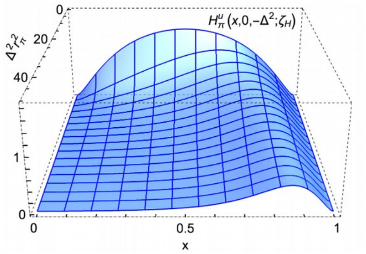
Kaon and pion properties from generalized parton distributions

Khépani Raya Montaño



Revealing pion and kaon structure via generalised parton distributions

Khepani Raya[1], Zhu-Fang Cui[2], Lei Chang[3], Jose-Manuel Morgado[4], Craig Roberts[5] and Jose Rodriguez-Quintero[4]

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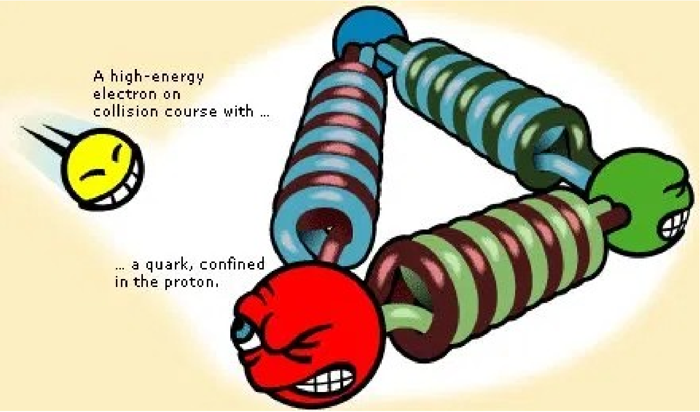
QCD: Basic Facts

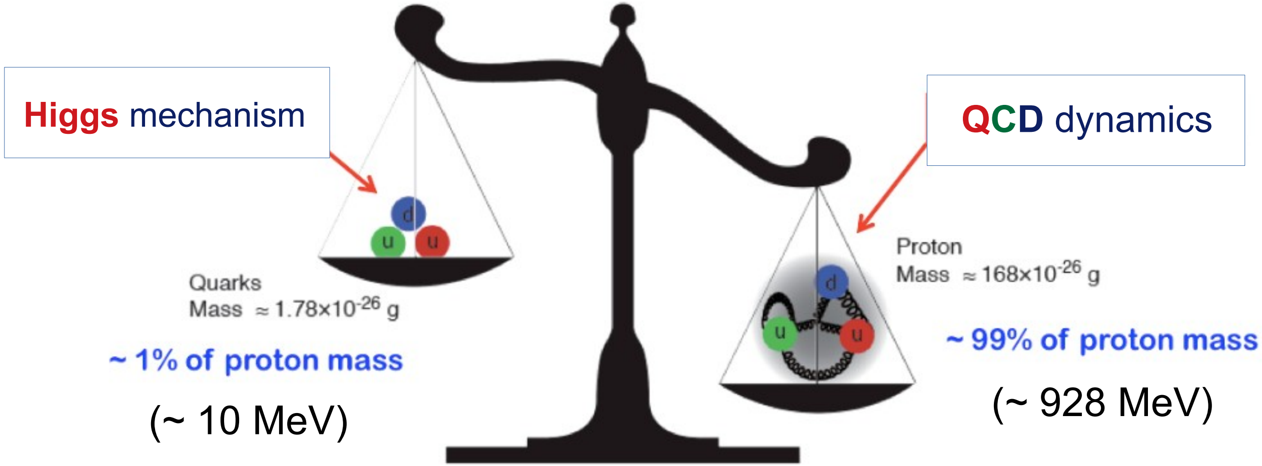
- **QCD** is characterized by two **emergent** phenomena: confinement and dynamical generation of mass (DGM).

$$\mathcal{L}_{\rm QCD} = \sum_{j=u,d,s,\ldots} \bar{q}_j[\gamma_\mu D_\mu + m_j]q_j + \frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu},$$
$$D_\mu = \partial_\mu + ig\frac{1}{2}\lambda^a A^a_\mu\,,$$
$$G^a_{\mu\nu} = \partial_\mu A^a_\nu + \partial_\nu A^a_\mu - \underline{g f^{abc}A^b_\mu A^c_\nu},$$

- Quarks and gluons not *isolated* in nature.
  - Formation of colorless bound states: "**Hadrons**"
  - **1-fm scale** size of hadrons?

- Emergence of hadron masses (EHM) from QCD **dynamics**





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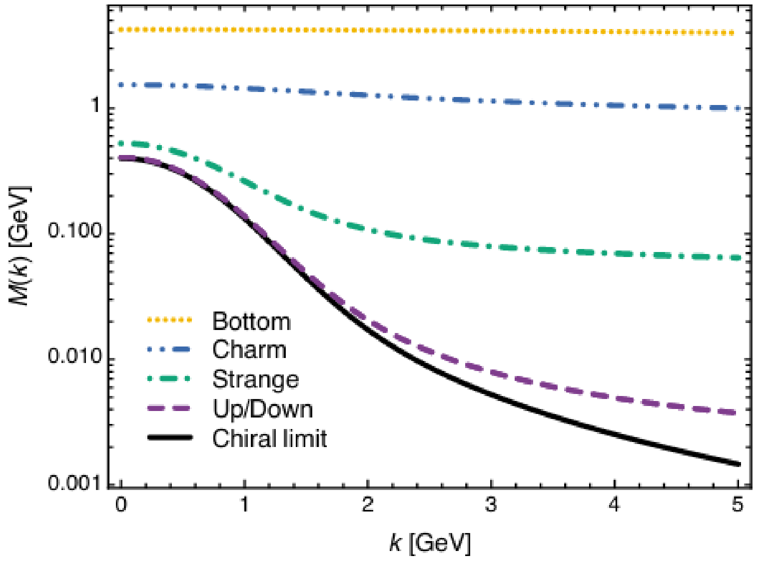
Can we trace them down to fundamental d.o.f?

- Emergence of hadron masses (EHM) from QCD **dynamics**

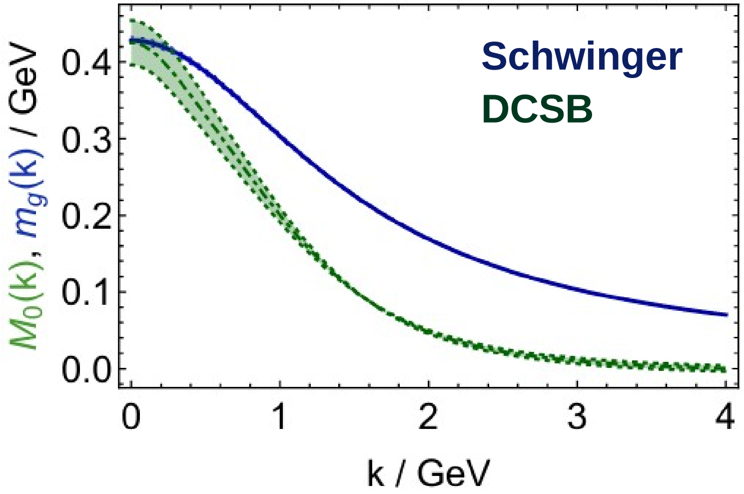
Dynamical masses

(Dynamical Chiral Symmetry Breaking)



"Higgs" masses

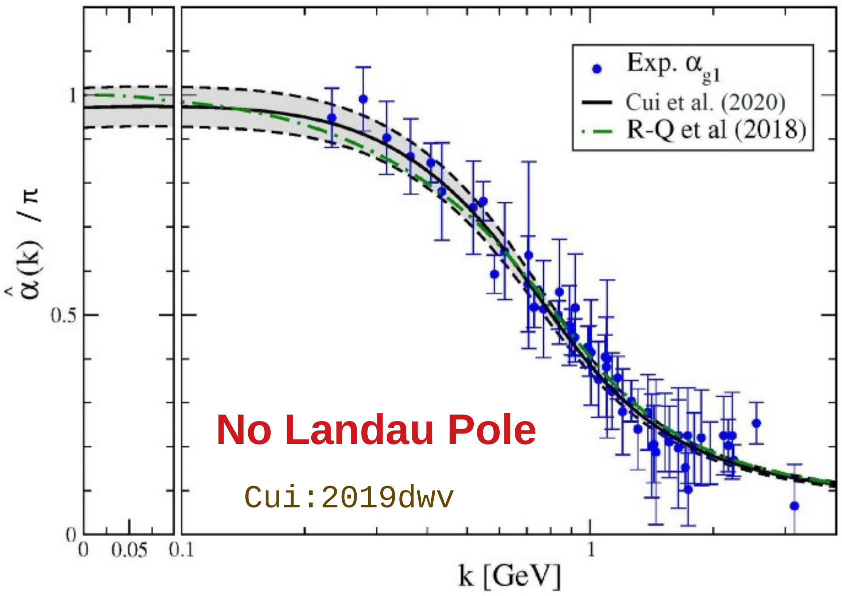
$$S_f^{-1}(p) = Z_f^{-1}(p^2)(i\gamma\cdot p + \mathbf{M_f(p^2)})$$

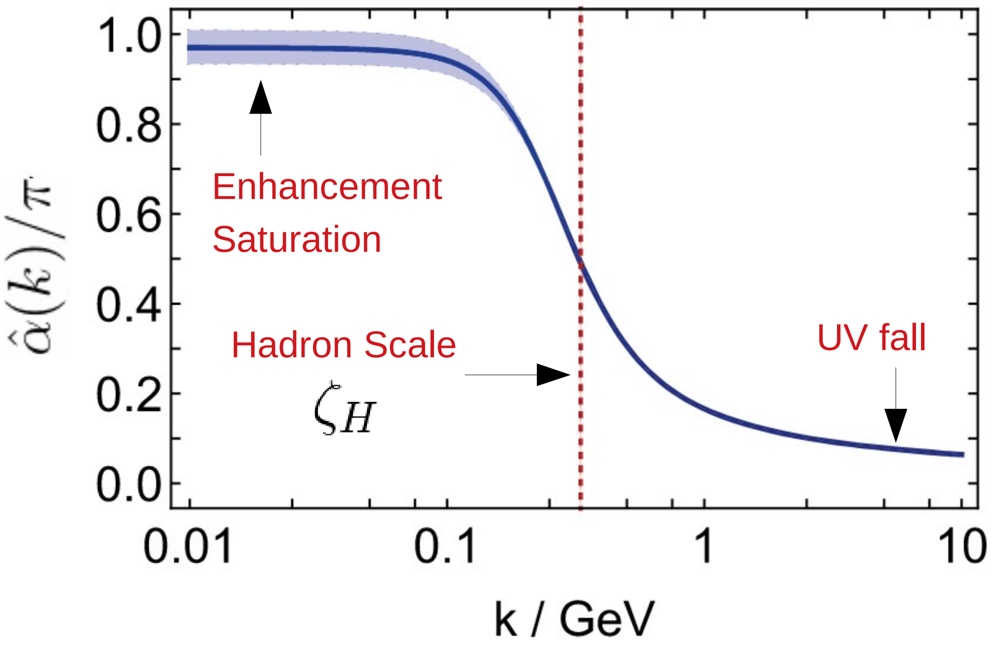


Gluon and quark *running masses*

QCD: Basic Facts

➢ **Confinement** and the **EHM** are tightly connected with **QCD's running coupling**.





Modern picture of **QCD** coupling. 'Effective Charge'

Combined continuum + QCD lattice analysis

ζ_H: Fully **dressed valence** quarks express all hadron's properties

QCD: Basic Facts

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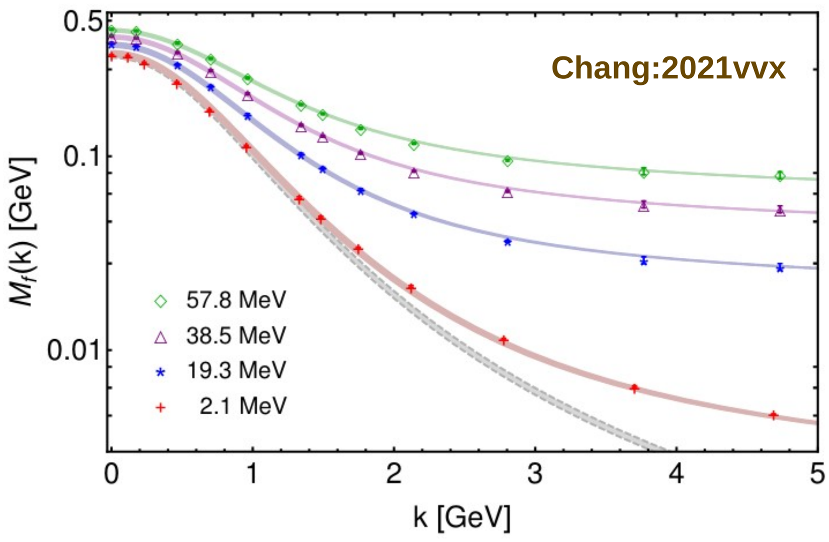
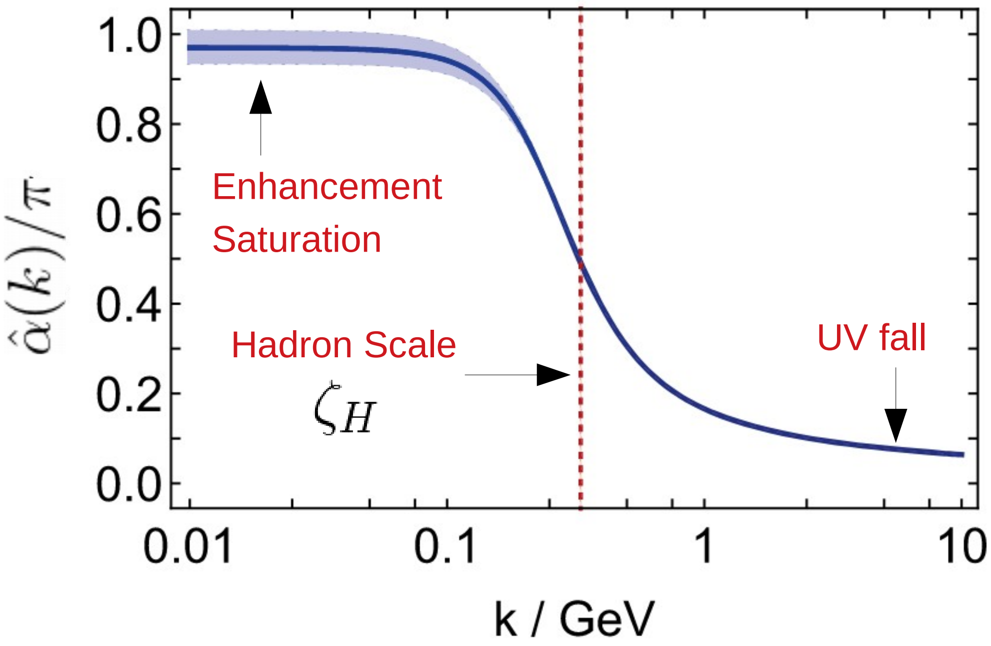




The **Effective Charge** connects **Lattice QCD** and **continuum** mass functions.

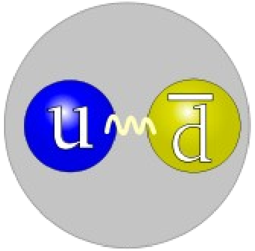
Same **charge** we shall use for **DGLAP** evolution.

… and defines ζ_H

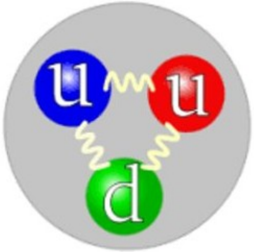
Why pions and Kaons?

- **Pions** and **kaons** emerge as (pseudo)-**Goldstone** bosons of **DCSB**.

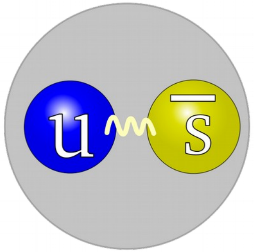
(besides being ‘simple’ bound states)



$m_\pi \approx 0.140$ GeV



$m_p \approx 0.940$ GeV



$m_K \approx 0.490$ GeV

‘Higgs’ masses

$m_{u/d} \approx 0.004$ GeV

$m_s \approx 0.095$ GeV

- Their study is crucial to understand the **EHM** and the ***hadron structure***:
  - Dominated by **QCD** dynamics

    Simultaneously explains the mass of the proton and the *masslessness* of the pion
  - Interplay between **Higgs** and **strong** mass generating mechanisms.

