Frame dependence of relativistic charge distributions

Based on [C.L., PRL125 (2020) 232002]

Cédric Lorcé

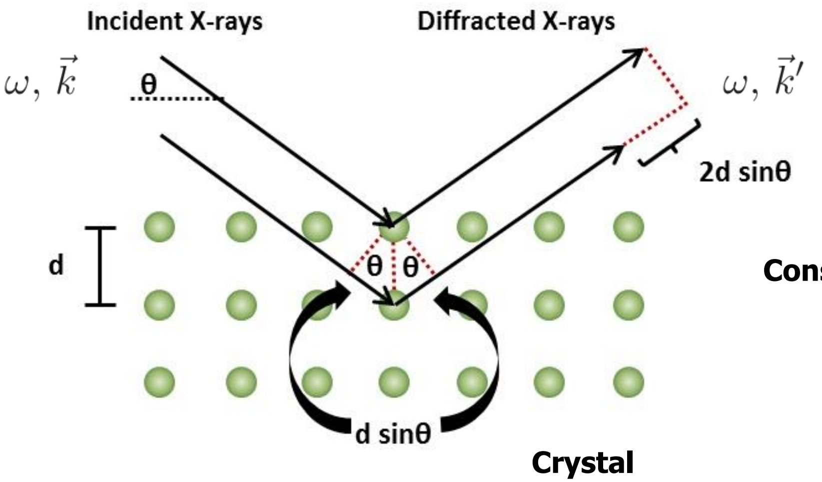
November 29, Jeju Island, South Korea

Outline

1. **Elastic scattering**
2. **Charge distributions in BF and IMF**
3. **Phase-space and EF charge distribution**

Spatial structure through elastic scattering

<u>Example:</u> X-ray diffraction

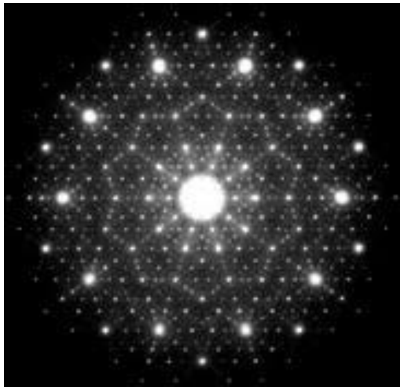


Constructive interference (Bragg's law)

$$2d\,\sin\theta = n\lambda$$

$$\boxed{\lambda \sim d}$$

Diffraction pattern



$\propto |A_{\mathrm{scatt}}|^2$

Scattered amplitude

$$A_{\mathrm{scatt}} \propto F(\vec{q}) = \int \mathrm{d}^3r\, e^{i\vec{q}\cdot\vec{r}}\,\rho(\vec{r}) \qquad \vec{q} = \vec{k} - \vec{k}'$$

Form factor

Scatterer distribution

Nuclear elastic scattering

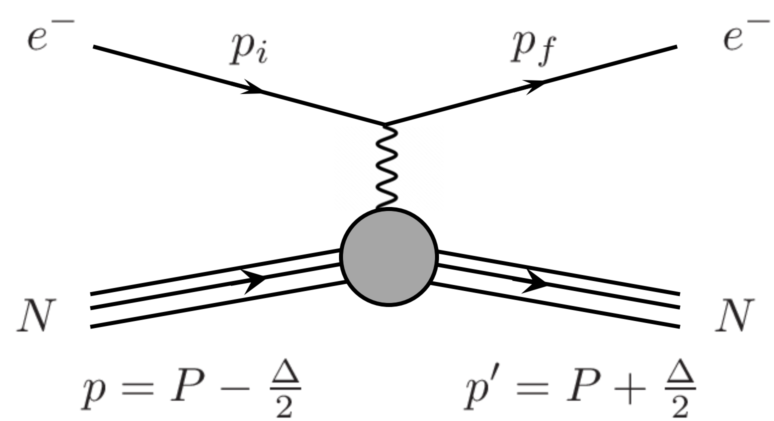
Crystals, atoms $d \approx 10^{-10}\,\mathrm{m} \quad \Rightarrow \quad \hbar\omega \approx 10^{4}\,\mathrm{eV}$ ➡ **X-rays**

Nuclei, nucleons $d \approx 10^{-15}\,\mathrm{m} \quad \Rightarrow \quad \hbar\omega \approx 10^{9}\,\mathrm{eV}$ ➡ **High-energy electron beams**

⚠ **Large recoil for light nuclei!**

Relativistic treatment

in Born approximation



Spin-0 target

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \Big/ \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\Big|_{\mathrm{pointlike}} = [F(Q^2)]^2,$$

Spin-1/2 target

$$= \left\{ \left[G_E(Q^2)\right]^2 + \frac{\tau}{\epsilon}\left[G_M(Q^2)\right]^2 \right\} \frac{1}{1+\tau}$$

