Confinement in Two-Dimensional QCD and the Infinitely Long Pion
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- Recent work by Li &Vary, de Teramond & Brodsky, Ahemedy et al., Chabysheva & Hiller intended to include non-vanishing quark masses and extend LF Holography from (1+2) to (1+3) dimensions

- Scheckler & I 2101.00100 realized it was necessary to include longitudinal dynamics to obtain complete set of states in (1+3)

- Paper was written because I wanted to understand what was going on

Summary of LF Holography

- Brodsky et al. Phys. Rept. 584, 1– 105 (2015), arXiv:1407.8131 [hep-ph].

Two-parton LF equation

$$\left[\frac{m_1^2}{x} + \frac{m_2^2}{1-x} + \frac{k_\perp^2}{x(1-x)} + V_{\text{eff}}\right]\psi = M_h^2\psi$$

Chiral limit $m_1, m_2 \to 0$ $\zeta = \mathbf{b}_\perp\sqrt{x(1-x)}$ $V_{\text{eff}} \to U_\perp(\zeta)$

$$\left(-\frac{d^2}{d\zeta^2} + \frac{L^2}{\zeta^2} + U_\perp(\zeta)\right)\varphi(\zeta) = M^2\varphi(\zeta)$$

Equation of motion in soft-wall model

$$U_\perp(\zeta) = \kappa^4\zeta^2 + 2\kappa^2(J-1)$$

Spectroscopy & massless pion ✓

x is held fixed, need longitudinal confining equation

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Longitudinal dynamics

$$V_{\text{eff}} = U_\perp(\zeta) + V_\parallel(x)$$

$$\left[\frac{m_1^2}{x} + \frac{m_2^2}{1-x} + V_\parallel\right] X_n(x) = M_\parallel^2 X_n(x)\,.$$

$$\psi(x, \mathbf{b}) = \frac{\varphi(\zeta)}{\sqrt{\zeta}} X_n(x)$$

$$\int_0^1 \frac{|X_n(x)|^2}{x(1-x)}\, dx = 1$$

$$\chi_n(x) = X_n(x)/\sqrt{x(1-x)}$$

Early applications ($m_{1,2} = 0, V_\parallel = 0$):

$$X(x) = \sqrt{x(1-x)}, \quad \chi = 1$$

Comment: QCD potential is not a sum of two independent terms. Product wave functions form a basis

Longitudinal dynamics with Hermitian confining potential ($H_{\|}$)

$H_{\|}\chi_n = M_n^2 \chi_n$ 'tHooft, Callen et al, Brower et al,Ellis …

$$\int dx\, \chi_n^*(x)\chi_m(x) = \delta_{nm}$$

$$\langle n \,|\, H_{\|} \,|\, m \rangle = \int dx\, dy\, \chi_n^*(x) H_{\|}(x,y) \chi_m(y)$$

$$H_{\|}(x,y) = \frac{m^2}{x(1-x)}\delta(x-y) + V_{||}(x,y)$$

Confining potential is off-diagonal in momentum, because it depends on the relative spatial coordinate $\tilde{z}$ (Miller & Brodsky 2019), canonically conjugate to momentum fraction x

$$\langle n \,|\, H_{\|} \,|\, m \rangle = \int dx \frac{m^2}{x(1-x)} \chi_n^*(x)\chi_m(x) + \int dx\, dy \chi_n^*(x) V_{||}(x,y) \chi_m(y)\,.$$

Three $V_{||}$

1 $$V_{\mathrm{LV}}(x)\chi_n(x) = -\sigma^2\partial_x\, x(1-x)\partial_x\chi_n(x)$$

2 $$(V_{\mathrm{tH}}\,\chi_n)(x) = \frac{g^2}{\pi}P\int_0^1 dy\frac{\chi_n(x)-\chi_n(y)}{(x-y)^2}$$

$$P\frac{f(x,y)}{(x-y)^2} \equiv \frac{1}{2}\left[\frac{f(x,y)}{(x-y+i\epsilon)^2}+\frac{f(x,y)}{(x-y-i\epsilon)^2}\right] \qquad (\epsilon\to 0)$$

P Comes from confining potential proportional to $|\tilde{z}|\,e^{-\epsilon|\tilde{z}|}$

Includes quark self-energy, m is current quark mass

3 $$\chi(x) = \mathcal{N}\exp[-1/(2\kappa^2)(-m_1^2/x + m_2^2/(1-x))]$$

Invariant mass wave function (IMWF) (Brodsky:2014yha)

1,2 use $\displaystyle\int dx\,\chi_n^*(x)\chi_m(x) = \delta_{nm}$

Using $$\int_0^1 \frac{|X_n(x)|^2}{x(1-x)}\,dx = 1$$

$$X_n(x) \equiv \sqrt{x(1-x)}\chi_n(x)$$

$$\langle n|H_{||}|m\rangle = \int \frac{dx}{x(1-x)}\tilde{X}_n(x)\frac{m^2}{x(1-x)}X_m(x) + \langle n|V_{||}|m\rangle$$

$$\langle n|V_{||}|m\rangle = \int \frac{dx\,dy}{x(1-x)}\tilde{X}_n(x)\frac{\sqrt{x(1-x)}}{\sqrt{y(1-y)}}V_{||}(x,y)X_m(y)$$