The light-front wave function approach

“One ring to rule them all”

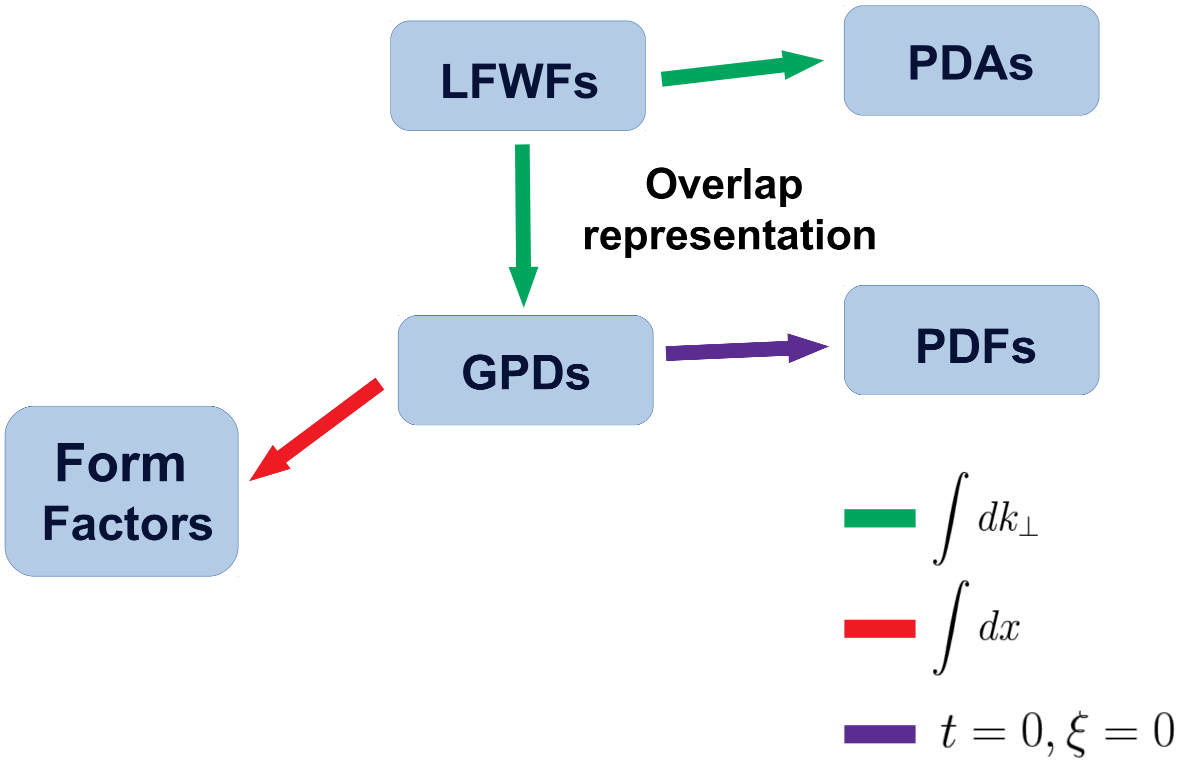
$$\psi_{\mathrm{M}}^{q}\left(x,k_{\perp}^{2}\right) = \mathrm{tr}\int_{dk_{\parallel}} \delta_{n}^{x}(k_{\mathrm{M}})\gamma_{5}\gamma\cdot n\,\chi_{\mathrm{M}}(k_{-},P)$$

Bethe-Salpeter wave function

- Yields a **variety** of **distributions**.

Light-front wave functions

- **Goal:** get a **broad picture** of the pion and Kaon structure.



The idea:
Compute ***everything*** from the **LFWF.**

LFWFs

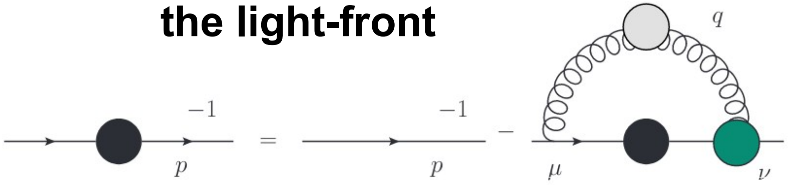
$$\psi_{\rm M}^{q}\left(x,k_{\perp}^{2}\right)=\mathrm{tr}\int_{dk_{\parallel}}\delta_{n}^{x}(k_{\rm M})\gamma_{5}\gamma\cdot n\,\chi_{\rm M}(k_{-},P)$$

- **Goal:** get a **broad picture** of the pion and Kaon structure.

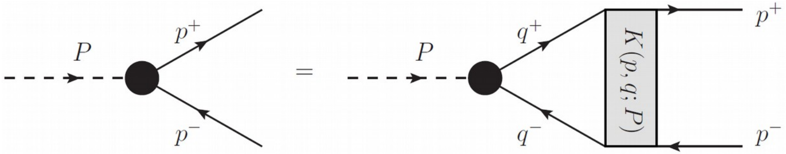
BSWFs → **LFWFs**

Project onto the light-front

Connection with Continuum Schwinger Methods (**CSM**)



Quark DSE



Meson BSE

Truncation required!

The idea:

Compute ***everything*** from the **LFWF.**

The inputs:

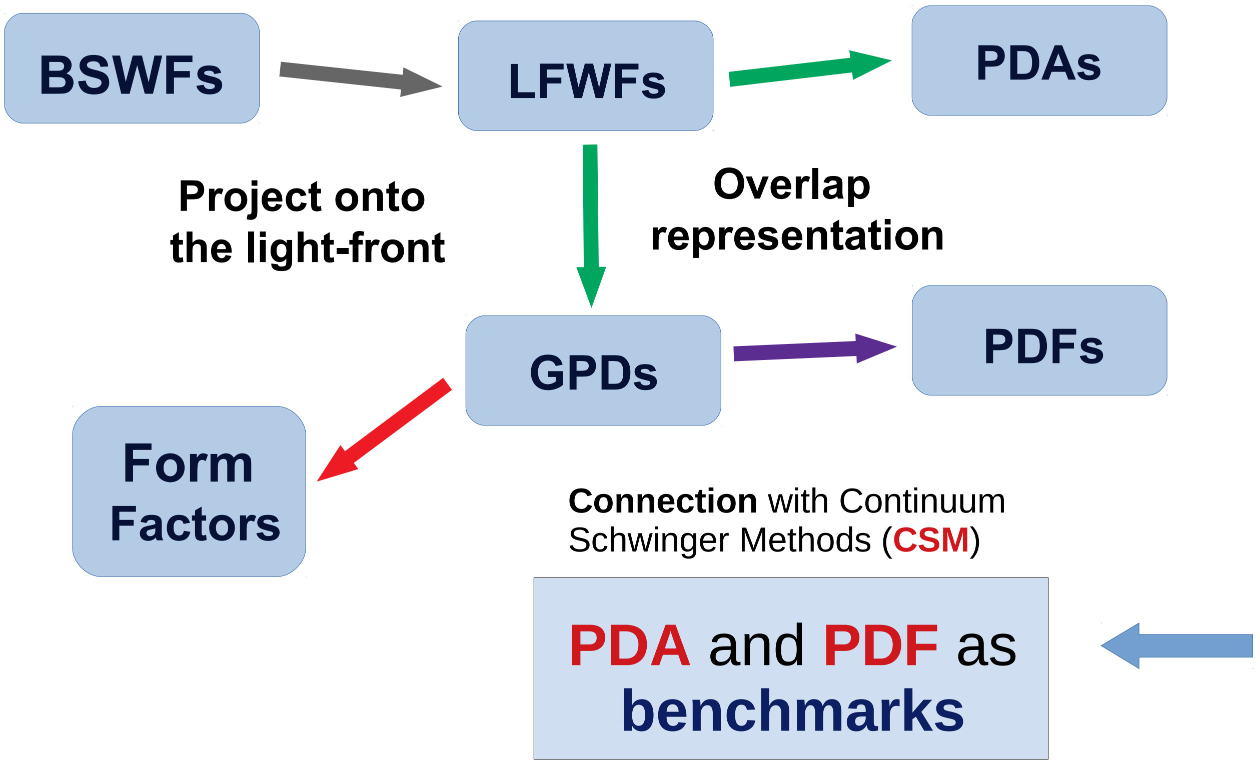
Solutions from quark **DSE** and meson **BSE**.

- Numerically challenging, but doable
- Already on the market: PDAs, PDFs, Form factors...

K. Raya *et al.*,
arXiv: 1911.12941 [nucl-th]

LFWFs

- **Goal:** get a **broad picture** of the pion and Kaon structure.



The idea:
Compute ***everything*** from the **LFWF.**

The inputs:
Solutions from quark **DSE** and meson **BSE**.

The alternative inputs:
Construct **BSWF** from realistic DSE ***predictions***.

LFWF: PTIR approach

- A perturbation theory integral representation for the **BSWF**:

$$n_K \chi_K(k_-^K, P_K) = \mathcal{M}(k,P) \int_{-1}^{1} dw\, \rho_K(w) \mathcal{D}(k,P)$$

(Kaon as example)

1 2 3

1: Matrix structure (leading BSA):

$$\mathcal{M}(k;P_K) = -\gamma_5[\gamma \cdot P_K M_u + \gamma \cdot k(M_u - M_s) + \sigma_{\mu\nu} k_\mu P_{K\nu}] \,,$$

Equivalent to considering the **leading** Bethe-Salpeter amplitude:

$$\Gamma_{\rm M}(q;P) = i\gamma_5 E_{\rm M}(q;P)$$

(from a total of **4**)

(others can be **incorporated** systematically)

S-S Xu et al., PRD 97 (2018) no.9, 094014.

LFWF: PTIR approach

- A perturbation theory integral representation for the **BSWF**:

$$n_K\chi_K(k_-^K, P_K) = \underbrace{\mathcal{M}(k,P)}_{1}\int_{-1}^{1} dw\, \underbrace{\rho_K(w)}_{2}\underbrace{\mathcal{D}(k,P)}_{3}$$

(Kaon as example)

1: Matrix structure (leading BSA):

$$\mathcal{M}(k;P_K) = -\gamma_5[\gamma\cdot P_K M_u + \gamma\cdot k(M_u - M_s) + \sigma_{\mu\nu}k_\mu P_{K\nu}]\,,$$

2: Spectral weight: Tightly connected with the meson properties.

3: Denominators: $\mathcal{D}(k;P_K) = \Delta(k^2, M_u^2)\Delta((k-P_K)^2, M_s^2)\hat{\Delta}(k_{\omega-1}^2, \Lambda_K^2)\,,$

where: $\Delta(s,t) = [s+t]^{-1},\ \hat{\Delta}(s,t) = t\Delta(s,t)\,.$

S-S Xu et al., PRD 97 (2018) no.9, 094014.

LFWF: PTIR approach

- Recall the expression for the **LFWF**:

$$\psi_{\rm M}^{q}\left(x,k_\perp^2\right) = \mathrm{tr}\int_{dk_\parallel} \delta_n^x(k_{\rm M})\gamma_5\gamma\cdot n\,\chi_{\rm M}(k_-,P)$$

$$<x>_{\rm M}^{q} := \int_0^1 dx\; x^m \psi_{\rm M}^{q}(x,k_\perp^2)$$

- Algebraic manipulations yield:

+ Uniqueness of Mellin moments

$$\Rightarrow \psi_{\rm M}^{q}(x,k_\perp) \sim \int dw\; \rho_{\rm M}(w)\cdots$$

- ✓ Compactness of this result is a merit of the AM.

- Thus, $\rho_{\rm M}(w)$ determines the profiles of, *e.g.* **PDA** and **PDF**: (it also works the **other way around**)

$$f_{\rm M}\phi_{\rm M}^{q}(x;\zeta_H) = \int \frac{d^2k_\perp}{16\pi^3}\psi_{\rm M}^{q}(x,k_\perp;\zeta_H)$$

$$q_{\rm M}(x;\zeta_H) = \int \frac{d^2k_\perp}{16\pi^3}|\psi_{\rm M}^{q}(x,k_\perp;\zeta_H)|^2$$

LFWF: Factorized case

- In the **chiral limit**, the **PTIR** reduces to:

$$\psi_{\rm M}^{q}(x,k_\perp^2;\zeta_H)\sim\tilde{f}(k_\perp)\phi_{\rm M}^{q}(x;\zeta_H)\sim f(k_\perp)[q_{\rm M}(x;\zeta_H)]^{1/2}$$

"Factorized model"

$$[\phi_{\rm M}^{q}(x;\zeta_H)]^2\sim q_{\rm M}(x;\zeta_H)$$

- Sensible assumption as long as:

$m_{\rm M}^2\approx 0$ (meson mass)

$M_{\bar h}^2-M_q^2\approx 0$ (*h*-antiquark, *q*-quark masses)

ζ_H

→ Produces **identical** results as PTIR model for pion

- Therefore:

$$\psi_{\rm M}^{q}(x,k_\perp^2;\zeta_H)=[q^{\rm M}(x;\zeta_H)]^{1/2}\left[4\sqrt{3}\pi\frac{M_q^3}{(k_\perp^2+M_q^2)^2}\right]$$

Single parameter!

$M_q\sim r_{\rm M}^{-1}$

(charge radius)

No need to determine the spectral weight !

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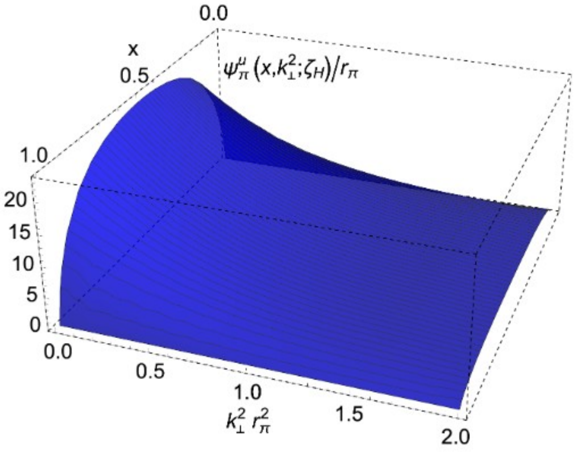
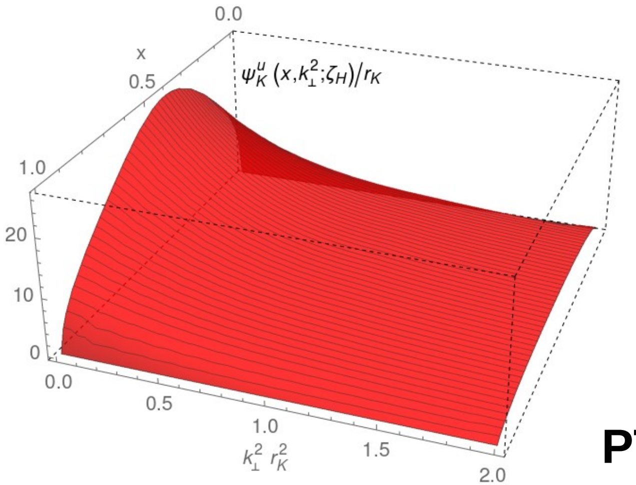
Single parameter!

$$M_u = 0.31\ \text{GeV} \Leftrightarrow r_\pi = 0.66\ \text{fm}$$

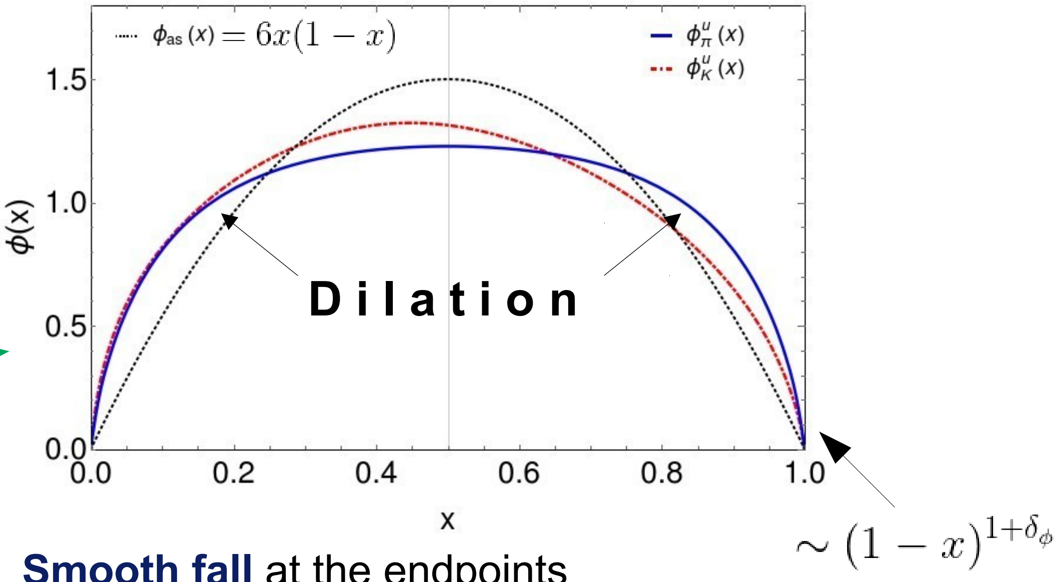
LFWFs and PDAs

$$f_M \phi_M^q(x;\zeta_H) = \int \frac{d^2 k_\perp}{16\pi^3} \psi_M^q(x, k_\perp; \zeta_H)$$