Electric form factor **Magnetic form factor**

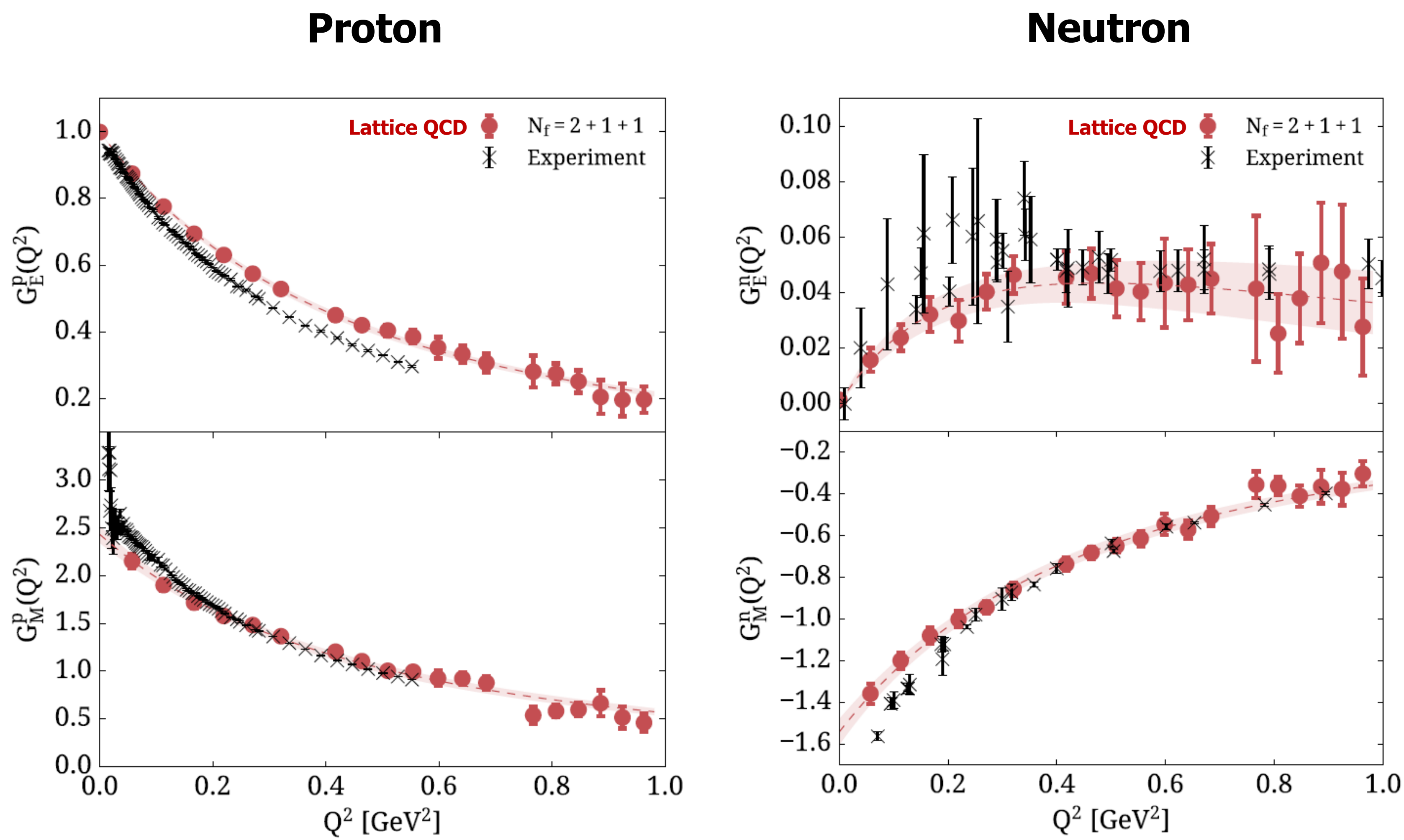
$Q^2 = -\Delta^2$

$\tau = Q^2/4M_N^2$

$\epsilon = (1 + 2(1+\tau)\tan^2\frac{\theta_e}{2})^{-1}$

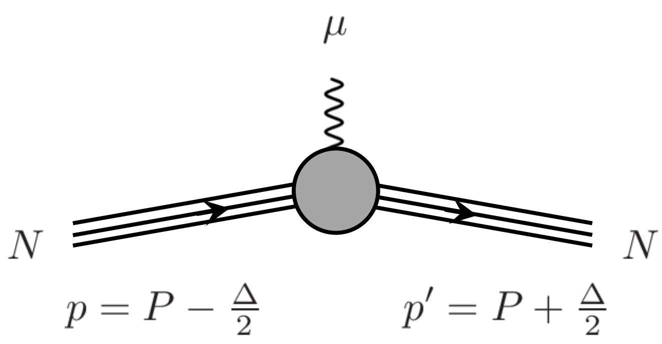
[Rosenbluth, PR79 (1950) 615]
[Hofstadter, RMP28 (1956) 214]
[Yennie, Lévy, Ravenhall, RMP29 (1957) 144]

Nucleon form factors



[Alexandrou *et al.*, PRD100 (2019) 014509]

Vector current matrix elements



$$\langle p', s' | J^\mu(0) | p, s \rangle = \overline{u}(p', s') \Gamma^\mu(P, \Delta) u(p, s)$$

$$\Gamma^\mu(P, \Delta) = \gamma^\mu F_1(Q^2) + \frac{i\sigma^{\mu\nu}\Delta_\nu}{2M_N} F_2(Q^2)$$

