}_\alpha(x^+,x^1,x^2,x^-),\frac{\partial}{\partial y^-}[\psi^\dagger_{(-)}]_\beta(y^+,y^1,y^2,y^-)\right\}\\ &= 2i\Lambda^-_{\alpha\beta}\frac{1}{4}\left[-\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^1}-\frac{\partial}{\partial x^2}\frac{\partial}{\partial x^2}+m^2\right]\frac{\partial}{\partial x^-} i\Delta(LF;x-y).\end{aligned} \tag{26.6}$$

$$\begin{aligned}&\left\{\frac{\partial}{\partial x^-}\psi^{(-)}_\mu(x^+,x^1,x^2,x^-),\frac{\partial}{\partial y^-}[\psi^\dagger_{(-)}]_\nu(x^+,y^1,y^2,y^-)\right\}\\ &= \frac{1}{4}\Lambda^-_{\mu\nu}\left[-\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^1}-\frac{\partial}{\partial x^2}\frac{\partial}{\partial x^2}+m^2\right]\delta(x^- - y^-)\delta(x^1-y^1)\delta(x^2-y^2).\end{aligned} \tag{26.7}$$

$$\begin{aligned}&\left\{[\psi_{(+)}]_\nu(x),[\psi^\dagger_{(-)}]_\sigma(y)\right\}\\ &= \tfrac{i}{8}\epsilon(x^- - y^-)[i(\gamma^-\gamma^1\partial^x_1+\gamma^-\gamma^2\partial^x_2)-m\gamma^-]_{\nu\sigma}\delta(x^1-y^1)\delta(x^2-y^2),\end{aligned} \tag{26.8}$$