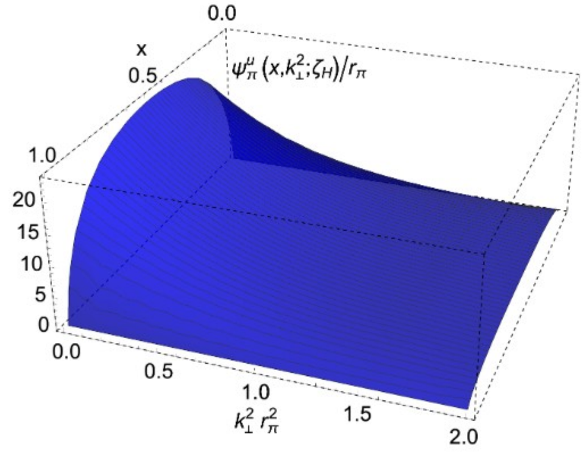
PTIR

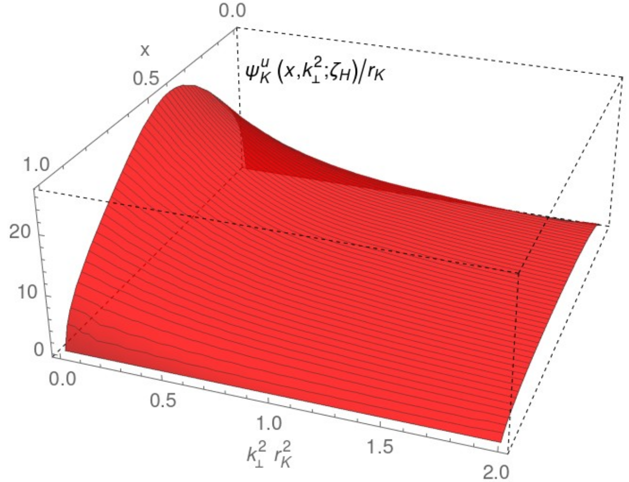


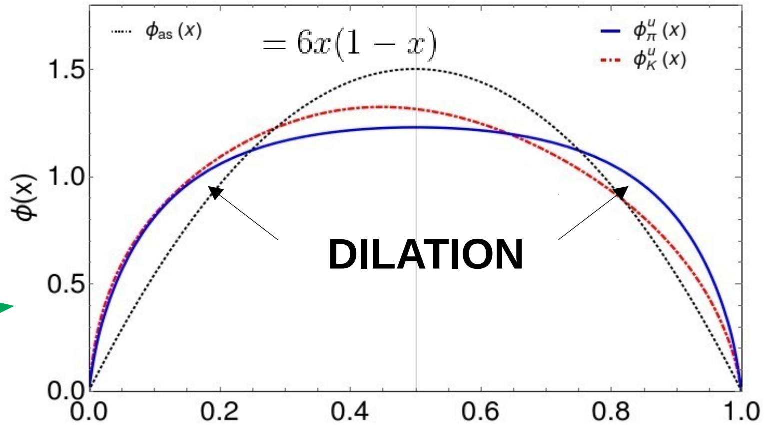
- **Smooth fall** at the endpoints
- **Broad** and concave functions of ***x***
  - Consequence of **DCSB**
- **Higgs** induced asymmetry for **Kaon**:
  - Moduled by the difference M_s - M_u

LFWFs and PDAs

$$f_{\rm M}\phi^q_{\rm M}(x;\zeta_H) = \int \frac{d^2k_\perp}{16\pi^3}\psi^q_{\rm M}(x,k_\perp;\zeta_H)$$

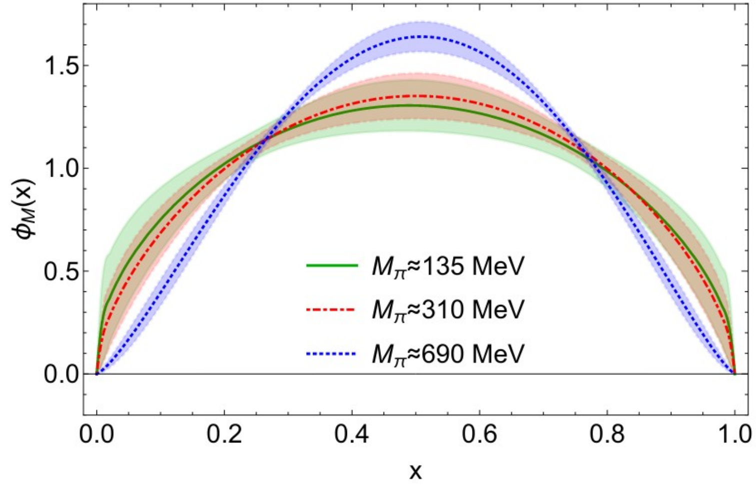






First **CSM** calculation of pion **PDA**:

Chang:2013pq



Patterns supported by **lattice**:

Zhang:2020gaj

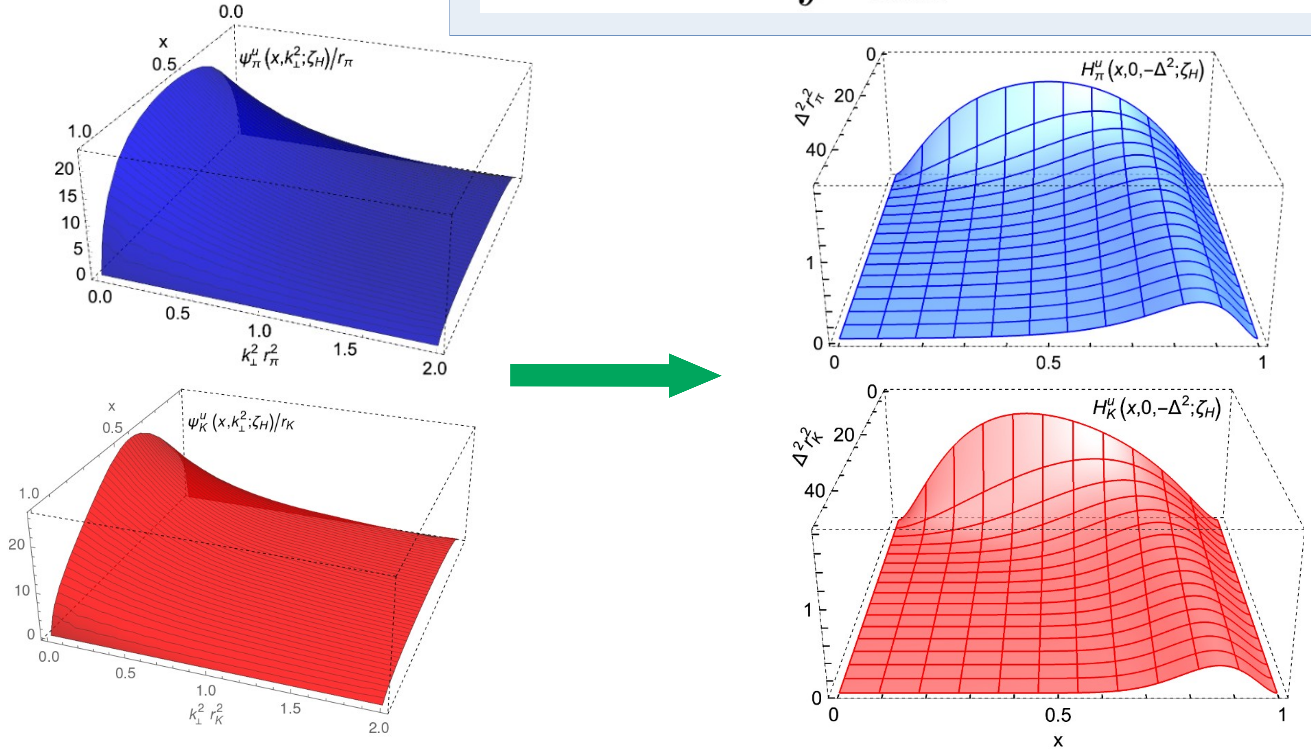
LFWFs and GPDs

LFWFs → GPDs

- In the **overlap representation**, the valence-quark **GPD** reads as:

$$H_{\rm M}^q(x,\xi,t) = \int \frac{d^2k_\perp}{16\pi^3}\psi_{\rm M}^{q*}\left(x^-,(\mathbf{k}_\perp^-)^2\right)\psi_{\rm M}^q\left(x^+,(\mathbf{k}_\perp^+)^2\right)$$

ζ_H



- ✓ ***Valid*** in the **DGLAP** region
- ✓ ***Positivity*** fulfilled
- ✓ Can be ***extended*** to the **ERBL** region $|x| \leq \xi$

`Chavez:2021llq`
`Chouika:2017dhe`

- ✓ ***Analytic*** in our factorized models.

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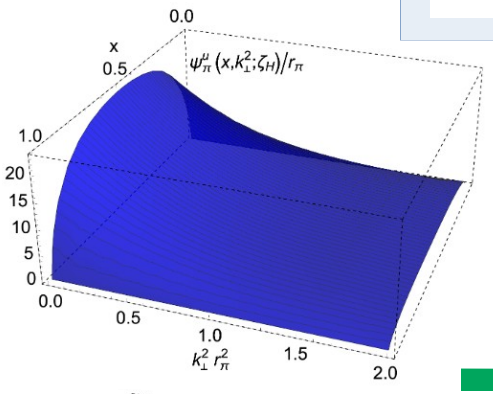
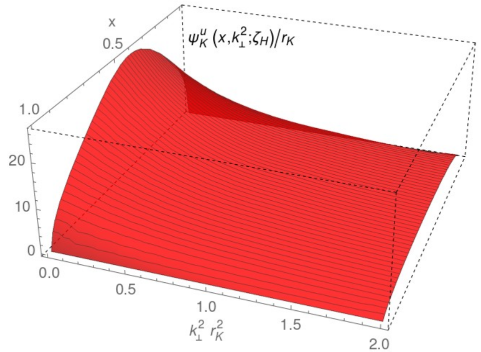
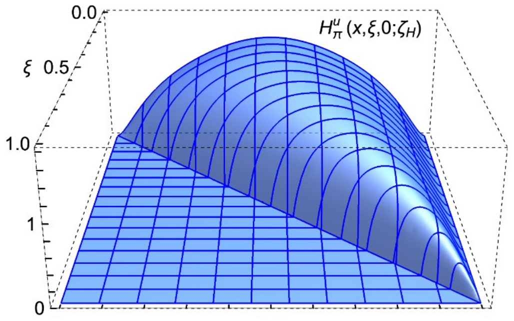
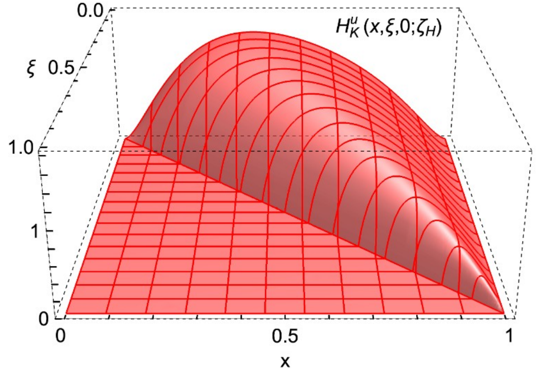
ζ_H









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Chavez:2021llq

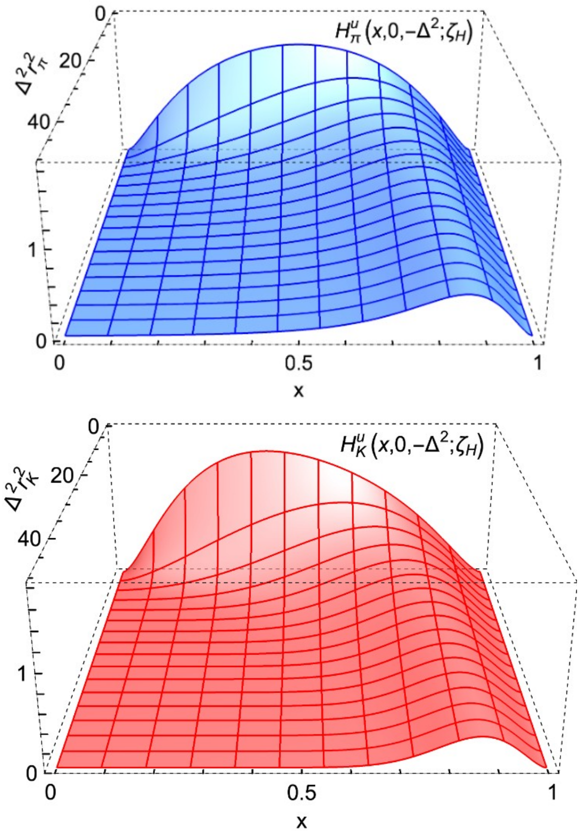
- ***Analytic*** in our factorized models.

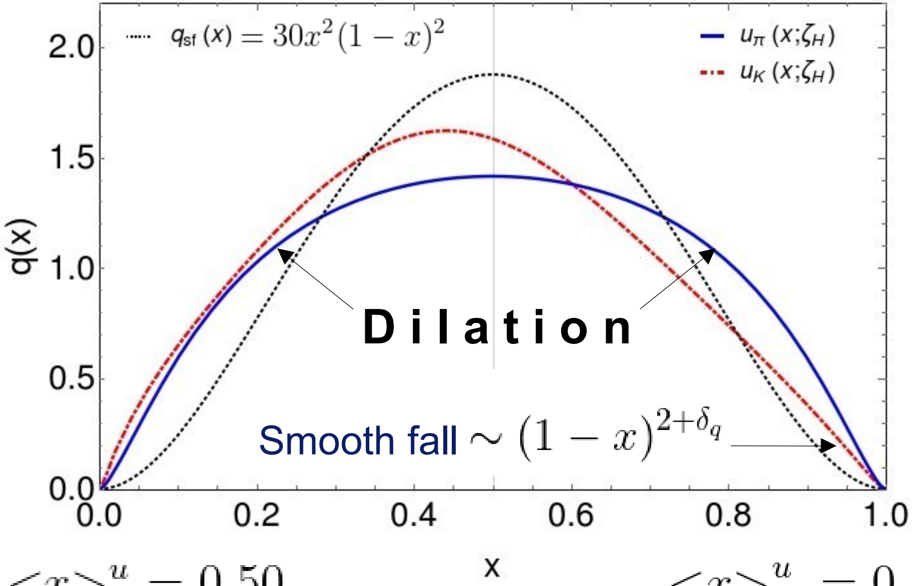
GPDs and PDFs

GPD → PDF

- The **PDF** is obtained from the **forward limit** of the **GPD**.

$$q(x) = H(x,0,0)$$

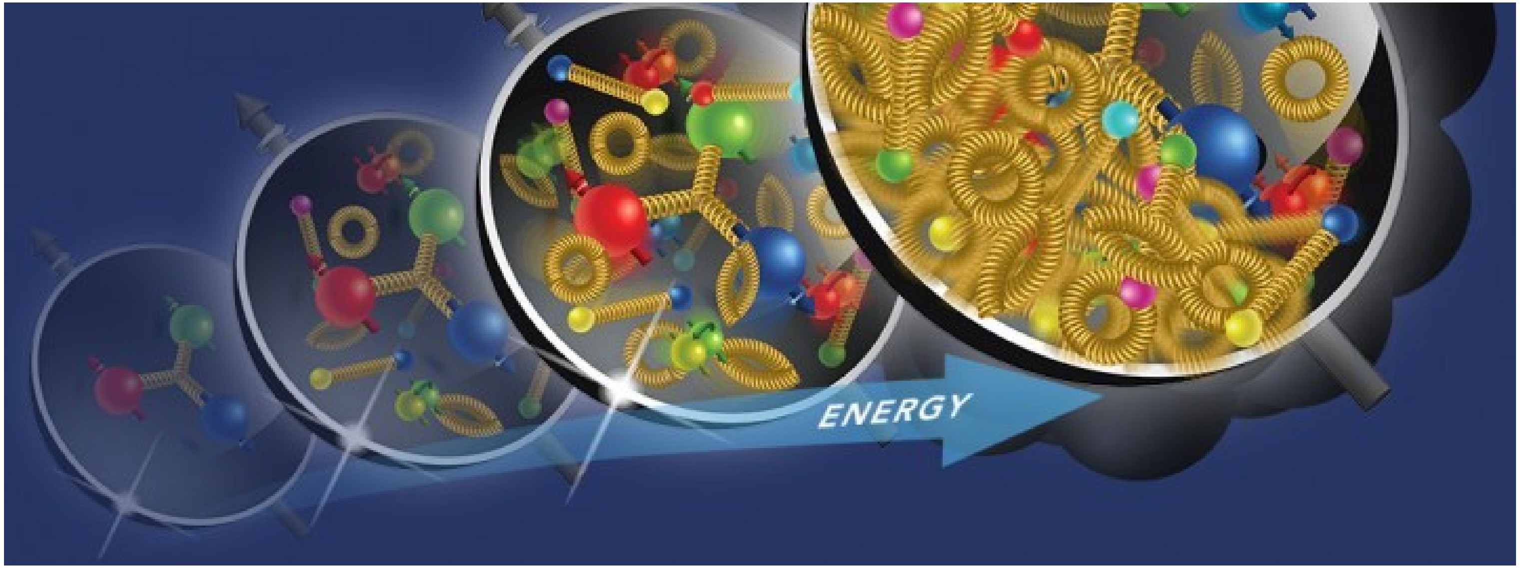




$<x>^u_\pi = 0.50$ $<x>^u_K = 0.47$

- ζ_H: meson properties determined by the fully-dressed valence-quarks.
- **Broad** + **Higgs**-induced *asymmetry*

All orders evolution...