Dirac form factor — **Pauli form factor**

$$F_1(0) = q_N, \qquad F_2(0) = \kappa_N$$

Electric charge — **Anomalous magnetic moment**

Sachs form factors

$$G_E(Q^2) = F_1(Q^2) - \tau F_2(Q^2)$$
$$G_M(Q^2) = F_1(Q^2) + F_2(Q^2)$$

[Foldy, PR87 (1952) 688]
[Ernst, Sachs, Wali, PR119 (1960) 1105]
[Sachs, PR126 (1962) 2256]

Non-relativistic interpretation

$$\langle \vec{x}\,'|\rho(\vec{r})|\vec{x}\rangle = \int \frac{\mathrm{d}^3 P}{(2\pi)^3}\,\frac{\mathrm{d}^3\Delta}{(2\pi)^3}\,e^{i\vec{P}\cdot(\vec{x}\,'-\vec{x})}\,e^{-i\vec{\Delta}\cdot(\vec{r}-\frac{\vec{x}\,'+\vec{x}}{2})}\,\langle \vec{P}+\tfrac{\vec{\Delta}}{2}|\rho(\vec{0})|\vec{P}-\tfrac{\vec{\Delta}}{2}\rangle$$

$$\vec{P} = (\vec{p}\,' + \vec{p})/2$$
$$\vec{\Delta} = \vec{p}\,' - \vec{p}$$

Galilean symmetry $\langle \vec{P}+\tfrac{\vec{\Delta}}{2}|\rho(\vec{0})|\vec{P}-\tfrac{\vec{\Delta}}{2}\rangle = \langle \tfrac{\vec{\Delta}}{2}|\rho(\vec{0})|-\tfrac{\vec{\Delta}}{2}\rangle$

$$\langle \vec{x}\,'|\rho(\vec{r})|\vec{x}\rangle = \delta^{(3)}(\vec{x}\,'-\vec{x})\,\underbrace{\rho(\vec{r}-\vec{x})}$$

$$= \int \frac{\mathrm{d}^3\Delta}{(2\pi)^3}\,e^{-i\vec{\Delta}\cdot(\vec{r}-\vec{x})}\,\langle \tfrac{\vec{\Delta}}{2}|\rho(\vec{0})|-\tfrac{\vec{\Delta}}{2}\rangle$$

$$\langle\rho\rangle_\psi(\vec{r}) = \langle\psi|\rho(\vec{r})|\psi\rangle = \int \mathrm{d}^3x\,|\psi(\vec{x})|^2\,\rho(\vec{r}-\vec{x})$$

Probabilistic interpretation

Relativistic interpretation (Sachs approach)

$$\langle j^\mu\rangle_\psi(\vec r)=\int\frac{\mathrm{d}^3P}{(2\pi)^3}\,\frac{\mathrm{d}^3\Delta}{(2\pi)^3}\,\psi^*(\vec P+\tfrac{\vec\Delta}{2})\psi(\vec P-\tfrac{\vec\Delta}{2})\,\langle\vec P+\tfrac{\vec\Delta}{2}|j^\mu(\vec r)|\vec P-\tfrac{\vec\Delta}{2}\rangle$$

$$\vec P=(\vec p\,'+\vec p)/2$$
$$\vec\Delta=\vec p\,'-\vec p$$

Lorentz symmetry $\langle\vec P+\tfrac{\vec\Delta}{2}|\rho(\vec 0)|\vec P-\tfrac{\vec\Delta}{2}\rangle\neq\langle\tfrac{\vec\Delta}{2}|\rho(\vec 0)|-\tfrac{\vec\Delta}{2}\rangle$

$$\langle j^\mu\rangle_\psi(\vec r)\approx\int\frac{\mathrm{d}^3P}{(2\pi)^3}\,|\psi(\vec P)|^2\,\underbrace{\rho_{\vec P}(\vec r)}$$

$$=\int\frac{\mathrm{d}^3\Delta}{(2\pi)^3}\,e^{-i\vec\Delta\cdot\vec r}\langle\vec P+\tfrac{\vec\Delta}{2}|j^\mu(\vec 0)|\vec P-\tfrac{\vec\Delta}{2}\rangle$$

Probabilistic interpretation

$$|\psi(\vec P)|^2\to(2\pi)^3\delta^{(3)}(\vec P)$$

$$\approx\rho_{\vec 0}(\vec r)$$

Validity domain $1/D\ll|\vec\Delta|\ll|\delta\vec p|\ll M$

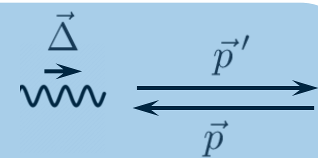
Hydrogen $M_HD_H\approx 10^5$

Nucleon $M_ND_N\approx 4$

[Sachs, PR126 (1962) 2256]
[Burkardt, PRD62 (2000) 071503]
[Belitsky, Ji, Yuan, PRD69 (2004) 074014]

Relativistic interpretation (Sachs approach)

Breit (aka brick-wall) **frame**



$$\vec{P} = \vec{0} \quad \Rightarrow \quad \Delta^0 = \frac{\vec{P}\cdot\vec{\Delta}}{P^0} = 0$$

$$\langle p', s' | J^0(0) | p, s \rangle \big|_{\rm BF} = 2M_N \delta_{s's}\, G_E(Q^2)$$
$$\langle p', s' | \vec{J}(0) | p, s \rangle \big|_{\rm BF} = i(\vec{\sigma}_{s's} \times \vec{\Delta})\, G_M(Q^2)$$

$$Q^2\big|_{\rm BF} = \vec{\Delta}^2$$

Same structure as in non-relativistic case!