The potential in red box is not Hermitian, thus tilde on X_n

With X-normalization 't Hooft eq. becomes

$$M_n^2 X_n(x) = \frac{m^2}{x(1-x)}X_n(x) - \frac{g^2}{\pi}P\int dy \frac{\left(X_n(x) - \frac{\sqrt{x(1-x)}}{\sqrt{y(1-y)}}X_n(y)\right)}{(x-y)^2}$$

- Chabysheva & Hiller solved this eq

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Differences between two potentials

1 $$V_{\rm LV}(x)\chi_n(x) = -\sigma^2\partial_x\, x(1-x)\partial_x\chi_n(x)$$

2 $$(V_{\rm tH}\,\chi_n)(x) = \frac{g^2}{\pi}P\int_0^1 dy\frac{\chi_n(x)-\chi_n(y)}{(x-y)^2}$$

Similarities first - both have same chiral limit ground state $\chi_n \to 1$

Both obey $M_{\|}^2\int_0^1 dx\,\chi_n(x) = \int_0^1 dx\,\chi_n(x)\left[\frac{m_1^2}{x}+\frac{m_2^2}{1-x}\right]$ χ_n vanishes at x=0,1

Potentials seem similar, but no: coordinate space

$V_{\rm tH}$ from confining potential proportional to $|\tilde z|e^{-\epsilon|\tilde z|}$

$$\langle\tilde z|V_{\rm LV}(x)|\tilde z'\rangle = \frac{\sigma^2}{2\pi}\tilde z\tilde z' e^{i\frac{(\tilde z'-\tilde z)}{2}}\frac{j_1(\frac{\tilde z'-\tilde z}{2})}{\tilde z'-\tilde z},$$

High energy spectrum is very different $M_{\rm LV}^2 \sim k^2, \quad M_{\rm tH}^2 \sim k$

Wave equation & spectrum for IMWF

Use $m_{1,2} = m, \quad y^2 \equiv \dfrac{1}{x(1-x)}$

If $$-\frac{\kappa^4}{m^2}\phi'' + m^2 y^2 \phi = M^2 \phi, \qquad \phi(y) = e^{\frac{-m^2 y^2}{2\kappa^2}}$$

Wave equation exists

$$M^2 = \kappa^2, 3\kappa^2, 5\kappa^2 \cdots \qquad \text{independent of } m$$

Not very reasonable

Small current quark masses

$$M_{\|}^2(\mathrm{LV}) = 2\sigma m + 4m^2 \qquad m = 15\,\mathrm{MeV},\ \sigma = 620\,\mathrm{MeV}$$

$$\mathrm{t'Hooft}\ \chi_0(x) \propto x^\beta(1-x)^\beta,\ \beta = \sqrt{\frac{3}{\pi}\frac{m}{g}}$$

$$M_{\|}^2(\mathrm{tH}) = 2\sqrt{\frac{\pi}{3}}gm + 4m^2\,.$$

$$m = 3.5\,\mathrm{MeV}\ \mathrm{g} = 2700\,\mathrm{MeV}$$

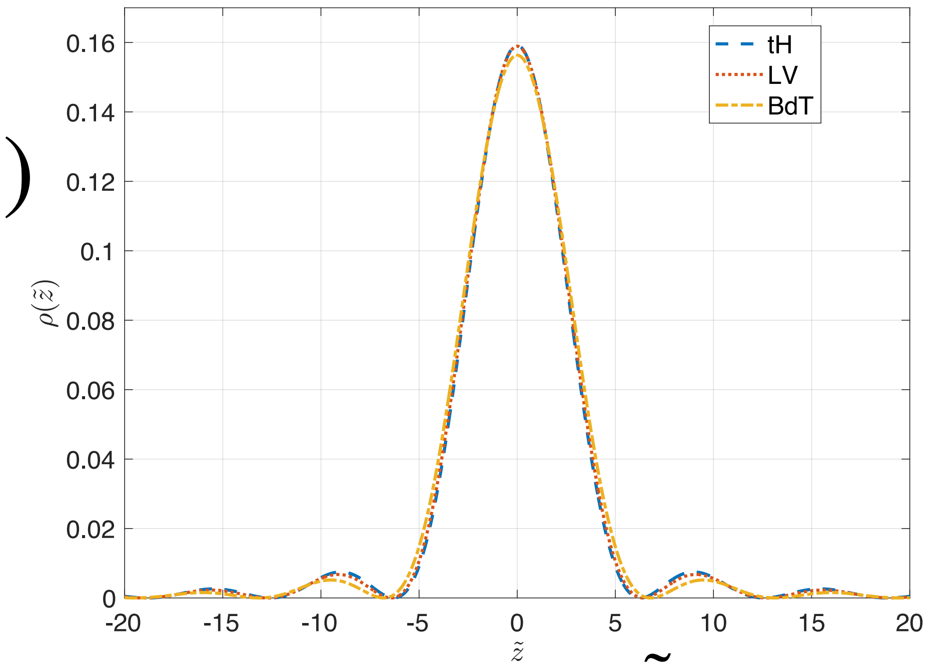
Both models have 1+1 dimensional version of Gell-Mann Oakes Renner