DGLAP: All orders evolution

Idea. Define an **effective** coupling such that:

"All orders evolution"

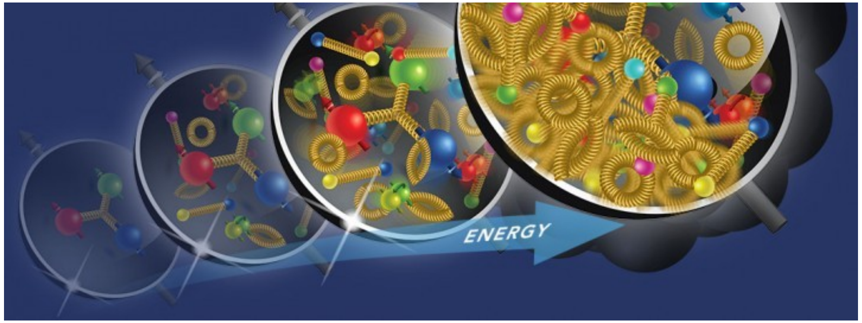
Raya:2021zrz

Cui:2020tdf

Starting from fully-dressed **quasiparticles**, at ζ_H

→ **Sea** and **Gluon** content unveils, as prescribed by **QCD**

$$\left\{\zeta^2\frac{d}{d\zeta^2}\int_0^1 dy\delta(y-x) - \boxed{\frac{\alpha(\zeta^2)}{4\pi}}\int_x^1\frac{dy}{y}\begin{pmatrix} P_{qq}^{\rm NS}\left(\frac{x}{y}\right) & 0 \\ 0 & \mathbf{P}^{\rm S}\left(\frac{\mathbf{x}}{\mathbf{y}}\right)\end{pmatrix}\right\}\begin{pmatrix} H_\pi^{\rm NS,+}(y,t;\zeta) \\ \mathbf{H}_\pi^{\rm S}(y,t;\zeta)\end{pmatrix} = 0$$



- **Not** the **LO** QCD coupling but an **effective** one.
- Making this equation **exact**.
- Connecting with the **hadron scale**, at which the **fully-dressed** valence-**quarks** express **all** of the hadron's properties.

(thus carrying all the momentum)

DGLAP: All orders evolution

Implication 1:

$$\langle x^n(\zeta_f)\rangle_q = \exp\left(-\frac{\gamma_{qq}^{(n)}}{4\pi} S(\zeta_H,\zeta_f)\right)\langle x^n(\zeta_H)\rangle_q$$

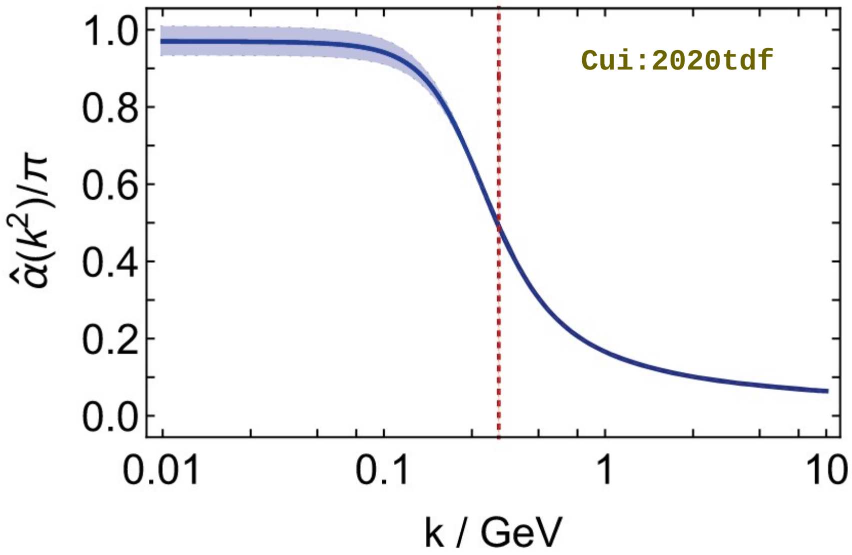
$$S(\zeta_0,\zeta_f) = \int_{2\ln(\zeta_0/\Lambda_{\rm QCD})}^{2\ln(\zeta_f/\Lambda_{\rm QCD})} dt\,\alpha(t)$$

Explicitly depending on the **effective charge**

$$\langle x^n(t;\zeta)\rangle_F = \int_0^1 dx\, x^n\, F(x,t;\zeta)$$

$$\gamma_{AB}^{(n)} = -\int_0^1 dx\, x^n P_{AB}^C(x)$$

- The **QCD PI effective charge** is our best candidate to accommodate our **all orders scheme**.



$$\hat{\alpha}(k^2) = \frac{\gamma_m \pi}{\ln\left[\frac{\mathcal{M}^2(k^2)}{\Lambda_{\rm QCD}^2}\right]} \Rightarrow \boxed{\zeta_H = 0.331\ \text{GeV}}$$

DGLAP: All orders evolution

$$\langle x^n(t;\zeta)\rangle_F = \int_0^1 dx\, x^n\, F(x,t;\zeta)$$

Implication 1:

$$\langle x^n(\zeta_f)\rangle_q = \exp\left(-\frac{\gamma_{qq}^{(n)}}{4\pi} S(\zeta_H,\zeta_f)\right)\langle x^n(\zeta_H)\rangle_q = \langle x^n(\zeta_H)\rangle_q \left(\frac{\langle x(\zeta_f)\rangle_q}{\langle x(\zeta_H)\rangle_q}\right)^{\gamma_{qq}^{(n)}/\gamma_{qq}^{(1)}}$$

$$S(\zeta_0,\zeta_f) = \int_{2\ln(\zeta_0/\Lambda_{\rm QCD})}^{2\ln(\zeta_f/\Lambda_{\rm QCD})} dt\, \alpha(t)$$

This contains, ***implicitly***, the information of the **effective charge**

- No actual ***need*** to know it. Assuming its existence is **sufficient**.
- ***Unambiguous*** definition of the **hadron scale**:

$$\langle x(\zeta_H)\rangle_q = 0.5 \;\Rightarrow\; \langle x^n(\zeta_f)\rangle_q = \langle x^n(\zeta_H)\rangle_q \left(\langle 2x(\zeta_f)\rangle_q\right)^{\gamma_{qq}^{(n)}/\gamma_{qq}^{(1)}}$$

(pion case)

DGLAP: All orders evolution

$$\langle x^n(t;\zeta)\rangle_F = \int_0^1 dx\, x^n F(x,t;\zeta)$$

Implication 1:

$$\langle x^n(\zeta_f)\rangle_q = \exp\left(-\frac{\gamma_{qq}^{(n)}}{4\pi} S(\zeta_H,\zeta_f)\right)\langle x^n(\zeta_H)\rangle_q = \langle x^n(\zeta_H)\rangle_q \left(\frac{\langle x(\zeta_f)\rangle_q}{\langle x(\zeta_H)\rangle_q}\right)^{\gamma_{qq}^{(n)}/\gamma_{qq}^{(1)}}$$

Information on the charge is here

- Details of the effective charge are **encoded** in the ratio of first moments.
- Natural connection with the **hadron scale**.

Implication 2:

$$\begin{aligned}
\langle 2x(\zeta_f)\rangle_q &= \exp\left(-\frac{8}{9\pi} S(\zeta_H,\zeta_f)\right), \qquad q = u, \bar{d}\,; \\
\langle x(\zeta_f)\rangle_{\text{sea}} &= \langle x(\zeta_f)\rangle_{\sum_q q+\bar{q}} - (\langle x(\zeta_f)\rangle_u + \langle x(\zeta_f)\rangle_{\bar{d}})\,, \\
&= \frac{3}{7} + \frac{4}{7}\langle 2x(\zeta_f)\rangle_u^{7/4} - \langle 2x(\zeta_f)\rangle_u \\
\langle x(\zeta_f)\rangle_g &= \frac{4}{7}\left(1 - \langle 2x(\zeta_f)\rangle_u^{7/4}\right);
\end{aligned}$$

- **Sea** and **gluon** determined from valence-quark moments

DGLAP: All orders evolution

$$\langle x^n(t;\zeta)\rangle_F = \int_0^1 dx\, x^n\, F(x,t;\zeta)$$

Implication 1:

$$\langle x^n(\zeta_f)\rangle_q = \exp\left(-\frac{\gamma_{qq}^{(n)}}{4\pi} S(\zeta_H,\zeta_f)\right)\langle x^n(\zeta_H)\rangle_q = \langle x^n(\zeta_H)\rangle_q \left(\frac{\langle x(\zeta_f)\rangle_q}{\langle x(\zeta_H)\rangle_q}\right)^{\gamma_{qq}^{(n)}/\gamma_{qq}^{(1)}}$$

Information on the charge is here

- Can **jump** from one scale to the other. (even downwards)
- Natural connection with the **hadron scale**.

Implication 2:

$$\begin{aligned}
\langle 2x(\zeta_f)\rangle_q &= \exp\left(-\frac{8}{9\pi} S(\zeta_H,\zeta_f)\right), \qquad q = u, \bar{d};\\
\langle x(\zeta_f)\rangle_{\text{sea}} &= \langle x(\zeta_f)\rangle_{\sum_q q+\bar{q}} - \left(\langle x(\zeta_f)\rangle_u + \langle x(\zeta_f)\rangle_{\bar{d}}\right),\\
&= \frac{3}{7} + \frac{4}{7}\langle 2x(\zeta_f)\rangle_u^{7/4} - \langle 2x(\zeta_f)\rangle_u\\
\langle x(\zeta_f)\rangle_g &= \frac{4}{7}\left(1 - \langle 2x(\zeta_f)\rangle_u^{7/4}\right);
\end{aligned}$$

- **Sea** and **gluon** determined from valence-quark moments
- **Asymptotic** (massless) limits are evident.

DGLAP: All orders evolution

$$\langle x^n(t;\zeta)\rangle_F = \int_0^1 dx\, x^n\, F(x,t;\zeta)$$

Implication 1:

$$\langle x^n(\zeta_f)\rangle_q = \exp\left(-\frac{\gamma_{qq}^{(n)}}{4\pi}S(\zeta_H,\zeta_f)\right)\langle x^n(\zeta_H)\rangle_q = \langle x^n(\zeta_H)\rangle_q \left(\frac{\langle x(\zeta_f)\rangle_q}{\langle x(\zeta_H)\rangle_q}\right)^{\gamma_{qq}^{(n)}/\gamma_{qq}^{(1)}}$$

Information on the charge is here

- Can **jump** from one scale to the other. (even downwards)
- Natural connection with the **hadron scale**.

Implication 2:

$$\begin{aligned}
\langle 2x(\zeta_f)\rangle_q &= \exp\left(-\frac{8}{9\pi}S(\zeta_H,\zeta_f)\right), \qquad q=u,\bar{d}\,; \\
\langle x(\zeta_f)\rangle_{\text{sea}} &= \langle x(\zeta_f)\rangle_{\sum_q q+\bar{q}} - (\langle x(\zeta_f)\rangle_u + \langle x(\zeta_f)\rangle_{\bar{d}})\,, \\
&= \frac{3}{7} + \frac{4}{7}\langle 2x(\zeta_f)\rangle_u^{7/4} - \langle 2x(\zeta_f)\rangle_u \\
\langle x(\zeta_f)\rangle_g &= \frac{4}{7}\left(1-\langle 2x(\zeta_f)\rangle_u^{7/4}\right)\,;
\end{aligned}$$

- **Sea** and **gluon** determined from valence-quark moments
- **Asymptotic** (massless) limits are evident.
- And, of course, the momentum **sum rule**:

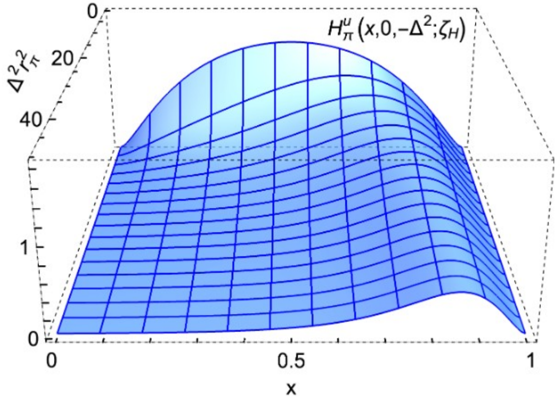
$$\langle 2x(\zeta_f)\rangle_q + \langle x(\zeta_f)\rangle_{\text{sea}} + \langle x(\zeta_f)\rangle_g = 1$$

Evolved **GPDs**

$\zeta_2 := 2\text{ GeV}$

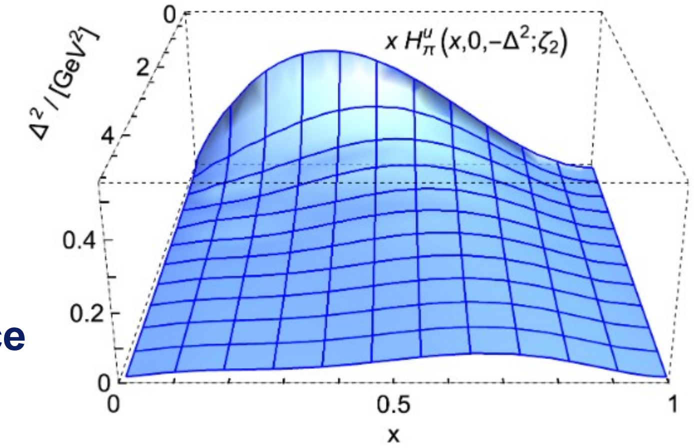
- Starting with **valence** distributions, at *hadron scale*, generate **gluon** and **sea** distributions via **all orders** evolution equations.

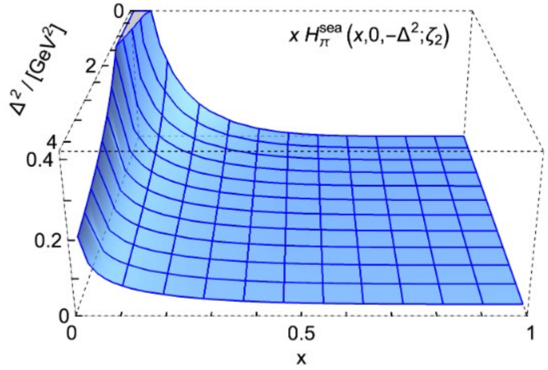
Glue and Sea GPDs !!!



