Multiresolution quantum field theory in light-front coordinates

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Subjects

W. N. Polyzou. In: *Phys. Rev. D* 101 (9 2020), p. 096004

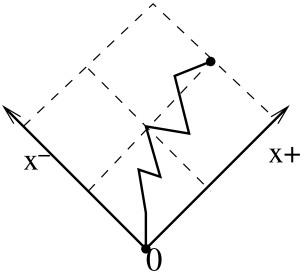
This paper

arXiv:2106.15706

$$S = \int dx^+ d\tilde{x} L(\phi(x^+, \tilde{x})),$$

$$x^+ = \frac{t+x}{\sqrt{2}}, x^- = \frac{t-x}{\sqrt{2}}$$

- Light-cone QFT
- Wavelet-based QFT
- Causality
- Measurement



Why scale-dependent functions?

$L^2(\mathbb{R})$ or not $L^2(\mathbb{R})$?

$\phi(x) \to \phi_a(x), \quad dx \to \frac{dxda}{a}$

- To localize a particle in an interval Δx the measuring device requests a momentum transfer of order $\Delta p \sim \hbar/\Delta x$. $\phi(x)$ at a point x has no experimental meaning. What is meaningful, is vacuum expectation of product of fields in a region around x

[MA *Phys. Rev. D* **81**(2010)125003]

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- If the particle, described by $\phi(x)$, have been initially prepared on the interval $(x - \frac{\Delta x}{2}, x + \frac{\Delta x}{2})$, the probability of registering it on this interval is ≤ 1: for the registration depends on the strength of interaction and the ratio of typical scales related to the particle and to the equipment.

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- Statement of existence: if a measuring equipment with a given resolution a fails to register an object, prepared on spatial interval of width Δx with certainty, then tuning the equipment to *all* possible resolutions a' would lead to the registration. $\int |\phi_a(x)|^2 d\mu(a,x) = 1$

[MA *Phys. Rev. D* **81**(2010)125003]

Continuous Wavelet Transform

[Carey, 1976,Bull. Austr. Math. Soc. **15**, 12; Duflo and Moore, 1976, J. Func. Anal., **21**, 209]:
Let $\mathcal{H}$ be a Hilbert space, G be a locally compact Lie group acting on $\mathcal{H}$, $d\mu(\nu), \nu \in G$ be a left-invariant measure on G, with a representation $U(\nu)$. $\forall|\phi\rangle \in \mathcal{H}$

$$|\phi\rangle = \frac{1}{C_\chi} \int_G U(\nu)|\chi\rangle d\mu(\nu)\langle\chi|U^*(\nu)|\phi\rangle$$

$|\chi\rangle \in \mathcal{H}$ is the *basic wavelet*, which satisfies the admissibility condition $C_\chi = \frac{1}{\|\chi\|^2} \int_G |\langle\chi|U(\nu)|\chi\rangle|^2 d\mu(\nu) < \infty$. $\langle\chi|U^*(\nu)|\phi\rangle$ are the *coefficients of wavelet decomposition*

Let

$$G : x' = ax + b, x, b \in \mathbb{R}^d, a \in \mathbb{R}_+,$$

be the affine group $\mathbb{R}^d$, with

$$U(a,b)\chi(x) = \frac{1}{a^d}\chi\left(\frac{x-b}{a}\right)$$

being its (L^1-normalized) representation. Then

$$\phi_a(b) = \int_{\mathbb{R}^d} \frac{1}{a^d}\overline{\chi\left(\frac{x-b}{a}\right)}\phi(x)d^dx$$

are the wavelet coefficients of the function $\phi \in L^2(\mathbb{R}^d)$ with respect to the basic wavelet χ.

$$\phi(x) = \frac{1}{C_\chi}\int \frac{1}{a^d}\chi\left(\frac{x-b}{a}\right)\phi_a(b)\frac{d^dbda}{a}$$

Multiresolution analysis (MRA)

Mallat sequence

Increasing sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}, V_j \in L^2(\mathbb{R})$:

1. $\ldots \subset V_0 \subset V_1 \subset V_2 \subset \ldots \subset L^2(\mathbb{R})$
2. $\mathrm{clos}_{L^2} \cup_{j\in\mathbb{Z}} V_j = L^2(\mathbb{R})$
3. $\cap_{j\in\mathbb{Z}} V_j = \emptyset$
4. V_j and V_{j+1} are "similar":

$$f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}.$$

If a set of functions $\varphi^0_k \equiv \varphi(x-k)$ forms a basis in V_0, then the *scaling functions*

$$\varphi^j_k = 2^{\frac{j}{2}}\varphi(2^j x - k)$$

form a basis in V_j.