Spectra of two models different because of parameters
With this 't Hooft model preserves spectrum of original LF holography because excited states of very high mass in unobserved region of spectra

Coordinate-space confinement

$$\chi(\tilde{z}) \equiv \int_0^1 \frac{dx}{\sqrt{2\pi}} e^{ix\tilde{z}} \chi(x) . \qquad \rho(\tilde{z}) \equiv |\chi(\tilde{z})|^2$$

$\rho(\tilde{z})$



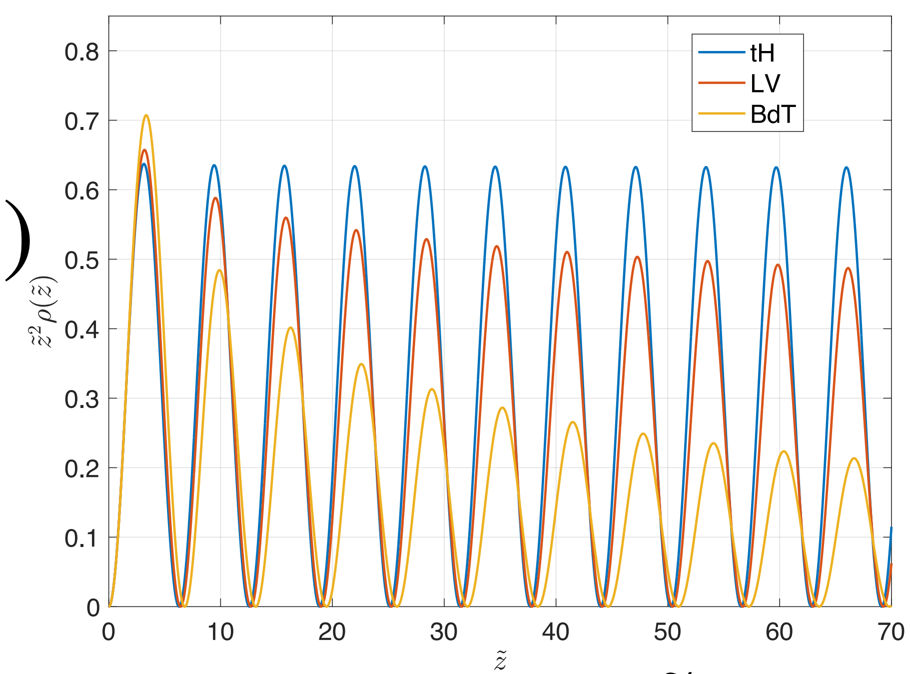
$\tilde{z}$

(a)

Chiral limit $\rho_\chi(\tilde{z}) = \frac{2}{\pi} \frac{\sin^2 \tilde{z}/2}{\tilde{z}^2}$

$$\langle \chi | \tilde{z}^2 | \chi \rangle = \frac{2}{\pi} \int_{-\infty}^{\infty} dz \, \sin^2(\tilde{z}/2) = \infty$$

$\tilde{z}^2 \rho(\tilde{z})$



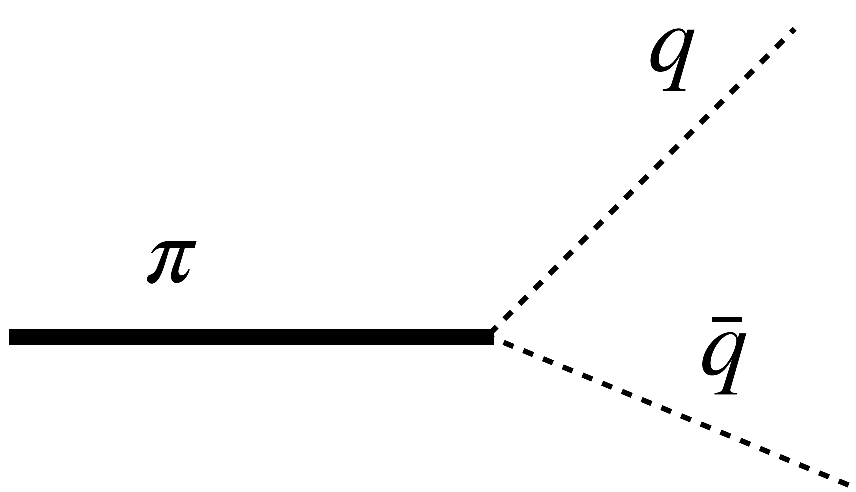
$\tilde{z}$

(b)

Longitudinal size is infinite

Infinitely long pion

- No problem for form factor, only transverse momentum transfer
- Infinite size like Ioffe time , ΔE for pion to $q\bar{q} = 0$



Summary

- The three models of longitudinal confinement are very different, LV ~ harmonic oscillator, tH linear, IMWF- potential depends on quark mass
- Consequences of hermiticity have been explored
- Product wave functions are basis states, better to use Hermitian, slight conflict with holography,
- Pion has infinite longitudinal size in chiral limit