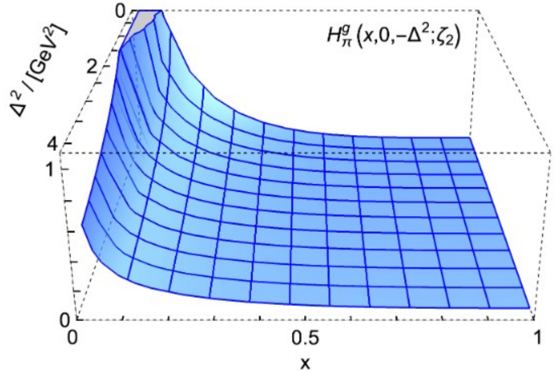
$\zeta_H = 0.331\text{ GeV}$

Valence





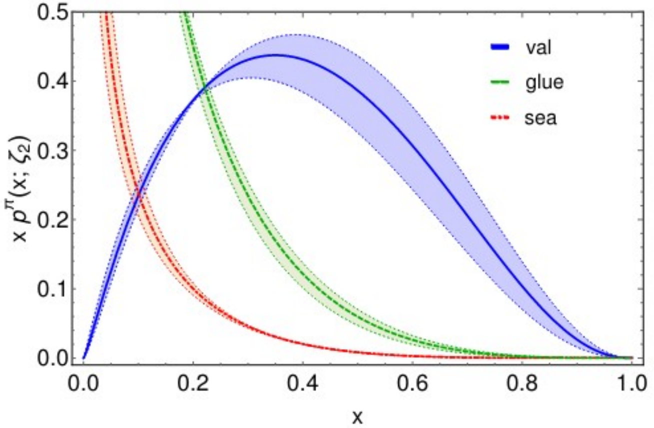
Sea

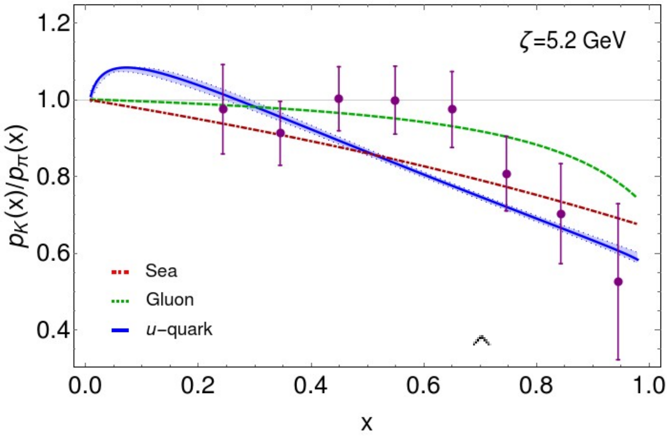


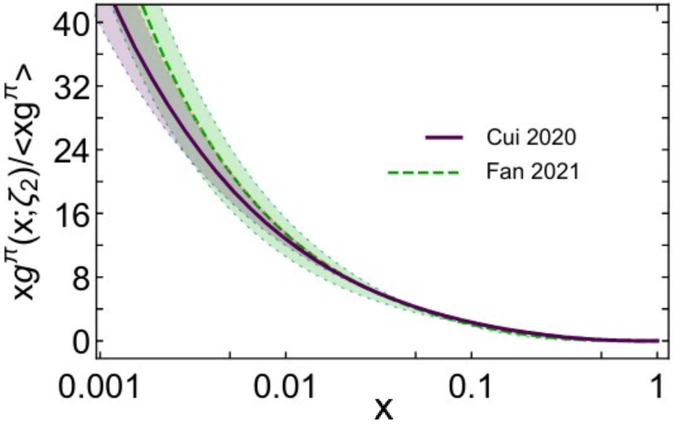
Gluon

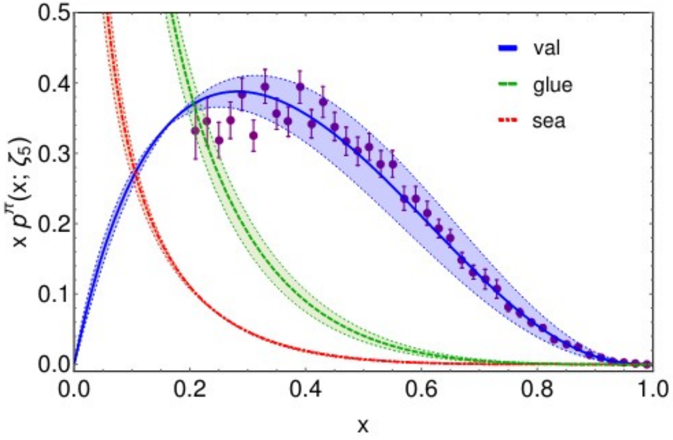
Evolved **PDFs**

GPD → PDF









- In **agreement** with:
  - ✔ **ASV** analysis — Aicher:2010cb
  - ✔ **Lattice** CS — Sufian:2020vzb, Sufian:2019bol
  - ✔ **xFitter** fits — Novikov:2020snp
  - ✔ **DSEs** — Cui:2020tdf

- **Gluon** in pion: Chang:2021utv
  - ✔ **Lattice** MSU — Fan:2021bcr

$$<\mathbf{x}>_{\pi}^{\mathrm{val}} = 0.41(4)$$

$$<\mathbf{x}>_{K}^{\mathrm{val}} = 0.43(4)$$

Going Off-forward...

$$H^q_{\mathrm{M}}(x,\xi,t) = \int \frac{d^2k_\perp}{16\pi^3}\,\psi^{q*}_{\mathrm{M}}\left(x^-,(\mathbf{k}^-_\perp)^2\right)\psi^q_{\mathrm{M}}\left(x^+,(\mathbf{k}^+_\perp)^2\right)$$

Electromagnetic FFs

GPD → FFs

- **Electromagnetic** form factor is obtained from the **t-dependence** of the **0-th moment**:

$$F^q_M(-t=\Delta^2) = \int_{-1}^{1} dx\, H^q_M(x,\xi,t)$$

Can safely take **ξ = 0**

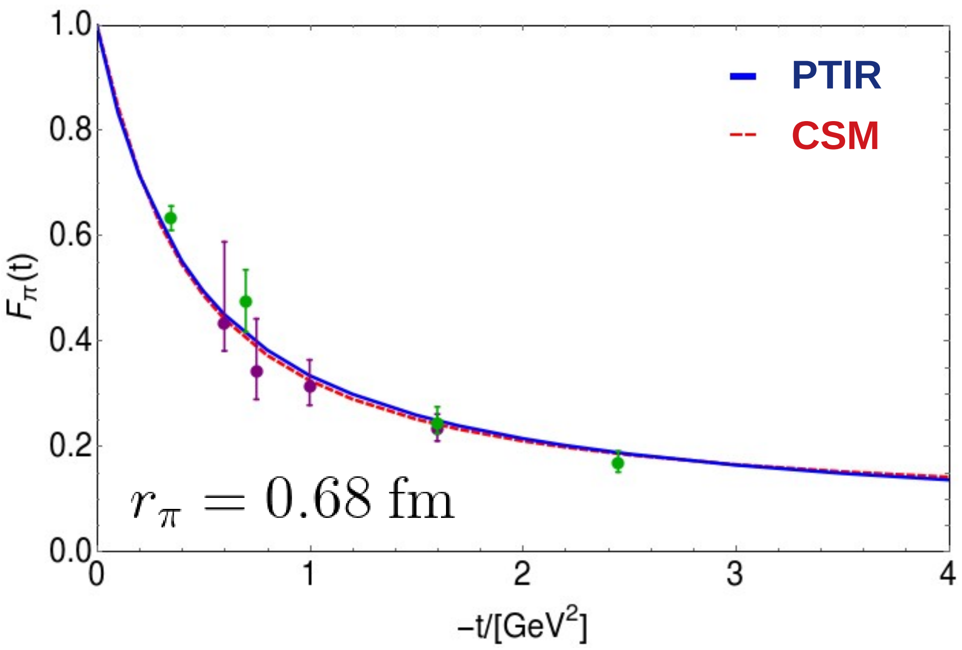
"Polinomiality"

$$F_M(\Delta^2) = e_u F^u_M(\Delta^2) + e_{\bar{f}} F^{\bar{f}}_M(\Delta^2)$$

Weighed by electric charges

- **Isospin symmetry**

$$\Rightarrow\quad F_{\pi^+}(-t) = F^u_{\pi^+}(-t)$$



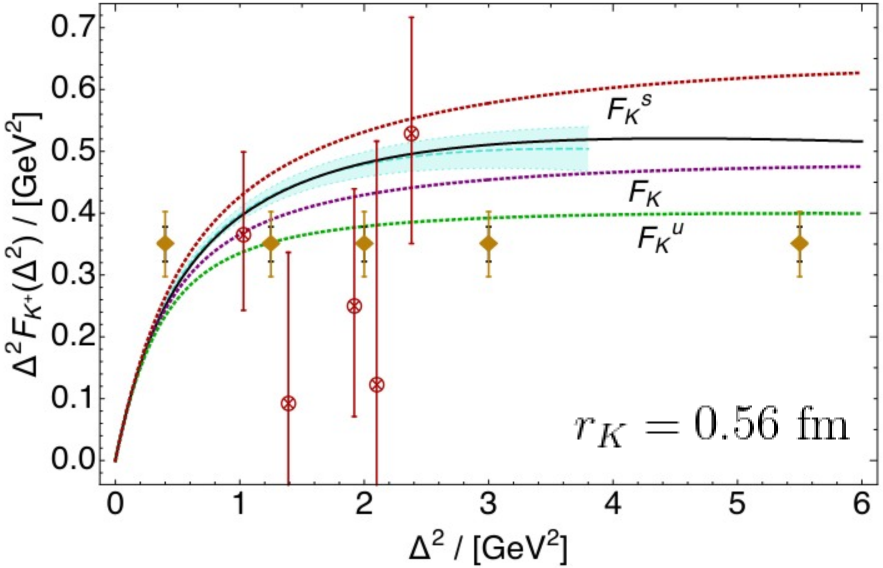
Data: G.M. Huber *et al*. PRC 78 (2008) 045202

CSM: L. Chang *et al*. PRL 111 (2013) 14, 141802

Kaon EFF

GPD → FFs

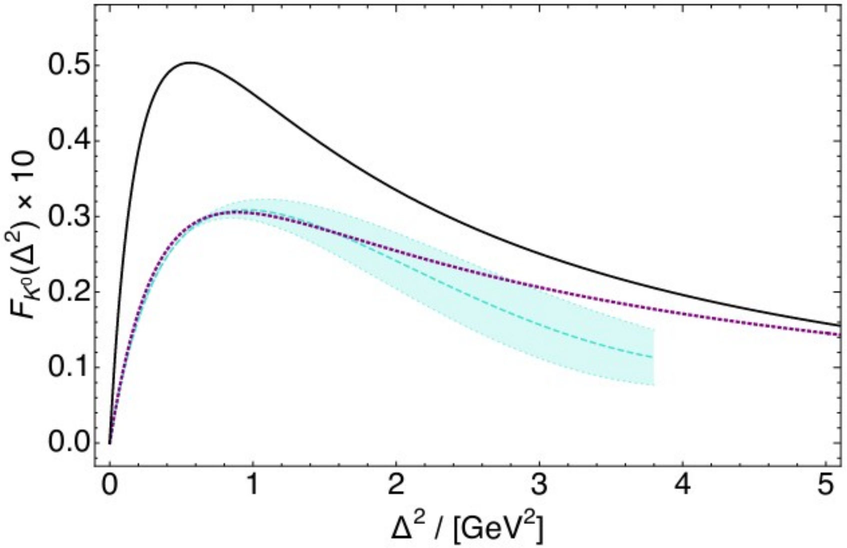
- **Electromagnetic** form factor: **charged** and **neutral** kaon



CSM - K⁺: Gao:2017mmp, Eichmann:2019bqf

CSM - K⁰: Gao:2017mmp

Lattice: Davies:2018zav



Kaon is more **compressed**

$$r_K^j \approx 0.85\, r_\pi^j$$

j = *mech, charge, mass*

Pion Gravitational FFs

GPD → FFs

- **Gravitational** form factors are obtained from the **t-dependence** of the **1-st moment**:

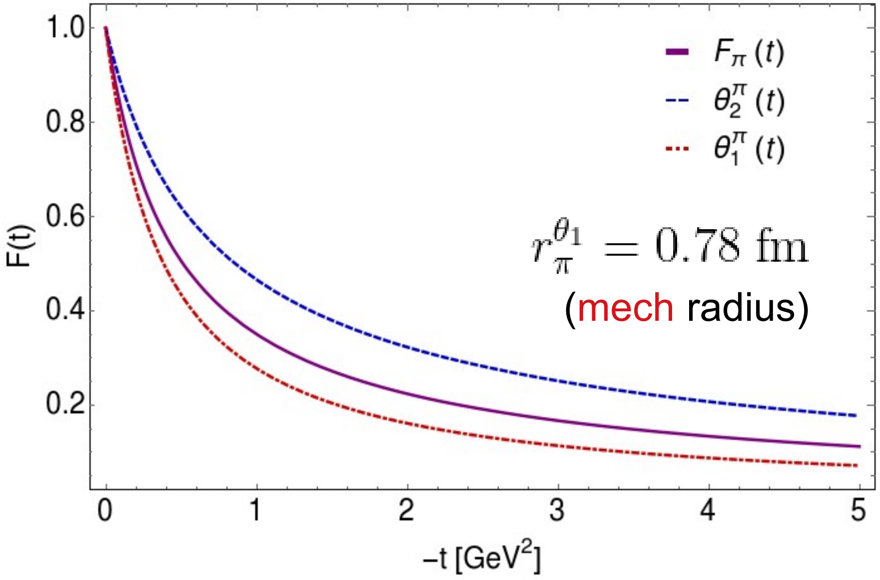
$$J_M(t,\xi) = \int_{-1}^{1} dx\, x H_M(x,\xi,t) = \Theta_2^M(t) - \xi^2 \Theta_1^M(t)$$

  - ✓ Directly obtained if **ξ = 0**
  - ✓ Only **DGLAP** GPD is required
  - ✗ **ERBL** GPD needed

- Sophisticated techniques exist.
- But a sound expression can be constructed:

$$\theta_1^{P_q}(\Delta^2) = c_1^{P_q}\theta_2^{P_q}(\Delta^2) + \int_{-1}^{1} dx\, x \left[H_P^q(x,1,0)P_{M_q}(\Delta^2) - H_P^q(x,1,-\Delta^2)\right]$$

"Soft pion theorem"



$r_\pi^E = 0.68$ fm , $r_\pi^{\theta_2} = 0.56$ fm

(charge radius) (mass radius)

Zhang:2021mtn

Charge and mass distributions

$$\rho_{\rm P}(b) = \frac{1}{2\pi}\int_0^\infty d\Delta\,\Delta J_0(\Delta\, b) F_{\rm P}(\Delta^2)$$

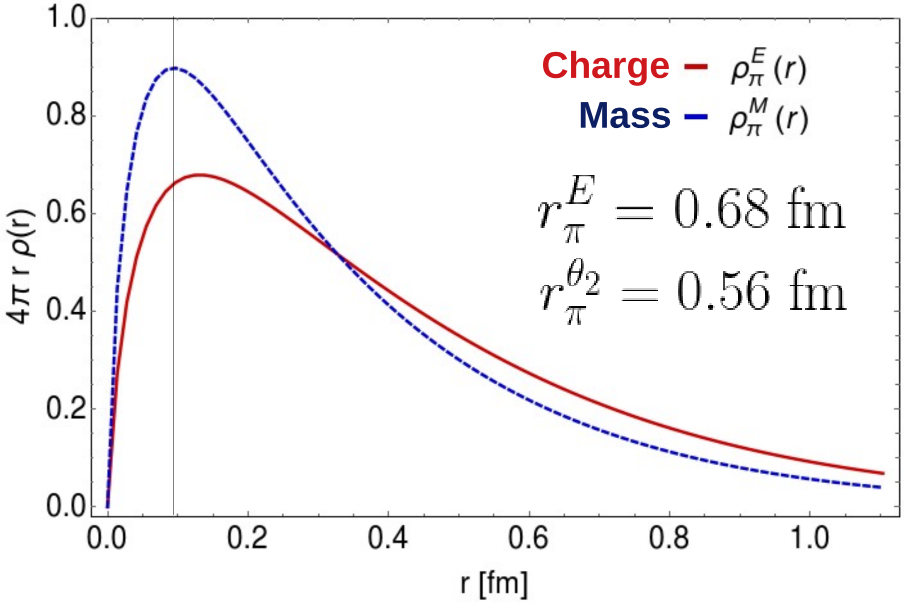
- **Intuitively**, we expect the meson to be localized at a **finite space**.