3D charge distribution

$$\rho_E^{\rm BF}(\vec{r}) = \int \frac{\mathrm{d}^3\Delta}{(2\pi)^3}\, e^{-i\vec{\Delta}\cdot\vec{r}}\, \frac{G_E(Q^2)}{\sqrt{1+\tau}}$$

Relativistic recoil corrections?

$$P^0\big|_{\rm BF} = M_N\sqrt{1+\tau}$$

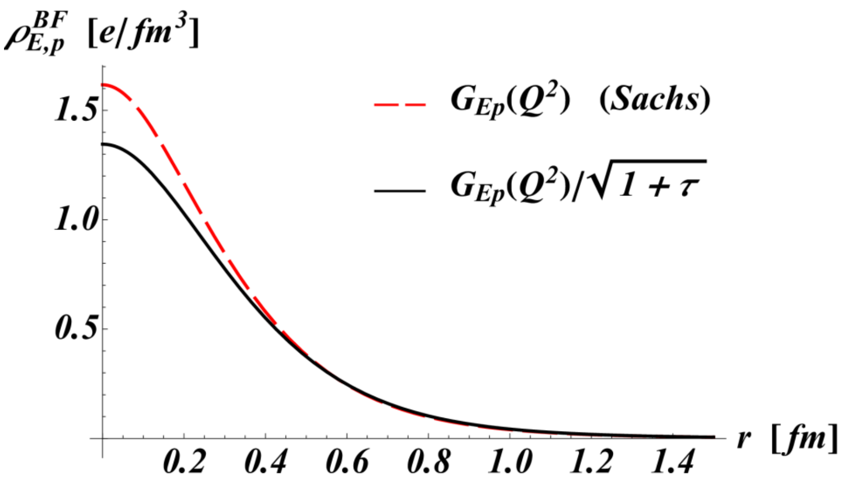
responsible for the Darwin term in the non-relativistic expansion

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\Big/\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\Big|_{\rm pointlike} = \left\{\left[G_E(Q^2)\right]^2 + \frac{\tau}{\epsilon}\left[G_M(Q^2)\right]^2\right\}\frac{1}{1+\tau}$$

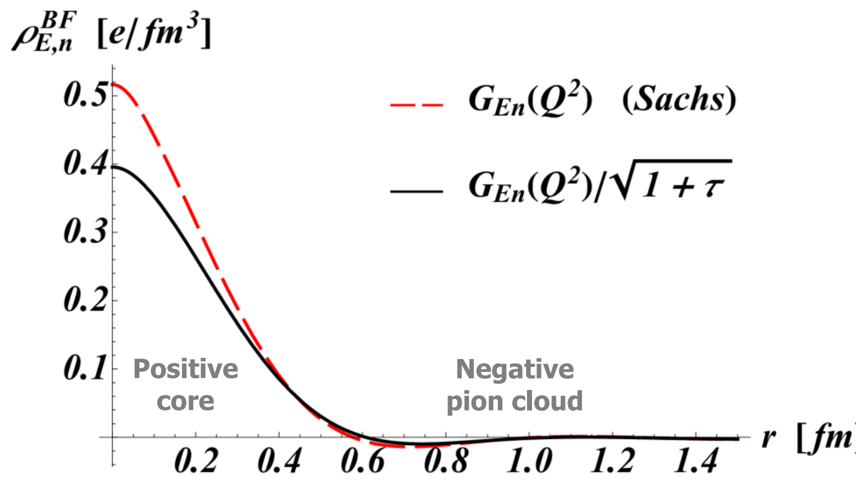
[Sachs, PR126 (1962) 2256]
[Friar, Negele, In *Adv. Nucl. Phys., Vol.8* (1975) 219]

Breit frame distributions

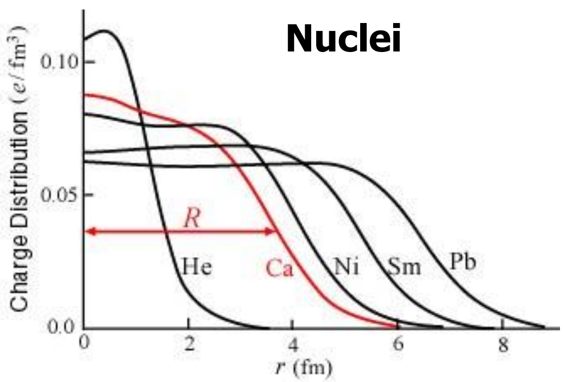
Proton



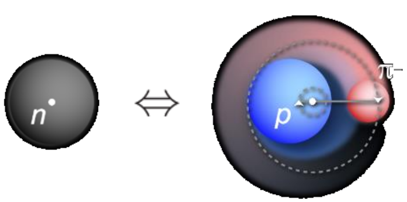
Neutron



Nuclei



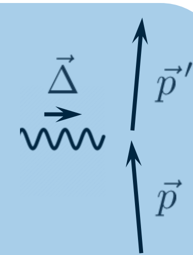
Proton-pion fluctuation



[C.L., PRL125 (2020) 232002]

Relativistic interpretation (Soper approach)

Infinite-momentum frame



$$P_z \to \infty \quad \Rightarrow \quad \Delta^0 \approx \Delta_z \ll P^0$$

$$\langle p', \lambda' | J^0(0) | p, \lambda \rangle \big|_{\text{IMF}} = 2P^0 \left[\delta_{\lambda'\lambda} \, F_1(Q^2) + \frac{i(\vec{\sigma}_{s's} \times \vec{\Delta})_z}{2M_N} \, F_2(Q^2) \right] \qquad Q^2\big|_{\text{IMF}} = \vec{\Delta}_\perp^2$$

Galilean symmetry under finite boosts!