Any function $f(x) \in V_0$ can be written as a sum of basic functions from V_1:

$$f(x) = \sum_k c_k 2^{\frac{1}{2}}\varphi(2x-k).$$

Thus $V_1 = V_0 \oplus W_0$, where $W_j := V_{j+1} \setminus V_j$:

$$V_2 = V_1 \oplus W_1 = V_0 \oplus W_0 \oplus W_1,$$

and so on. The basic functions in orthogonal complements W_j are referred to as *wavelet functions*

$$\chi^j_k(x) = 2^{\frac{j}{2}}\chi(2^j x - k).$$

Requirements of the orthonormality of basic functions and compactness of their support on $[0, 2N-1]$ for some $N \in \mathbb{N}$ enables the iterative construction of the basic wavelets from the scaling equation:

$$\varphi(x) = \sqrt{2}\sum_k h_k \varphi(2x-k),$$

from where the basic wavelet functions are derived I. Daubechies. "Orthonormal bases of compactly supported wavelets". In: *Comm. Pure. Apl. Math.* 41 (1988), pp. 909–996:

$$\chi(x) = \sqrt{2}\sum_{k=0}^{2N-1} g_k \varphi(2x-k), \quad g_k = (-1)^k h_{2N+1-k}.$$

I. Daubechies. *Ten lectures on wavelets*. Philadelphie: S.I.A.M., 1992

If $\{V_j\}$ chain is bounded from above by the best resolution space V_M, we can decompose this data into projections on

$$W_{M-1} \oplus \ldots \oplus W_2 \oplus W_1 \oplus W_0 \oplus V_0$$

by applying a pair of filters (h, g):

$$c_i^{j-1} = \sum_{k=0}^{2N-1} h_k c_{k+2i}^j,$$

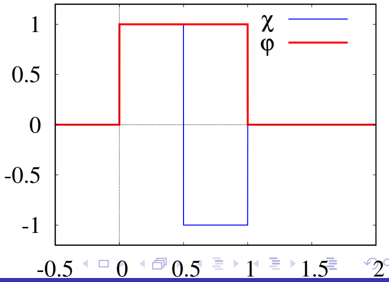
$$d_i^{j-1} = \sum_{k=0}^{2N-1} g_k c_{k+2i}^j,$$

where c_i^j are coefficients of the projection on V_j; d_i^j – on W_j.

Haar wavelet: $N = 1$, $h_0 = h_1 = \frac{1}{\sqrt{2}}$

$$\varphi(x) = \begin{cases} 1: & 0 \le x \le 1, \\ 0: & \text{otherwise} \end{cases},$$

$$\chi(x) = \begin{cases} +1, & 0 \le x < 1/2 \\ -1, & 1/2 \le x < 1 \\ 0, & \text{otherwise} \end{cases}.$$



Euclidean scale-dependent QFT

Euclidean QFT (ϕ^4): $S_E[\phi] = \int_{\mathbb{R}^d} \left[\frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4\right] d^d x$

$$Z[J] = \mathcal{N} \int \mathcal{D}\phi \exp\left(-S_E[\phi] + \int J(x)\phi(x)d^d x\right), \quad \phi(x) := \langle x|\phi\rangle$$

If we want the fields to depend on scale (and other parameters) of observation, we need $\phi_{a\theta}(x) := \langle x, a, \theta; \chi|\phi\rangle$.
In isotropic [$SO(d)$-invariant] case

$$\phi(x) \to \phi_a(x) \equiv \langle x, a, \chi|\phi\rangle, \quad d^d x \to \frac{da d^d x}{C_\chi a}$$

$$Z_W[J_a(x)] = \int \mathcal{D}\phi_a(x) \exp\left(-S_W[\phi_a(x)] + \int \phi_a(x) J_a(x) \frac{da d^d x}{C_\chi a}\right)$$

M. V. Altaisky. "Quantum field theory without divergences". In: *Phys. Rev. D* 81 (2010), p. 125003.

Feynman diagrams in multiscale QFT

1. Each field $\tilde\phi(k)$ is substituted by the scale component $\tilde\phi(k) \to \tilde\phi_a(k) = \overline{\tilde\chi(ak)}\tilde\phi(k)$.
2. Each integration in the momentum variable is accompanied by the corresponding scale integration
$$\frac{d^dk}{(2\pi)^d} \to \frac{d^dk}{(2\pi)^d}\frac{da}{a}\frac{1}{C_\chi}.$$
3. Each interaction vertex is substituted by its wavelet transform; for the N-th power interaction vertex, this gives multiplication by the factor $\displaystyle\prod_{i=1}^{N}\tilde\chi(a_ik_i)$.