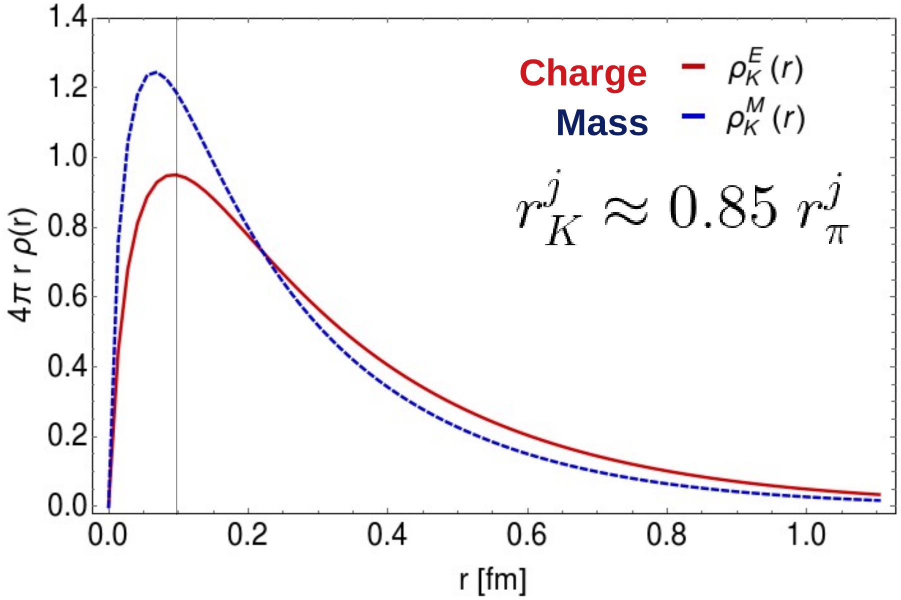
$$F_{\rm P}^{E}(\Delta^2) \to \rho_{\rm P}^{E}(b)$$

$$\theta_2^{\rm P}(\Delta^2) \to \rho_{\rm P}^{M}(b)$$

- **Charge** effect span over a **larger domain** than **mass** effects.

More **massive** hadron → More **compressed**





Pressure distributions

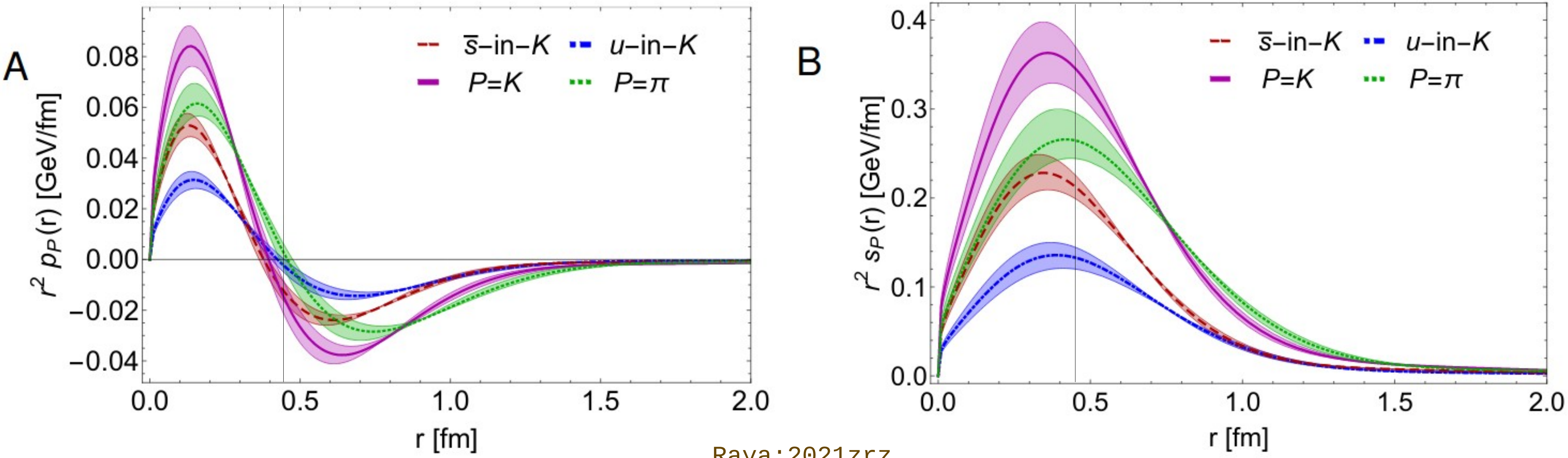
$$p_K^u(r) = \frac{1}{6\pi^2 r}\int_0^\infty d\Delta\, \frac{\Delta}{2E(\Delta)}\, \sin(\Delta r)[\Delta^2 \theta_1^{K_u}(\Delta^2)]\,,$$

$$s_K^u(r) = \frac{3}{8\pi^2}\int_0^\infty d\Delta\, \frac{\Delta^2}{2E(\Delta)}\, j_2(\Delta r)\,[\Delta^2 \theta_1^{K_u}(\Delta^2)]\,,$$

"Pressure" Quark attraction/repulsion

CONFINEMENT

"Shear" *Deformation* QCD forces



Raya:2021zrz

Impact parameter space GPDs

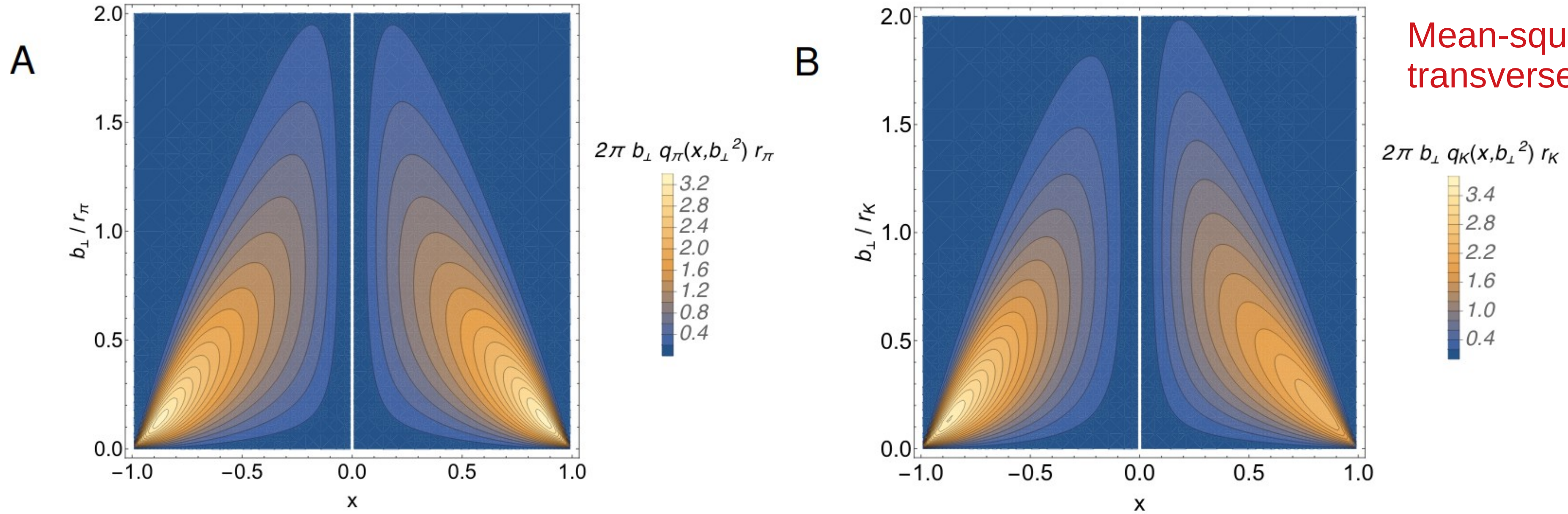
$$u^{\mathsf{P}}(x, b_\perp^2; \zeta_{\mathcal{H}}) = \int_0^\infty \frac{d\Delta}{2\pi} \Delta J_0(b_\perp \Delta)\, H_{\mathsf{P}}^u(x, 0, -\Delta^2; \zeta_{\mathcal{H}})$$

Algebraic derivation!

$$\langle b_\perp^2(\zeta_{\mathcal{H}})\rangle_u^\pi = \frac{2}{3} r_\pi^2 = \langle b_\perp^2(\zeta_{\mathcal{H}})\rangle_{\bar{d}}^\pi ,$$

$$\langle b_\perp^2(\zeta_{\mathcal{H}})\rangle_u^K = 0.71 r_K^2 , \langle b_\perp^2(\zeta_{\mathcal{H}})\rangle_{\bar{s}}^K = 0.58 r_K^2 .$$

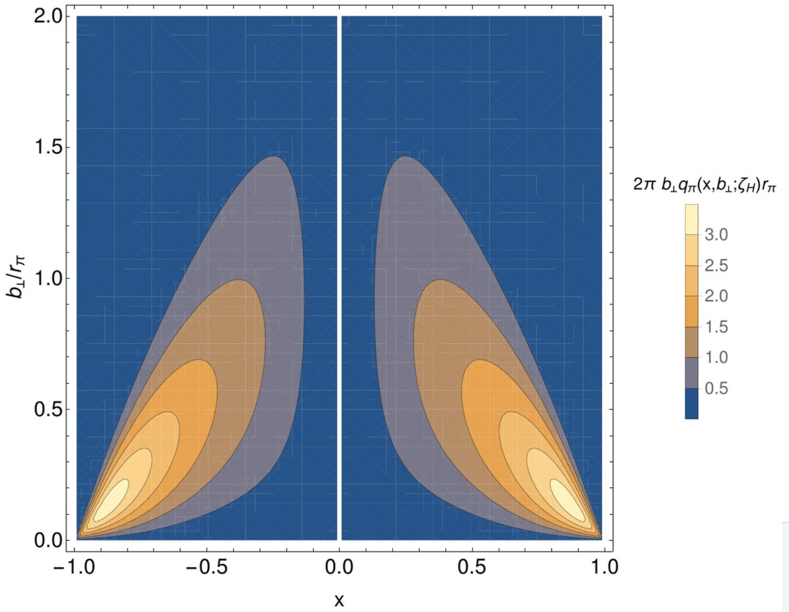
Mean-square transverse extent



- Likelihood of finding a valence-**quark** with **momentum fraction x**, at **position b**.

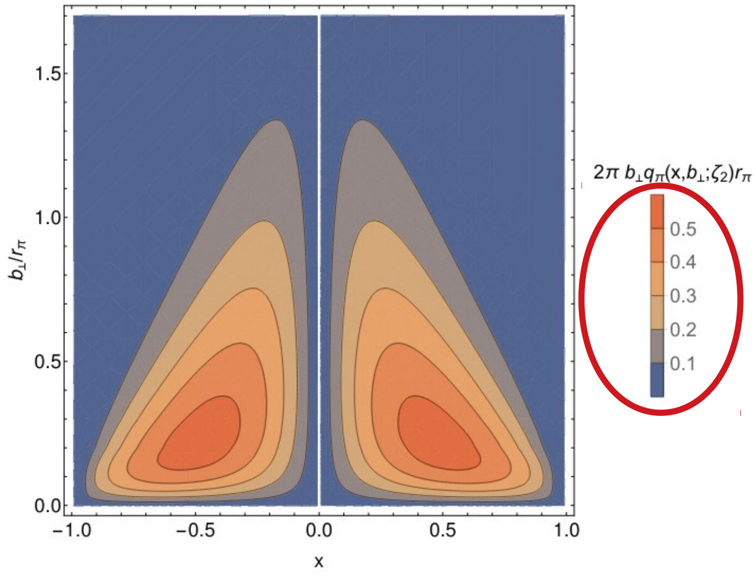
Evolved IPS-GPD: Pion Case

$$u^{\mathsf{P}}(x, b_\perp^2; \zeta_{\mathcal{H}}) = \int_0^\infty \frac{d\Delta}{2\pi} \Delta J_0(b_\perp \Delta)\, H^u_{\mathsf{P}}(x, 0, -\Delta^2; \zeta_{\mathcal{H}})$$



- **Likelihood** of finding a parton with LF momentum **x** at transverse position **b**

$\zeta_H \to \zeta = 2$ GeV

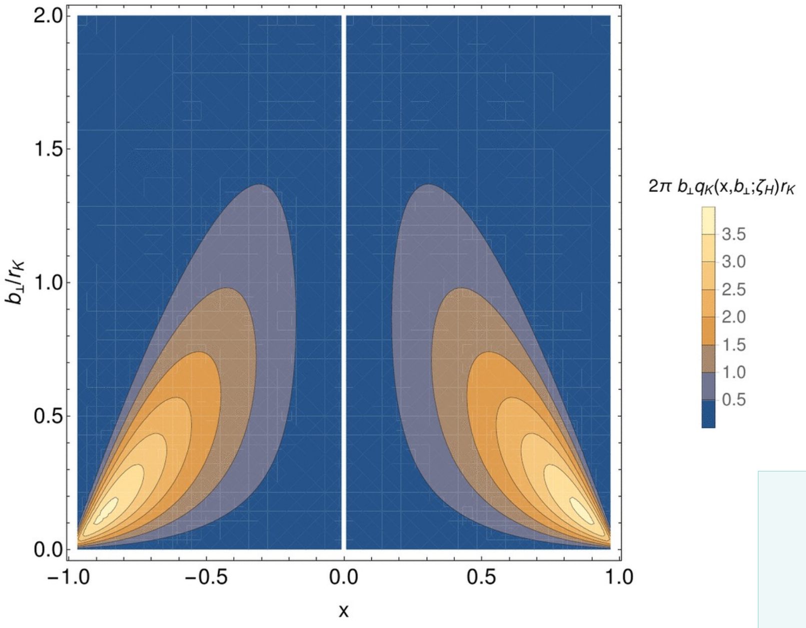


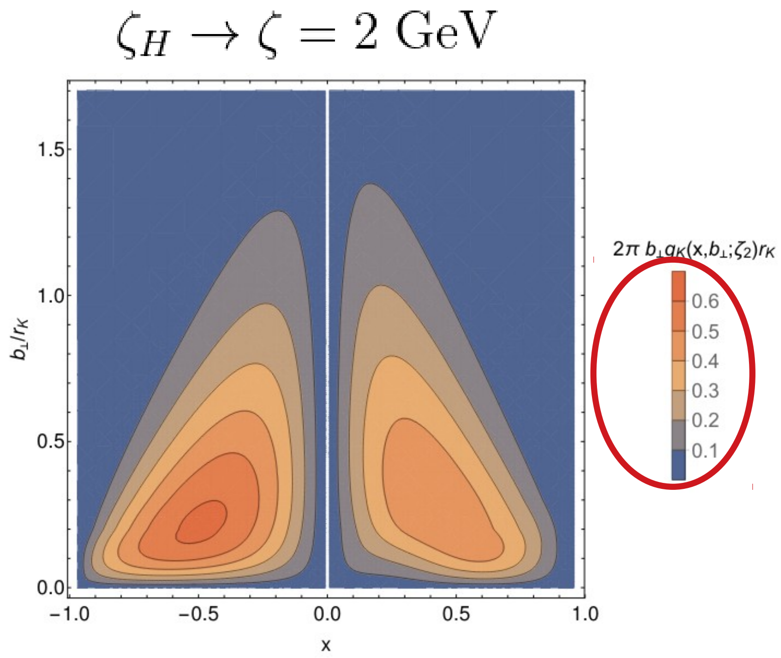
- Peaks **broaden** and **maximum drifts:**

$$\max: \ 3.29 \to 0.55$$
$$(|x|, b) = (0.88, 0.13) \to (0.47, 0.23)$$

Evolved IPS-GPD: Kaon Case

$$u^{\mathsf{P}}(x, b_\perp^2; \zeta_{\mathcal{H}}) = \int_0^\infty \frac{d\Delta}{2\pi} \Delta J_0(b_\perp \Delta)\, H_{\mathsf{P}}^u(x, 0, -\Delta^2; \zeta_{\mathcal{H}})$$