2D charge distribution

$$\rho_E^{\text{IMF}}(\vec{b}_\perp) = \int \frac{\mathrm{d}^2\Delta_\perp}{(2\pi)^2} \, e^{-i\vec{\Delta}_\perp \cdot \vec{b}_\perp} \, F_1(Q^2) - \frac{(\vec{S} \times \vec{\nabla})_z}{M_N} \int \frac{\mathrm{d}^2\Delta_\perp}{(2\pi)^2} \, e^{-i\vec{\Delta}_\perp \cdot \vec{b}_\perp} \, F_2(Q^2)$$

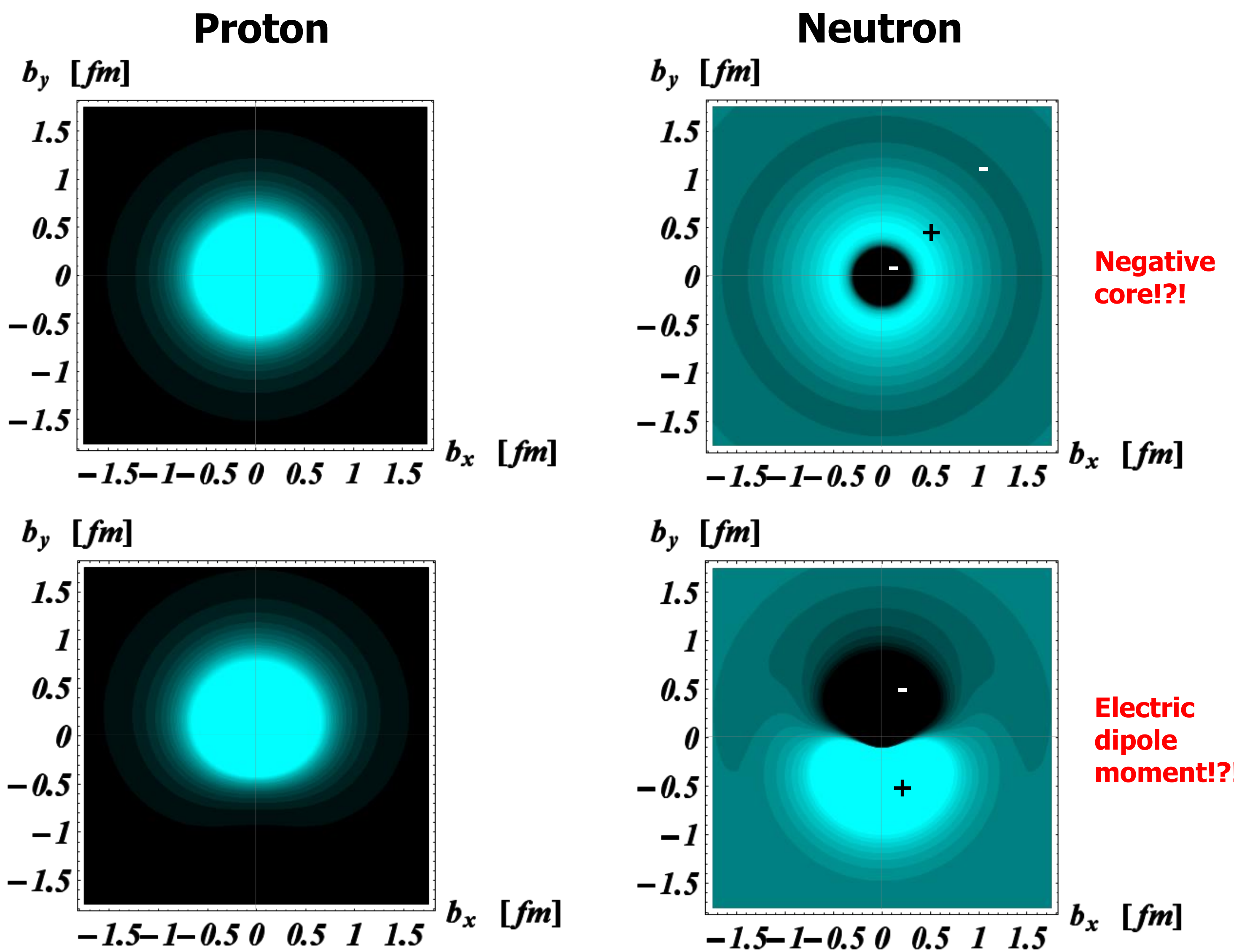
No recoil correction!