The finiteness of the loop integrals is provided by the following rule: *There should be no scales a_i in internal lines smaller than the minimal scale of all external lines* ($A = \min_{k\in E} a_k$):
$$\int_A^\infty |\tilde\chi(a_ip)|^2\frac{da_i}{C_\chi a_i} \times \int_A^\infty |\tilde\chi(a_jp)|^2\frac{da_j}{C_\chi a_j},$$

Multiscale Green functions

The Green functions

$$\langle \phi_{a_1}(x_1)\cdots\phi_{a_n}(x_n)\rangle_c = \left.\frac{\delta^n \ln Z_W[J_a]}{\delta J_{a_1}(x_1)\ldots\delta J_{a_n}(x_n)}\right|_{J=0},$$

are cumulants of the field $\phi_a(x)$.
The bare Green function in wavelet representation takes the form

$$G_0^{(2)}(a_1,a_2,p) = \frac{\tilde\chi(a_1p)\tilde\chi(-a_2p)}{p^2+m^2}.$$

The integration over the internal scale variables a_i results in a squared wavelet cutoff factors $f^2(Ap)$ in each diagram line, where

$$f(x) = \frac{1}{C_\chi}\int_x^\infty |\tilde\chi(a)|^2\frac{da}{a}, \quad \boxed{\tilde\chi_1(k) = -\imath k e^{-\frac{k^2}{2}}, f_{\chi_1}(x) = e^{-x^2}}$$

for isotropic wavelets. Normalization condition $f(0) = 1$ corresponds to the divergent theory in the infinite resolution limit $A\to 0$.

Scale-dependent vertex functions

As usual in functional renormalization group technique [C. Wetterich. "Exact evolution equation for the effective potential". In: *Phys. Lett. B* 301.1 (1993), pp. 90 –94], we can introduce the effective action functional

$$\Gamma[\phi_a(x)] = -\ln Z_W[J_a(x)] + \int J_a(x)\phi_a(x)\frac{da\,d^dx}{C_\chi a},$$

the functional derivatives of which are the vertex functions. We can express it in a form of perturbation expansion:

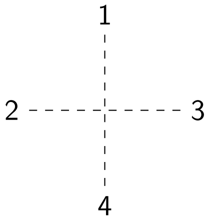
$$\Gamma_{(A)}[\phi_a] = \Gamma^{(0)}_{(A)} + \sum_{n=1}^{\infty}\int \Gamma^{(n)}_{(A)}(a_1, b_1, \ldots, a_n, b_n)\times$$

$$\times\, \phi_{a_1}(b_1)\ldots\phi_{a_n}(b_n)\frac{da_1 d^d b_1}{C_\chi a_1}\ldots\frac{da_n d^d b_n}{C_\chi a_n}$$

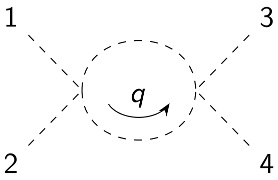
The subscript (A) indicates the presence in the theory of minimal scale – the observation scale.

Vertex renormalization ϕ^4 [MA *Phys. Rev.* D**93**(2016)105043]

$\Gamma^{(4)} = -$



$- \frac{3}{2}$



Using $\tilde{\chi}_1(k) = -\imath k e^{-\frac{k^2}{2}}$ wavelet results in the cutoff factor $f_{\chi_1}(x) = e^{-x^2}$. In four dimensions in the relativistic limit $s^2 \gg 4m^2$ we get the following scaling equation for the coupling constant $\lambda = \lambda^{eff}(A)$:

$$\frac{\partial \lambda}{\partial \mu} = \frac{3\lambda^2}{16\pi^2} \frac{2\alpha^2 + 1 - e^{\alpha^2}}{\alpha^2} e^{-2\alpha^2},$$

where $\mu = -\ln A + const$, $\quad \alpha = As, \quad s = p_1 + p_2.$

Wavelet renormalization [No renormalization of fields is required]