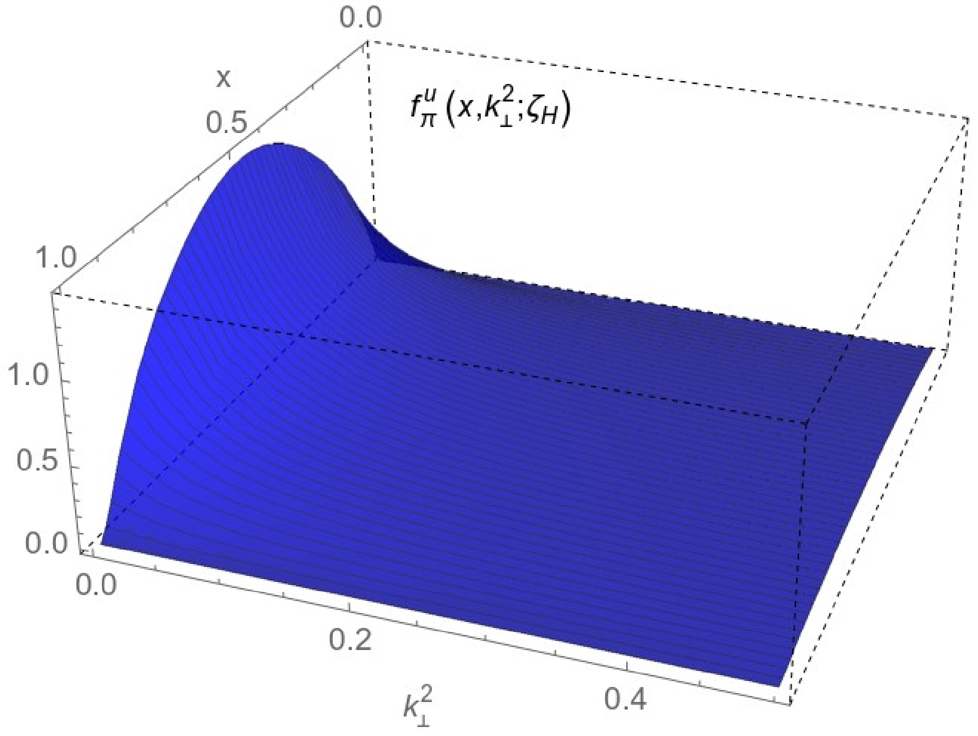
$$\max_{(s,u)} : (3.61, 2.38) \to (0.61, 0.49)$$

$$(x, b)_u = (0.84, 0.17) \to (0.41, 0.28)$$

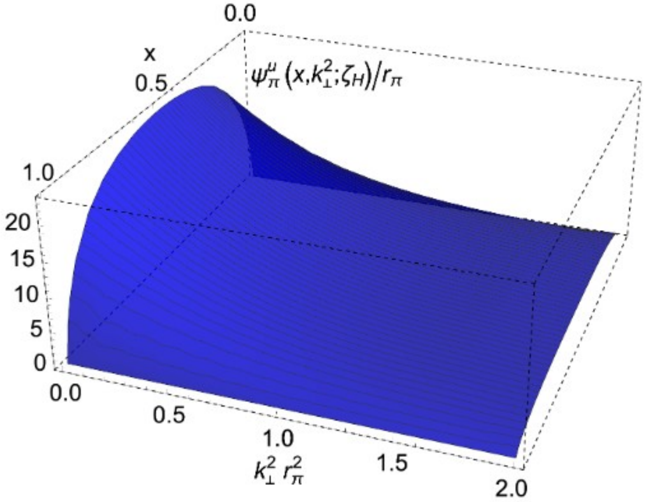
$$(x, b)_s = (-0.87, 0.13) \to (-0.48, 0.22)$$

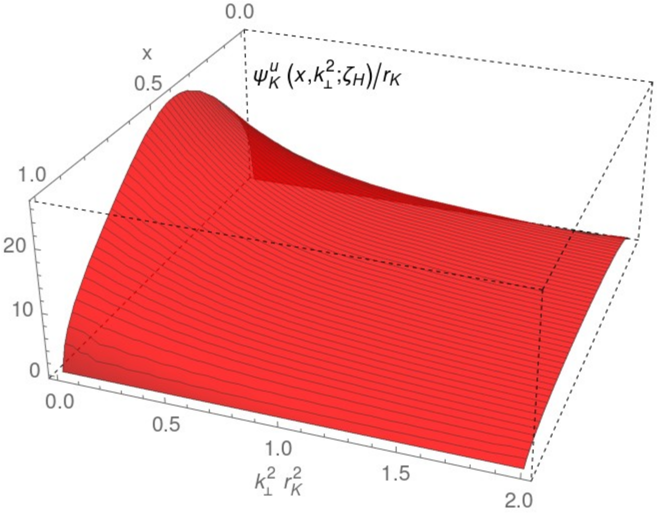
- **Likelihood** of finding a parton with LF momentum **x** at transverse position **b**

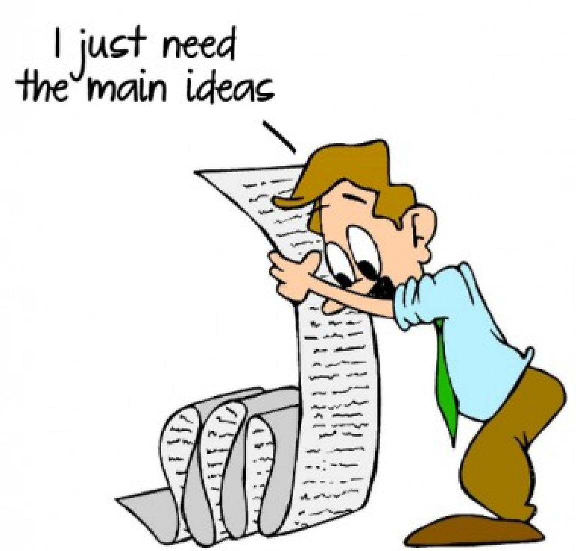
Transverse Momentum Dependent PDFs



Summary and Highlights







Summary

- Focusing on the **pion** and **Kaon**, we discussed a variety of **parton distributions**:

Light-front **wave function**
(encodes many properties)

Parton distribution amplitudes
(momentum distribution)

Inputs from **Continuum Schwinger Methods**
(PDF, PDA)

Generalized parton distributions
(leads to many other objects)

Impact parameter space GPD
(spatial distributions)

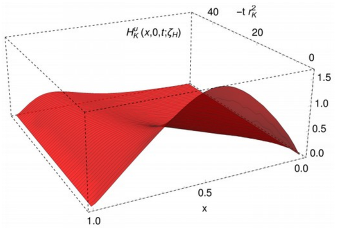
- ✓ Parton distribution functions
  (momentum distribution, valence-sea-gluon)
- ✓ Electromagnetic form factors
  (charge and momentum distributions)
- ✓ Gravitational form factors
  (mass distribution, mechanical properties)

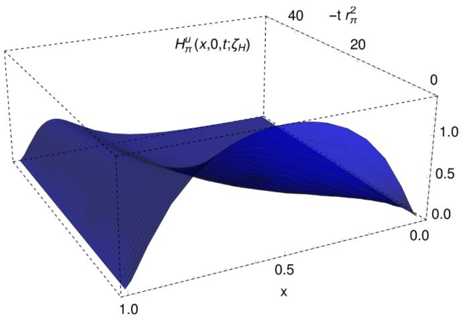
The role of **evolution:**
Generate **sea** and **glue** from pure **valence**

Pressure distributions
(spatial distribution, mechanical properties)

Highlights

- **QCD's EHM** produce **broad π-K** distributions.

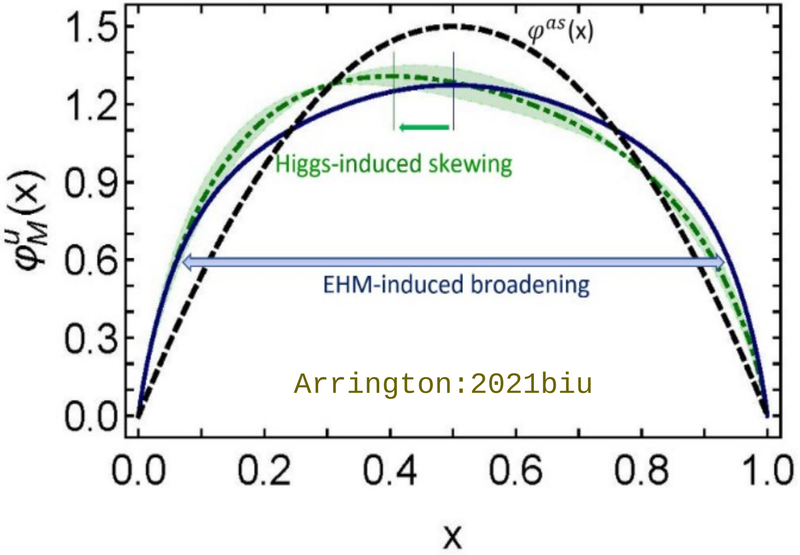




- **Interplay** between **QCD** and **Higgs** mass generation:
  - Slightly *skewed* Kaon distributions.
- The **ordering** of **radii**: $r_\pi^{\theta_1} > r_\pi^{E} > r_\pi^{\theta_2}$ $\quad r_K^j \approx 0.85\, r_\pi^j$
- **Gluon** and **sea** revealed through **evolution**.
  - **Definition** of ζ_H

    Valence Picture

  '**All orders**' scheme
  - **QCD** effective charge.



> - Mass, gluon/sea, pressure, charge **distributions** addressed through **LFWFs** and **GPDs**
>
>   **... TMDs are within reach**


I'M GONNA NEED
BACKUP....
memegenerator.net

On the Radii: Factorized Models

$$H_{\rm M}^q(x,\xi,t) = \int \frac{d^2k_\perp}{16\pi^3} \psi_{\rm M}^{q*}\left(x^-,(\mathbf{k}_\perp^-)^2\right)\psi_{\rm M}^q\left(x^+,(\mathbf{k}_\perp^+)^2\right)$$

Overlap representation

Factorized LFWF

$$\psi_{\rm M}^q(x,k_\perp^2;\zeta_H) = \left[q^{\rm M}(x;\zeta_H)\right]^{1/2} \widetilde{\psi}_{\rm M}(k_\perp^2;\zeta_H)$$

(recall PTIR ~ Factorized in Chiral Limit)

PDF controls (mostly) the *x*-dependence

$$H_{\rm M}^q(x,\xi,t;\zeta_H) = \theta(x_-)\left[q^{\rm M}(x_-;\zeta_H)q^{\rm M}(x_+;\zeta_H)\right]^{1/2}\Phi_{\rm M}(z;\zeta_H)$$

$$\Phi_{\rm M}(z;\zeta_H) = \int \frac{d^2k_\perp}{16\pi^3}\widetilde{\psi}_{\rm M}(k_\perp^2;\zeta_H)\widetilde{\psi}_{\rm M}\left((k_\perp - s_\perp)^2;\zeta_H\right)$$

$$x_\pm = \frac{x\pm\xi}{1\pm\xi}$$

$$z = s_\perp^2 = \frac{-t(1-x)^2}{1-\xi^2}$$

t-dependence, evaluated **analytically**

On the Radii: FM

GPD → FFs

$$H^u_{\mathrm{P}}(x,\xi,t;\zeta_H) = \theta(x_-)\left[u^{\mathrm{P}}(x_-;\zeta_H)u^{\mathrm{P}}(x_+;\zeta_H)\right]^{1/2}\Phi_{\mathrm{P}}(z;\zeta_H)$$

- In the **factorized** models:

$$\frac{\partial^n}{\partial^n z}\Phi^u_{\mathrm{P}}(z;\zeta_{\mathcal{H}})\Big|_{z=0} = \frac{1}{\langle x^{2n}\rangle^{\zeta_{\mathcal{H}}}_{\bar{h}}}\frac{d^n F^u_{\mathrm{P}}(\Delta^2)}{d(\Delta^2)^n}\Big|_{\Delta^2=0}$$

PDF moments — Derivatives of **EFF**

$$\frac{\partial}{\partial z}\Phi^u_{\mathrm{P}}(z;\zeta_{\mathcal{H}})\Big|_{z=0} = -\frac{r^2_{\mathrm{P}}}{4\chi^2_{\mathrm{P}}(\zeta_{\mathcal{H}})},$$

$$\frac{\partial}{\partial z}\Phi^{\bar{h}}_{\mathrm{P}}(z;\zeta_{\mathcal{H}})\Big|_{z=0} = (1-d_{\mathrm{P}})\frac{\partial}{\partial z}\Phi^u_{\mathrm{P}}(z;\zeta_{\mathcal{H}})\Big|_{z=0}$$

Asymmetry term = **0** for **pion**

GPD can be built from:

- Distribution **amplitude** / Distribution **function**
- Derivatives of the electromagnetic **form factor**

Reminder:

$$[\phi^q_{\mathrm{M}}(x;\zeta_H)]^2 \sim q_{\mathrm{M}}(x;\zeta_H)$$

On the Radii: FM

GPD → FFs

$$H_{\rm P}^{u}(x,\xi,t;\zeta_H) = \theta(x_-)\left[u^{\rm P}(x_-;\zeta_H)u^{\rm P}(x_+;\zeta_H)\right]^{1/2}\Phi_{\rm P}(z;\zeta_H)$$

- In the **factorized** models:

$$\frac{\partial^n}{\partial^n z}\Phi_{\rm P}^u(z;\zeta_{\mathcal{H}})\Big|_{z=0} = \frac{1}{\langle x^{2n}\rangle_{\bar h}^{\zeta_{\mathcal{H}}}}\frac{d^n F_{\rm P}^u(\Delta^2)}{d(\Delta^2)^n}\Big|_{\Delta^2=0}$$

PDF moments Derivatives of **EFF**

$$\frac{\partial}{\partial z}\Phi_{\rm P}^u(z;\zeta_{\mathcal{H}})\Big|_{z=0} = -\frac{r_{\rm P}^2}{4\chi_{\rm P}^2(\zeta_{\mathcal{H}})},$$

$$\frac{\partial}{\partial z}\Phi_{\rm P}^{\bar h}(z;\zeta_{\mathcal{H}})\Big|_{z=0} = (1-d_{\rm P})\frac{\partial}{\partial z}\Phi_{\rm P}^u(z;\zeta_{\mathcal{H}})\Big|_{z=0}$$

Asymmetry term = **0** for **pion**

- In the **Chiral M1** model:

$$\frac{{r_{\rm P}}^2}{6\langle x^2\rangle_{\zeta_H}} = \frac{3}{5{M_q}^2}$$

Clear **connection**:

- Constituent mass ***M***
- Charge **radius**
- PDF **moment**

(at hadron scale)

Sensible values

$$M_u = 0.31\ \text{GeV}$$
$$\Leftrightarrow r_\pi = 0.66\ \text{fm}$$

On the Radii: FM

GPD → FFs

$$H^u_{\mathrm{P}}(x,\xi,t;\zeta_H) = \theta(x_-)\left[u^{\mathrm{P}}(x_-;\zeta_H)u^{\mathrm{P}}(x_+;\zeta_H)\right]^{1/2}\Phi_{\mathrm{P}}(z;\zeta_H)$$

- In the **factorized** models:

$$\frac{\partial^n}{\partial^n z}\Phi^u_{\mathrm{P}}(z;\zeta_{\mathcal{H}})\Big|_{z=0} = \frac{1}{\langle x^{2n}\rangle^{\zeta_{\mathcal{H}}}_{\bar{h}}}\left.\frac{d^n F^u_{\mathrm{P}}(\Delta^2)}{d(\Delta^2)^n}\right|_{\Delta^2=0}$$

PDF moments — Derivatives of **EFF**

$$\frac{\partial}{\partial z}\Phi^u_{\mathrm{P}}(z;\zeta_{\mathcal{H}})\Big|_{z=0} = -\frac{r^2_{\mathrm{P}}}{4\chi^2_{\mathrm{P}}(\zeta_{\mathcal{H}})},$$

$$\frac{\partial}{\partial z}\Phi^{\bar{h}}_{\mathrm{P}}(z;\zeta_{\mathcal{H}})\Big|_{z=0} = (1-d_{\mathrm{P}})\frac{\partial}{\partial z}\Phi^u_{\mathrm{P}}(z;\zeta_{\mathcal{H}})\Big|_{z=0}$$

Asymmetry term = **0** for **pion**

- Therefore, the **mass *radius***:

$$r^{\theta_2 2}_{\mathrm{P}_u} = \frac{3r^2_{\mathrm{P}}}{2\chi^2_{\mathrm{P}}}\langle x^2(1-x)\rangle_{\mathrm{P}_{\bar{h}}},$$

$$r^{\theta_2 2}_{\mathrm{P}_{\bar{h}}} = \frac{3r^2_{\mathrm{P}}}{2\chi^2_{\mathrm{P}}}(1-d_{\mathrm{P}})\langle x^2(1-x)\rangle_{\mathrm{P}_u}$$

$$\left(\frac{r^{\theta_2}_\pi}{r^E_\pi}\right)^2 = \frac{<x^2(1-x)>^q_{\zeta_H}}{<x^2>^q_{\zeta_H}} \approx \left(\frac{4}{5}\right)^2$$

Determined from **PDF** moments!