[Soper, PRD15 (1977) 1141]
[Burkardt, PRD62 (2000) 071503]

Relativistic interpretation (Soper approach)

$\vec{S} = \frac{\hbar}{2}\,\vec{e}_z$

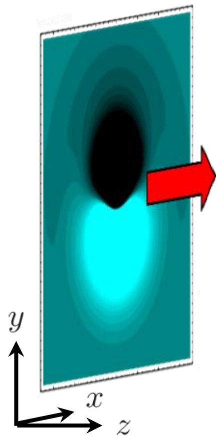
$\vec{S} = \frac{\hbar}{2}\,\vec{e}_x$



[Miller, PRL99 (2007) 11200]
[Carlson, Vanderhaeghen, PRL100 (2008) 032004]

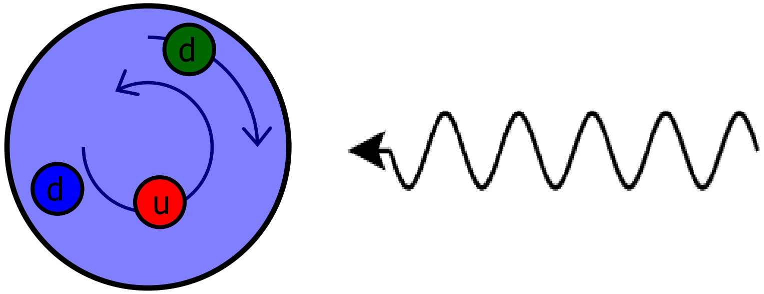
Infinite-momentum frame artifacts

$$J^0_{\rm IMF} \propto J^0_{\rm rest} + J^3_{\rm rest}$$



$J^3_{\rm rest} < 0$

$J^3_{\rm rest} > 0$



$\vec{S} \otimes$

$$\kappa_n = F_{2,n}(0) = -1.91$$

Anomalous magnetic moment

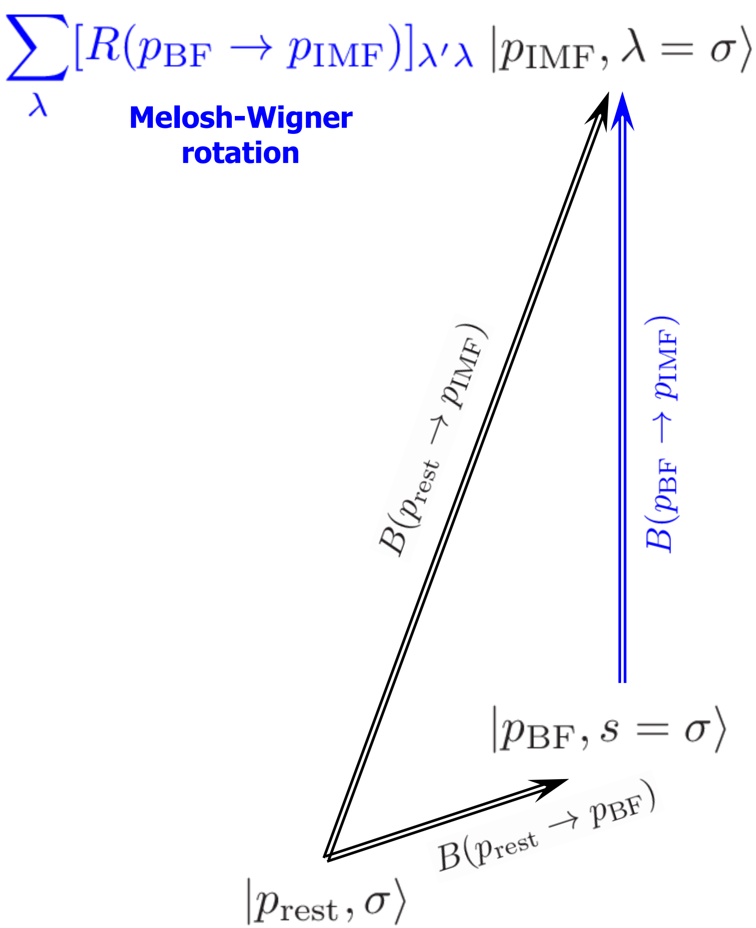
$$\vec{E}' = \gamma(\vec{E} + \vec{v} \times \vec{B}) \quad \Rightarrow \quad \vec{d}' = \gamma \vec{v} \times \vec{\mu}$$

Induced electric dipole moment

[Burkardt, IJMPA18 (2003) 173]
[Carlson, Vanderhaeghen, PRL100 (2008) 032004]

Infinite-momentum frame artifacts

Relativistic boosts do not commute! $[K^i, K^j] = -i\epsilon^{ijk} J^k$



	Spin independent	*Spin dependent*
BF	G_E	G_M
IMF	$F_1 = \frac{G_E + \tau G_M}{1+\tau}$	$F_2 = \frac{G_M - G_E}{1+\tau}$

Which set is the « physical » one?