ϕ^4: M. V. Altaisky. "Unifying renormalization group and the continuous wavelet transform". In: *Phys. Rev. D* 93 (10 2016), p. 105043

Quantum electrodynamics: M. V. Altaisky and R. Raj. "Wavelet regularization of Euclidean QED". In: *Phys. Rev. D* 102 (12 2020), p. 125021

Quantum chromodynamics: M. V. Altaisky. "Wavelet regularization of gauge theories". In: *Phys. Rev. D* 101 (10 2020), p. 105004

Stochastic dynamics: M. V. Altaisky, M. Hnatich, and N. E. Kaputkina. "Renormalization of viscosity in wavelet-based model of turbulence". In: *Phys. Rev. E* 98 (3 2018), p. 033116

Wavelet bases in Minkowski space

E. Gorodnitskiy and M.Perel, *J. Math. Phys.* **45**(2012)385203

In the Minkowski space we cannot define wavelet transform using a single mother wavelet. This is because the group $SO(1,1)$ of Lorentz transformations is not a simply-connected group, but includes 4 connected components

$$\begin{pmatrix} \mathrm{ch}\,\eta & \mathrm{sh}\,\eta \\ \mathrm{sh}\,\eta & \mathrm{ch}\,\eta \end{pmatrix}, \begin{pmatrix} \mathrm{ch}\,\eta & -\mathrm{sh}\,\eta \\ \mathrm{sh}\,\eta & -\mathrm{ch}\,\eta \end{pmatrix}, \begin{pmatrix} -\mathrm{ch}\,\eta & \mathrm{sh}\,\eta \\ -\mathrm{sh}\,\eta & \mathrm{ch}\,\eta \end{pmatrix}, \begin{pmatrix} -\mathrm{ch}\,\eta & -\mathrm{sh}\,\eta \\ -\mathrm{sh}\,\eta & -\mathrm{ch}\,\eta \end{pmatrix}$$

parametrized by the rapidity $\mathrm{th}\,(\eta) = v/c$. Wavelet transform in $\mathbb{R}^{1,1}$ requires 4 separate wavelets

$$\chi_j(x) = \int_{A_j} \frac{d\omega dk}{(2\pi)^2} \tilde{\chi}(k) e^{-\imath(\omega t - kx)},$$

different from each other by their support in momentum space:

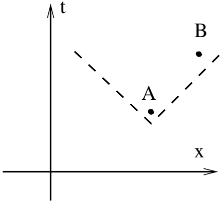
$$A_1 : |\omega| > |k|, \omega > 0, \qquad A_2 : |\omega| > |k|, \omega < 0,$$
$$A_3 : |\omega| < |k|, \omega > 0, \qquad A_4 : |\omega| < |k|, \omega < 0.$$

Goal

What we want is a Lorentz-invariant theory with the spacetime regions being spanned by some wavelet basis in a way totally symmetric with respect to the space and the time variables.

This goal is not easy to achieve because of the causality issues.

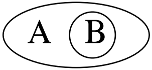
There is only one causality relation ($\prec$) in local QFT, but two ($\prec, \subset$) in scale-dependent QFT



Event A can causally affect event B only within the future-directed light cone

$\prec$ – signal causality

$\subset$ – the whole – the part causality



M. V. Altaisky and N. E. Kaputkina. "Continuous wavelet transform in quantum field theory". In: *Phys. Rev. D* 88 (2 2013), p. 025015 M. Altaisky and N. Kaputkina. "On the wavelet decomposition in light cone variables". In: *Russian Physics Journal* 55.10 (2013), pp. 1177–1182

Definition

A set of regions $A, B, C, \ldots \in \mathcal{Z}$ with two partial orders, such that:

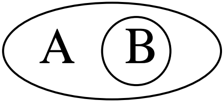
1. The subset relation $\subset$ is a partial order on the set of regions: $A \subset B \wedge B \subset C \implies A \subset C$; $A \subset A$, $A \subset B \wedge B \subset A \implies A = B$
2. The partial order $\subset$ has a minimum element: $\forall A, \emptyset \subset A$
3. The partial order $\subset$ has unions: $A \subset A \cup B$, $B \subset A \cup B$; if $A \subset C \wedge B \subset C \implies A \cup B \subset C$
4. Relation $\prec$ induces a strict partial order on the non-empty regions: $A \prec B \wedge B \prec C \implies A \prec C$; $A \not\prec A$.
5. $\forall A, B, C$:
   $A \subset B \wedge B \prec C \implies A \prec C$,
   $A \subset B \wedge C \prec B \implies C \prec A$,
   $A \prec C \wedge B \prec C \implies A \cup B \prec C$

is called a causal site

Measurements and probability

Probability theory:

$$P(\phi_B) = \int P(\phi_B|\phi_A)P(\phi_A)\mathcal{D}\phi_A$$

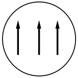


In Euclidean space

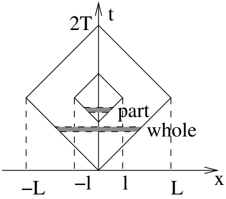
$$\langle \phi_{a_1}(x_1)\ldots\phi_{a_n}(x_n)\rangle$$

is well defined.

The measurement of $S_z = +\frac{3}{2}$



rules out either of $s_i = -\frac{1}{2}$



In Minkowski space we need light-front variables

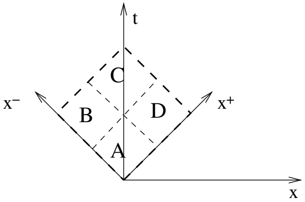
Light front variables

$$x^+ = \frac{t+x}{\sqrt{2}}, x^- = \frac{t-x}{\sqrt{2}},$$
$$x^2 = 2x^+x^- - (x^\perp)^2$$

The standard (signal) ordering implies a partial order on the set $\Theta = A \cup B \cup C \cup D$:

$$A \prec B \prec C, A \prec D \prec C.$$

How can we define functional integration on Θ?



The regions, B and D, being separated by a space-like interval, are not causally ordered. In the picture B and D are simultaneous, but in other Lorentz frames it may be either $B \prec D$, or $D \prec B$.

In standard approach x^+ is taken as a 'time', so that $B \prec D$.

Polyzou, W.N. *Phys. Rev. D* **101**(2020)096004

Path integration over event regions: one dimension

Transition amplitude:

$$\langle Q'|Q\rangle \propto \int \mathcal{D}q e^{\frac{\imath}{\hbar}\int_0^T L[q(t)]dt}$$

Feynman measure

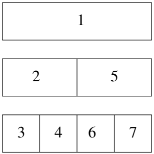
$$\mathcal{D}q = \prod_{i=1}^{n} dq(t_i), \quad \max_i (t_i - t_{i-1}) \to 0.$$

Discrete wavelet transform

$$d_n^m := \int_0^T 2^{-\frac{m}{2}} \bar{\chi}\left(2^{-m}t - nb_0\right) q(t)dt.$$

Integral over the scale variable $\frac{da}{a}$ becomes discrete sum $\sum_m d_n^m \chi_{mn}(t)$

$$q_0, q_1, q_2, q_3 \to d_0^1, d_1^1, d_0^2, c_0^2.$$



$$c_0^2 = \frac{q_0 + q_1 + q_2 + q_3}{2},$$

$$d_0^2 = \frac{q_0 + q_1 - q_2 - q_3}{2},$$

$$d_0^1 = \frac{q_0 - q_1}{\sqrt{2}},$$

$$d_1^1 = \frac{q_2 - q_3}{\sqrt{2}}$$

Wavelet transform on $[0,T]\otimes[0,T]$ in (x^+,x^-)-plane

To store the information of each 4 points of the j hierarchy level $(c^j_{2k,2m}, c^j_{2k+1,2m}, c^j_{2k,2m+1}, c^j_{2k+1,2m+1})$ we need 4 basic functions:

$$\begin{array}{cc} \varphi(x^+)\varphi(x^-) & \chi(x^+)\varphi(x^-) \\ \varphi(x^+)\chi(x^-) & \chi(x^+)\chi(x^-), \end{array}$$