LFWF: Spectral weight

- More **explicitly**:

$$\psi^q_{\rm M}(x,k_\perp^2;\zeta_H) = 12\left[M_q(1-x)+M_{\bar h}x\right]X_{\rm P}(x;\sigma_\perp^2)$$

$$\sigma_\perp = k_\perp^2 + \Omega_{\rm P}^2$$

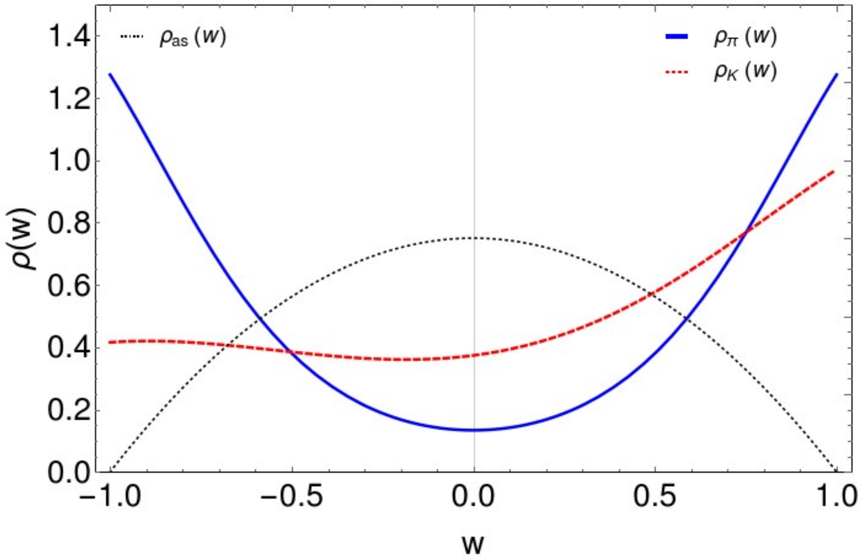
$$X_{\rm M}(x;\sigma_\perp^2) = \left[\int_{-1}^{1-2x} dw \int_{1+\frac{2x}{w-1}}^{1} dv + \int_{1-2x}^{1} dw \int_{\frac{w-1+2x}{w+1}}^{1} dv\right] \frac{\rho_{\rm M}(w)}{n_{\rm M}}\frac{\Lambda_{\rm M}^2}{\sigma_\perp^2}$$

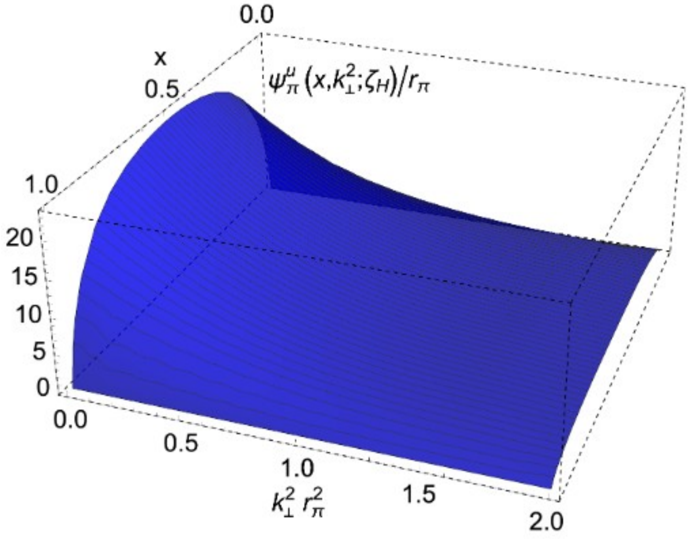
$$\Omega_{\rm M}^2 = vM_q^2 + (1-v)\Lambda_{\rm P}^2 + (M_h^2 - M_q^2)\left(x - \tfrac{1}{2}[1-w][1-v]\right) + \left(x[x-1] + \tfrac{1}{4}[1-v][1-w^2]\right) m_{\rm M}^2$$

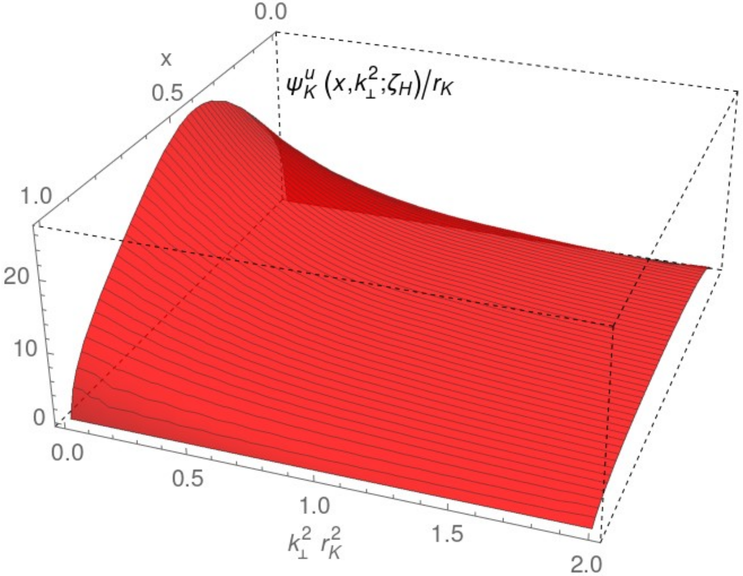
- Model **parameters**:

P	$m_{\rm P}$	M_u	M_h	$\Lambda_{\rm P}$	$b_0^{\rm P}$	$\omega_0^{\rm P}$	$v_{\rm P}$
π	0.14	0.31	M_u	M_u	0.275	1.23	0
K	0.49	0.31	$1.2M_u$	$3M_s$	0.1	0.625	0.41

$$\rho_{\rm P}(\omega) = \frac{1+\omega\, v_{\rm P}}{2a_{\rm P}b_0^{\rm P}}\left[\operatorname{sech}^2\left(\frac{\omega-\omega_0^{\rm P}}{2b_0^{\rm P}}\right) + \operatorname{sech}^2\left(\frac{\omega+\omega_0^{\rm P}}{2b_0^{\rm P}}\right)\right]$$







- Model **parameters**:

P	m_P	M_u	M_h	Λ_P	b_0^P	ω_0^P	v_P
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More General: DGLAP

In **Mellin** space, the **all orders** evolution prescription entails:

$$\frac{\langle x^n H_{\mathsf{P}}^u \rangle_{\zeta}^{\Delta^2}}{\langle x^n H_{\mathsf{P}}^u \rangle_{\zeta_H}^{\Delta^2}} = \left(\frac{\langle x H_{\mathsf{P}}^u \rangle_{\zeta}^{\Delta^2}}{\langle x H_{\mathsf{P}}^u \rangle_{\zeta_H}^{\Delta^2}} \right)^{\gamma_0^n/\gamma_0^1} ;$$

$$\begin{pmatrix} \langle x^n H_{\mathsf{P}}^{\mathrm{S}} \rangle_{\zeta}^{\Delta^2} \\ \langle x^n H_{\mathsf{P}}^{g} \rangle_{\zeta}^{\Delta^2} \end{pmatrix} = [W_n \mathcal{E}_n W_n^{-1}] \begin{pmatrix} \langle x^n H_{\mathsf{P}}^{\mathrm{S}} \rangle_{\zeta_H}^{\Delta^2} \\ \langle x^n H_{\mathsf{P}}^{g} \rangle_{\zeta_H}^{\Delta^2} \end{pmatrix} , \quad (64a)$$

$$\mathcal{E}_n = \begin{pmatrix} \left[\frac{\langle x H_{\mathsf{P}}^u \rangle_{\zeta}^{\Delta^2}}{\langle x H_{\mathsf{P}}^u \rangle_{\zeta_H}^{\Delta^2}} \right]^{\lambda_+^n/\gamma_0^1} & 0 \\ 0 & \left[\frac{\langle x H_{\mathsf{P}}^u \rangle_{\zeta}^{\Delta^2}}{\langle x H_{\mathsf{P}}^u \rangle_{\zeta_H}^{\Delta^2}} \right]^{\lambda_-^n/\gamma_0^1} \end{pmatrix} , \quad (64b)$$

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The particular case for **PDFs**:

$$\langle x^n\rangle_u^{\zeta} = \langle x^n\rangle_u^{\zeta_H} \left(\langle 2x\rangle_u^{\zeta}\right)^{9\gamma_0^n/32}, \quad (67a)$$

$$\begin{pmatrix} \langle x^n\rangle_{\mathsf{S}}^{\zeta} \\ \langle x^n\rangle_{g}^{\zeta} \end{pmatrix} = W_n \begin{pmatrix} [\langle 2x\rangle_u^{\zeta}]^{\lambda_+^n/\gamma_0^1} & 0 \\ 0 & [\langle 2x\rangle_u^{\zeta}]^{\lambda_-^n/\gamma_0^1} \end{pmatrix} \times W_n^{-1} \begin{pmatrix} \langle 2x^n\rangle_u^{\zeta_H} \\ 0 \end{pmatrix}. \quad (67b)$$

$$\langle 2x(\zeta_f)\rangle_q + \langle x(\zeta_f)\rangle_{\text{sea}} + \langle x(\zeta_f)\rangle_g = 1$$

More General: DGLAP

In **Mellin** space, the **all orders** evolution prescription entails:

$$\frac{\langle x^n H^u_{\mathsf{P}}\rangle^{\Delta^2}_{\zeta}}{\langle x^n H^u_{\mathsf{P}}\rangle^{\Delta^2}_{\zeta_H}} = \left(\frac{\langle x H^u_{\mathsf{P}}\rangle^{\Delta^2}_{\zeta}}{\langle x H^u_{\mathsf{P}}\rangle^{\Delta^2}_{\zeta_H}}\right)^{\gamma_0^n/\gamma_0^1} ;$$

$$\begin{pmatrix} \langle x^n H^{\mathsf{S}}_{\mathsf{P}}\rangle^{\Delta^2}_{\zeta} \\ \langle x^n H^{g}_{\mathsf{P}}\rangle^{\Delta^2}_{\zeta} \end{pmatrix} = [W_n \mathcal{E}_n W_n^{-1}] \begin{pmatrix} \langle x^n H^{\mathsf{S}}_{\mathsf{P}}\rangle^{\Delta^2}_{\zeta_H} \\ \langle x^n H^{g}_{\mathsf{P}}\rangle^{\Delta^2}_{\zeta_H} \end{pmatrix}, \quad (64a)$$

$$\mathcal{E}_n = \begin{pmatrix} \left[\frac{\langle x H^u_{\mathsf{P}}\rangle^{\Delta^2}_{\zeta}}{\langle x H^u_{\mathsf{P}}\rangle^{\Delta^2}_{\zeta_H}}\right]^{\lambda_+^n/\gamma_0^1} & 0 \\ 0 & \left[\frac{\langle x H^u_{\mathsf{P}}\rangle^{\Delta^2}_{\zeta}}{\langle x H^u_{\mathsf{P}}\rangle^{\Delta^2}_{\zeta_H}}\right]^{\lambda_-^n/\gamma_0^1} \end{pmatrix}, \quad (64b)$$

The particular case for $\boldsymbol{\theta_2}$:

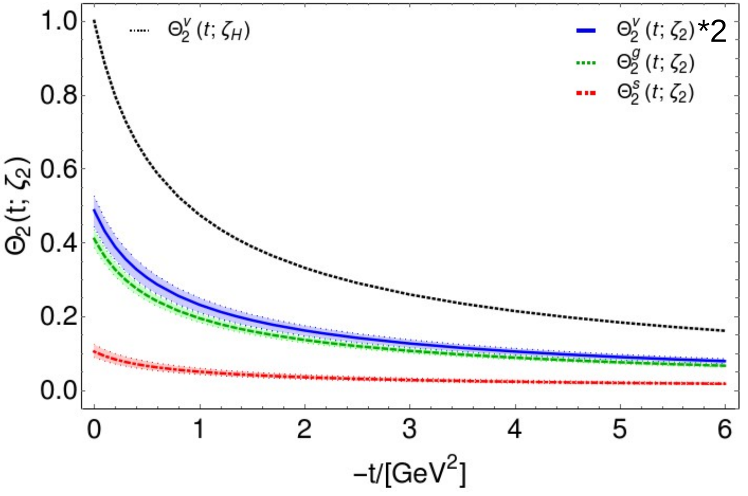
$$2\theta_2^{\pi \text{val}}(\Delta^2;\zeta) + \theta_2^{\pi \text{sea}}(\Delta^2;\zeta) = 2\theta_2^{\pi \text{val}}(\Delta^2;\zeta_{\mathcal{H}})\left[\frac{3}{7} + \frac{4}{7}(\langle 2x\rangle^{\zeta}_u)^{\frac{7}{4}}\right],$$

$$\theta_2^{\pi \text{g}}(\Delta^2;\zeta) = \frac{4}{7}2\theta_2^{\pi \text{val}}(\Delta^2;\zeta_{\mathcal{H}})\left[1 - (\langle 2x\rangle^{\zeta}_u)^{\frac{7}{4}}\right].$$

$$2\theta_2^{\pi \text{val}}(\Delta^2;\zeta) + \theta_2^{\pi \text{sea}}(\Delta^2;\zeta) + \theta_2^{\pi \text{g}}(\Delta^2;\zeta) = 2\theta_2^{\pi \text{val}}(\Delta^2;\zeta_{\mathcal{H}}) = \theta_2^{\pi}(\Delta^2).$$

More General: DGLAP

In **Mellin** space, the **all orders** evolution prescription entails:



$\zeta = \zeta_2$	mass-squared fraction			
	u	$\bar{h}$	g	sea
m_π^2	0.24(2)	0.24(2)	0.41(2)	0.11(2)
m_K^2	0.23(2)	0.27(2)	0.40(2)	0.10(2)

$$m_{\mathrm{P}}^2\,\theta_2^{\mathrm{P}}(\Delta^2 = 0) = m_{\mathrm{P}}^2$$

The particular case for **θ_2**:

$$2\theta_2^{\pi\,\mathrm{val}}(\Delta^2;\zeta) + \theta_2^{\pi\,\mathrm{sea}}(\Delta^2;\zeta)$$
$$= 2\theta_2^{\pi\,\mathrm{val}}(\Delta^2;\zeta_{\mathcal{H}})\left[\frac{3}{7} + \frac{4}{7}(\langle 2x\rangle_u^\zeta)^{\frac{7}{4}}\right],$$
$$\theta_2^{\pi\,\mathrm{g}}(\Delta^2;\zeta)$$
$$= \frac{4}{7}2\theta_2^{\pi\,\mathrm{val}}(\Delta^2;\zeta_{\mathcal{H}})\left[1 - (\langle 2x\rangle_u^\zeta)^{\frac{7}{4}}\right].$$

$$2\theta_2^{\pi\,\mathrm{val}}(\Delta^2;\zeta) + \theta_2^{\pi\,\mathrm{sea}}(\Delta^2;\zeta) + \theta_2^{\pi\,\mathrm{g}}(\Delta^2;\zeta)$$
$$= 2\theta_2^{\pi\,\mathrm{val}}(\Delta^2;\zeta_{\mathcal{H}}) = \theta_2^{\pi}(\Delta^2).$$

More General: DGLAP

The particular case for $\boldsymbol{\theta}_2$:

$$2\theta_2^{\pi\text{val}}(\Delta^2;\zeta) + \theta_2^{\pi\text{sea}}(\Delta^2;\zeta)$$
$$= 2\theta_2^{\pi\text{val}}(\Delta^2;\zeta_{\mathcal{H}})\left[\frac{3}{7} + \frac{4}{7}(\langle 2x\rangle_u^{\zeta})^{\frac{7}{4}}\right],$$
$$\theta_2^{\pi\text{g}}(\Delta^2;\zeta)$$
$$= \frac{4}{7}2\theta_2^{\pi\text{val}}(\Delta^2;\zeta_{\mathcal{H}})\left[1 - (\langle 2x\rangle_u^{\zeta})^{\frac{7}{4}}\right].$$

$$2\theta_2^{\pi\text{val}}(\Delta^2;\zeta) + \theta_2^{\pi\text{sea}}(\Delta^2;\zeta) + \theta_2^{\pi\text{g}}(\Delta^2;\zeta)$$
$$= 2\theta_2^{\pi\text{val}}(\Delta^2;\zeta_{\mathcal{H}}) = \theta_2^{\pi}(\Delta^2).$$