[Melosh, PRD9 (1974) 1095]
[Chung *et al.*, PRC37 (1988) 2000]
[Rinehimer, Miller, PRC80 (2009) 015201]

NEW: phase-space perspective

Phase-space (aka Wigner) **distribution**

$$\rho_\psi(\vec{R},\vec{P}) \equiv \int \mathrm{d}^3\ell\, e^{-i\vec{P}\cdot\vec{\ell}}\,\psi^*(\vec{R}-\tfrac{\vec{\ell}}{2})\psi(\vec{R}+\tfrac{\vec{\ell}}{2})$$
$$= \int \frac{\mathrm{d}^3\Delta}{(2\pi)^3}\, e^{-i\vec{\Delta}\cdot\vec{R}}\,\tilde{\psi}^*(\vec{P}+\tfrac{\vec{\Delta}}{2})\tilde{\psi}(\vec{P}-\tfrac{\vec{\Delta}}{2})$$

Wave packet

$$\int \mathrm{d}^3R\,\rho_\psi(\vec{R},\vec{P}) = |\tilde{\psi}(\vec{P})|^2$$
$$\int \frac{\mathrm{d}^3P}{(2\pi)^3}\,\rho_\psi(\vec{R},\vec{P}) = |\psi(\vec{R})|^2$$

Expectation value

$$\langle O\rangle_\psi = \int \frac{\mathrm{d}^3P\,\mathrm{d}^3R}{(2\pi)^3}\,\rho_\psi(\vec{R},\vec{P})\,\langle O\rangle_{\vec{R},\vec{P}}$$

$$\langle O\rangle_{\vec{R},\vec{P}} \equiv \int \frac{\mathrm{d}^3\Delta}{(2\pi)^3}\, e^{i\vec{\Delta}\cdot\vec{R}}\,\langle\vec{P}+\tfrac{\vec{\Delta}}{2}|O|\vec{P}-\tfrac{\vec{\Delta}}{2}\rangle$$

System « localized » in phase space

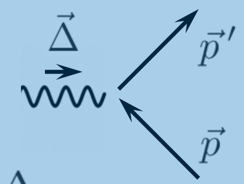
***Quasi*-probabilistic interpretation**

$$\langle\vec{p}\,'|\vec{p}\rangle = (2\pi)^3\,\delta^{(3)}(\vec{p}\,'-\vec{p})$$

[C.L., EPJC78 (2018) 9, 785]
[C.L., PRL125 (2020) 232002]

NEW: elastic frame interpretation

Elastic frame



$$\vec{P} = P_z \vec{e}_z \quad \Rightarrow \quad \Delta^0 = \frac{P_z \Delta_z}{P_0}$$

$$\Delta^0 = 0 \quad \Rightarrow \quad \Delta_z = 0 \quad \Leftrightarrow \quad \int \mathrm{d}z$$

2D charge distribution

$$\rho_E^{\mathrm{EF}}(\vec{b}_\perp; P_z) \equiv \int \mathrm{d}z \, \langle J^0(r) \rangle_{\vec{R}, P_z \vec{e}_z}$$

$$= \int \frac{\mathrm{d}^2 \Delta_\perp}{(2\pi)^2} \, e^{-i\vec{\Delta}_\perp \cdot \vec{b}_\perp} \, \frac{\langle p', s' | J^0(0) | p, s \rangle}{2P^0} \bigg|_{\mathrm{EF}}$$

Interpolates between BF and IMF

$$\vec{b}_\perp = \vec{r}_\perp - \vec{R}_\perp$$

$$\langle p', s' | p, s \rangle = \delta_{s's} \, 2P^0 (2\pi)^3 \delta^{(3)}(\vec{p}\,' - \vec{p})$$

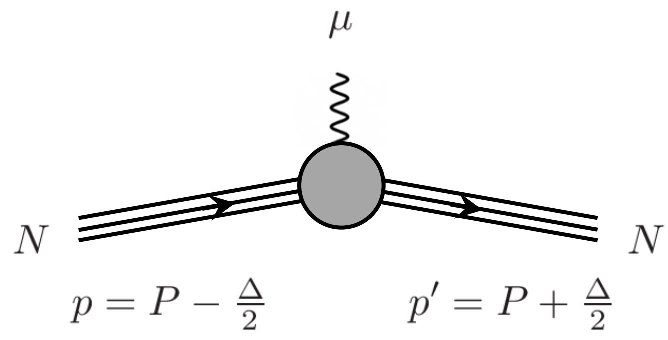
$$\rho_E^{\mathrm{EF}}(\vec{b}_\perp; 0) = \int \mathrm{d}z \, \rho_E^{\mathrm{BF}}(\vec{r})$$

$$\rho_E^{\mathrm{EF}}(\vec{b}_\perp; \infty) = \rho_E^{\mathrm{IMF}}(\vec{b}_\perp)$$

[C.L., Mantovani, Pasquini, PLB776 (2018) 38]
[C.L., PRL125 (2020) 232002]

NEW: elastic frame interpretation

$$\langle p', s'|J^\mu(0)|p, s\rangle = \overline{u}(p', s')\Gamma^\mu(P, \Delta)u(p, s)$$



$$\Gamma^\mu(P, \Delta) = \gamma^\mu F_1(Q^2) + \frac{i\sigma^{\mu\nu}\Delta_\nu}{2M_N} F_2(Q^2)$$

$$= \frac{MP^\mu}{P^2} G_E(Q^2) + \frac{i\epsilon^{\mu\alpha\beta\lambda}\Delta_\alpha P_\beta \gamma_\lambda \gamma_5}{2P^2} G_M(Q^2)$$