This gives 4 different wavelet coefficients:

$$c^{j+1}_{k,m} = \frac{c^j_{2k,2m} + c^j_{2k,2m+1} + c^j_{2k+1,2m} + c^j_{2k+1,2m+1}}{2},$$
$$d^{(1),j+1}_{k,m} = \frac{c^j_{2k,2m} - c^j_{2k,2m+1} + c^j_{2k+1,2m} - c^j_{2k+1,2m+1}}{2},$$
$$d^{(2),j+1}_{k,m} = \frac{c^j_{2k,2m} + c^j_{2k,2m+1} - c^j_{2k+1,2m} - c^j_{2k+1,2m+1}}{2},$$
$$d^{(3),j+1}_{k,m} = \frac{c^j_{2k,2m} - c^j_{2k,2m+1} - c^j_{2k+1,2m} + c^j_{2k+1,2m+1}}{2}.$$

Light front QFT in $(1+1)$ dimensions

Considering the square domain $D = [0, T] \otimes [0, T]$ in the (x^+, x^-) plane, and the action functional

$$S[\phi] = \int_0^T dx^+ \int_0^T dx^- [\frac{\partial \phi}{\partial x^+} \frac{\partial \phi}{\partial x^-} - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4],$$

originated from the standard Lagrangian of ϕ^4 theory, We can formally decompose the field $\phi(x^+, x^-)$ into the scale components

$$\phi(x^+, x^-) = \sum d^{m_1,m_2}_{j,k_1,k_2} \chi^{m_1}_{j,k_1}(x^+) \chi^{m_2}_{j,k_2}(x^-),$$

where the upper indices $m_1, m_2 \in \{h, g\}$ designate the type of basic function: $\chi^h \equiv \varphi, \chi^g \equiv \chi$. Similar decomposition can be written for a full four-dimensional case of $\phi(x^+, x^-, \boldsymbol{x}_\perp)$.

Correspondence to the ordinary theory

DWT-based generating functional

$$Z[J] = \int \mathcal{D}d^{m_1,m_2,\ldots}_{j,k_1,k_2,\ldots} e^{\frac{\imath}{\hbar}S[d^{m_1,m_2,\cdots}_{j,k_1,k_2,\ldots}]+\imath d^{m_1,m_2,\cdots}_{j,k_1,k_2,\ldots} J^{m_1,m_2,\cdots}_{j,k_1,k_2,\ldots}}$$

Mass term: $\frac{m^2}{2}\int \phi^2 dx^+ dx^- \to \frac{m^2}{2}\sum |d^{m_1 m_2}_{j,k_1 k_2}|^2$
Source term: $\int J(x)\phi(x)dx^+ dx^- \to \sum J^{m_1,m_2}_{j,k_1,k_2} d^{m_1 m_2}_{j,k_1 k_2}$.
Kinetic term:

$$\int \frac{\partial\phi}{\partial x^+}\frac{\partial\phi}{\partial x^-}dx^+ dx^- = -d^{m_1',m_2'}_{j',k_1',k_2'} d^{m_1,m_2}_{j,k_1,k_2} \Omega^{m_1',m_1}_{j',k_1-k_1'} \Omega^{m_2',m_2}_{j,k_2-k_2'} =$$

$$= -\int d^{m_1',m_2'}_{j',k_1',k_2'} \chi^{m_1'}_{j',k_1'}(x^+)\chi^{m_2'}_{j',k_2'}(x^-) d^{m_1,m_2}_{j,k_1,k_2} \frac{\partial\chi^{m_1}_{j,k_1}(x^+)}{\partial x^+}\frac{\partial\chi^{m_2}_{j,k_2}(x^-)}{\partial x^-} dx^+ dx^-$$

Connection coefficients $\Omega^{m',m}_{j,k-k'} := \int dx \chi^{m'}_{j,k'}(x)\frac{\partial\chi^m_{j,k}(x)}{\partial x}$

J. M. Restrepo and G. K. Leaf. "Inner product computations using periodized Daubechies wavelets". In: *International Journal for Numerical Methods in Engineering* 40.19 (1997), pp. 3557–3578

Thank You for your attention !

$$W^i_{ab\eta\phi} = \int_{A_i} e^{\imath k_- b_+ + \imath k_+ b_- - \imath \boldsymbol{k}_\perp \boldsymbol{b}_\perp} \tilde{f}(k_-, k_+, \boldsymbol{k}_\perp)$$

$$\times \bar{\tilde{\chi}}(ae^{\eta} k_-, ae^{-\eta} k_+, aR^{-1}(\phi)\boldsymbol{k}_\perp) \frac{dk_+ dk_- d^2\boldsymbol{k}_\perp}{(2\pi)^4}$$