Reminiscent of $\vec{J} = \rho\vec{v} + \vec{\nabla} \times \vec{M}$

$$\rho_E^{\mathrm{EF}}(b; P_z) = \int_0^\infty \frac{\mathrm{d}Q}{2\pi} Q\, J_0(Qb) \left[\tilde{\rho}_E^{\mathrm{conv}}(Q; P_z) + \tilde{\rho}_E^{\mathrm{magn}}(Q; P_z)\right]$$

$$\tilde{\rho}_E^{\mathrm{conv}}(Q; P_z) = \frac{P^0 + M(1+\tau)}{(P^0 + M)(1+\tau)} G_E(Q^2)$$

$$\tilde{\rho}_E^{\mathrm{magn}}(Q; P_z) = \frac{\tau P_z^2}{P^0(P^0 + M)(1+\tau)} G_M(Q^2)$$

$$P^0 = \sqrt{M^2(1+\tau) + P_z^2}$$

BF

$$\tilde{\rho}_E^{\mathrm{conv}}(Q; 0) = \frac{G_E(Q^2)}{\sqrt{1+\tau}}$$

$$\tilde{\rho}_E^{\mathrm{magn}}(Q; 0) = 0$$

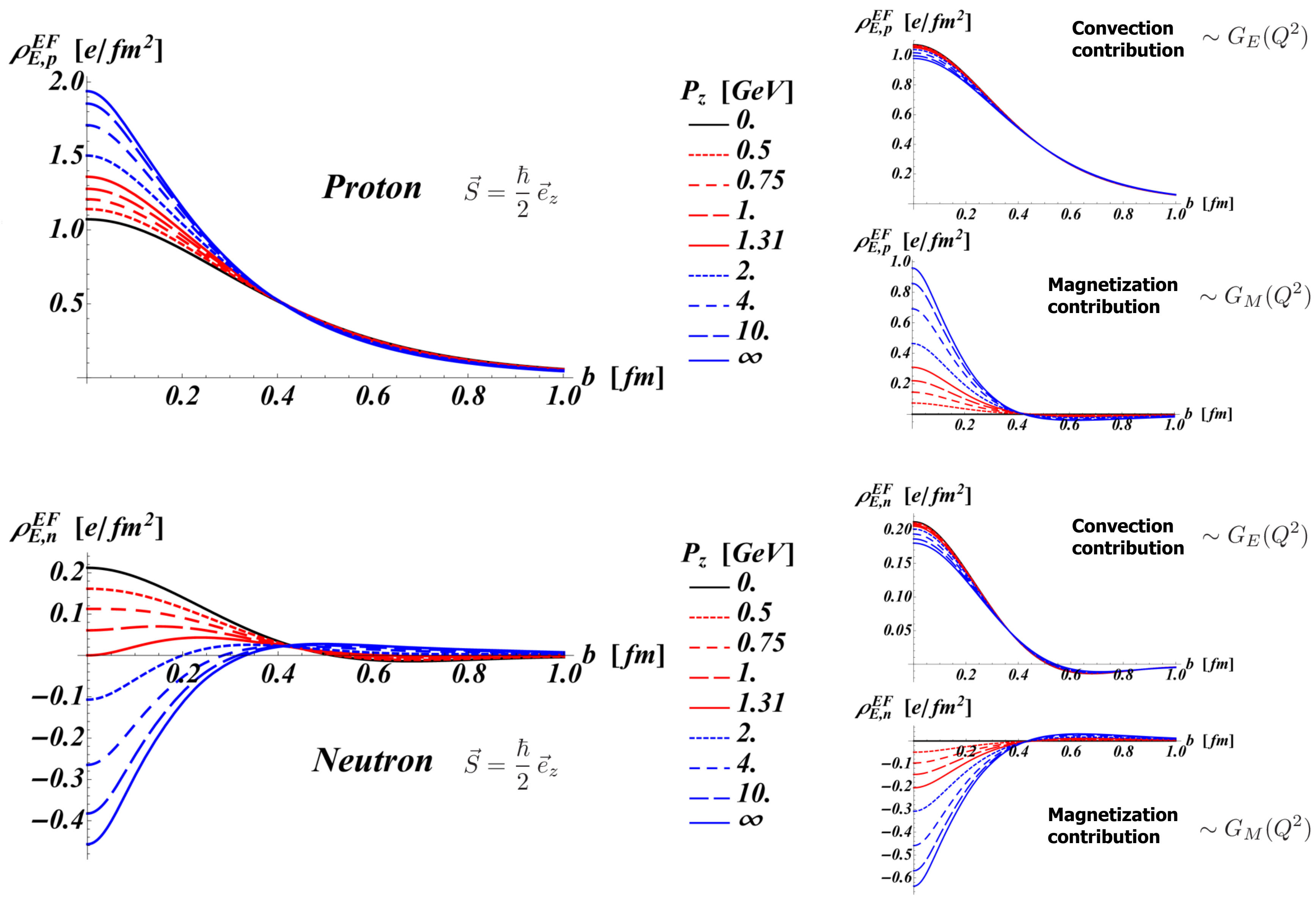
IMF

$$\tilde{\rho}_E^{\mathrm{conv}}(Q; \infty) = \frac{G_E(Q^2)}{1+\tau}$$

$$\tilde{\rho}_E^{\mathrm{magn}}(Q; \infty) = \frac{\tau G_M(Q^2)}{1+\tau}$$

[C.L., PRL125 (2020) 232002]

NEW: elastic frame interpretation



[C.L., PRL125 (2020) 232002]

Summary

- **Relativistic charge distributions are frame-dependent**
- **BF is the rest frame of a « localized » system**
- **EF interpolates between BF and IMF**
- **Relativistic boosts mix rest-frame charge distribution with magnetization**