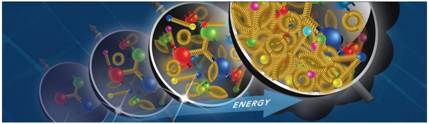
Going to the light front with contour deformations

GE, E. Ferreira, A. Stadler, in preparation

Gernot Eichmann
LIP & IST Lisboa

Light Cone 2021: Physics of Hadrons on the Light Front
Dec 1, 2021

Motivation



Brookhaven National Laboratory

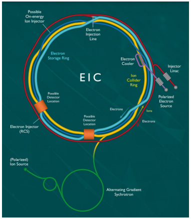
Quark-gluon structure of hadrons and nuclei

- Flavor structure of proton, pion and kaon
- Spin and orbital angular momentum
- 3D imaging of the nucleon and nuclei
- Origin of mass

Encoded in **parton distributions,** defined on the light front: PDFs, GPDs, TMDs, TDAs

Large experimental efforts

JLab, EIC, COMPASS/AMBER, PANDA, JPARC, LHC, ...



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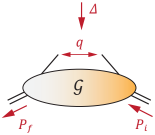
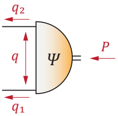
Parton distributions



Hadron-to-hadron correlator

$$\mathcal{G}(z, P, \Delta) = \langle P_f |\, \mathsf{T}\, \Phi(z)\, \mathcal{O}\, \Phi(0)\, | P_i \rangle$$



Bethe-Salpeter WF:
vacuum-to-hadron correlator

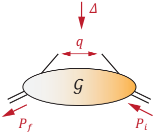
$$\Psi(z, P) = \langle 0 | \mathsf{T}\, \Phi(z)\, \Phi(0) | P \rangle$$

	$\mathcal{G}(q, P, \Delta = 0)$	$\mathcal{G}(q, P, \Delta)$	$\Psi(q, P)$
$\int dq^-$	TMD	GTMD	LFWF
$\int d^2\boldsymbol{q}_\perp \int dq^-$	PDF	GPD	PDA

Diehl, Phys. Rept. 388 (2003)
Belitsky, Radyushkin, Phys. Rept. 418 (2005)
Lorcé, Pasquini, Vanderhaeghen, JHEP 05 (2011)
...

Light-front wave functions:
coefficients of Fock expansion
in light-front quantization
Brodsky, Pauli, Pinsky, Phys. Rept. 301 (1998)

Parton distributions



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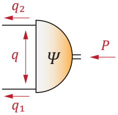

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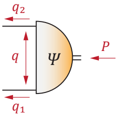
Progress in **lattice QCD:**
Quasi-PDFs, pseudo-PDFs, ...
Ji, PRL 110 (2013),
Radyushkin, PLB 767 (2017),
Lin et al., PPNP 100 (2018)
Constantinou et al., PPNP 121 (2021)

Nakanishi representation
Nakanishi 1963, 1969, 1988
Kusaka, Williams, PRD 51 (1995)
Sauli, Adam, PRD 67 (2003)
Karmanov, Carbonell, EPJ A 27 (2006)
Frederico, Salmè, Viviani, PRD 85 (2012)

Many continuum studies of
LFWFs, PDFs, GPDs, TMDs, ...
Tiburzi, Miller, PRD 65 (2002)
Nguyen, Bashir, Roberts, Tandy, PRC 83 (2011)
Chang et al., PRL 110 (2013)
Frederico, Salmè, Viviani, PRD 89 (2014)
Mezrag et al., PLB 741 (2015)
de Paula et al., PRD 94 (2016)
Mezrag, Segovia, Chang, Roberts, PLB 783 (2018)
Bednar, Cloet, Tandy, PRL 124 (2020)
Ding et al., PRD 101 (2020)
Serna et al., EPJ C 80 (2020)
Freese, Cloet, PRC 103 (2021)
Zhang et al., PLB 815 (2021)
Ydrefors, Frederico, 2108.02146
...

Light-front wave function



$q_1 = q + P/2$

$q_2 = -q + P/2$

LFWF = BSWF integrated over q^-

$$\psi(q^+, \boldsymbol{q}_\perp) = \mathcal{N} P^+ \int_{-\infty}^{\infty} \frac{dq^-}{2\pi}\, \Psi(q,P)$$

Light-front variables:

$$q^\pm = q^0 \pm q^3$$

$$\int d^4q = \tfrac{1}{2} \int d^2\boldsymbol{q}_\perp \int dq^+ \int dq^-$$

Introduce **momentum partitioning** $\alpha \in [-1,1]$ through

$$q = k + \tfrac{\alpha}{2} P \quad \Rightarrow \quad \begin{aligned} q_1 &= k + \tfrac{1+\alpha}{2} P \\ q_2 &= -k + \tfrac{1-\alpha}{2} P \end{aligned}$$

$k^+ = 0$

$\Rightarrow$ α plays role of longitudinal momentum fraction:

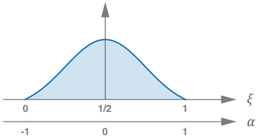
$$q_1^+ = \xi\, P^+ \qquad \xi = \tfrac{1+\alpha}{2}$$

$$q_2^+ = (1-\xi)\, P^+$$

Then:

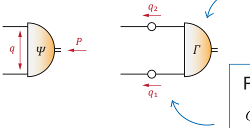
LFWF: $\psi(\alpha, \boldsymbol{k}_\perp) = \mathcal{N} P^+ \int_{-\infty}^{\infty} \frac{dq^-}{2\pi}\, \Psi(q,P)\Big|_{q^+ = \frac{\alpha}{2}P^+,\ \boldsymbol{q}_\perp = \boldsymbol{k}_\perp}$

PDA: $\phi(\alpha) = \dfrac{1}{16\pi^3 f} \int d^2\boldsymbol{k}_\perp\, \psi(\alpha, \boldsymbol{k}_\perp)$



Example: monopole

BSWF



BS amplitude: monopole

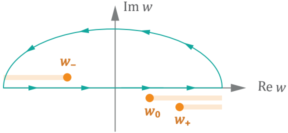
$$\Gamma(q,P) = -\frac{m^2}{q^2 - m^2\gamma + i\epsilon}$$

Propagators:

$$G_0(q,P) = \frac{i}{q_1^2 - m^2 + i\epsilon}\,\frac{i}{q_2^2 - m^2 + i\epsilon}$$

LFWF becomes

$$\psi(\alpha,x) = \frac{\mathcal{N}}{i\pi m^2}\,\frac{1}{\alpha\,(1-\alpha^2)}\int\limits_{-\infty}^{\infty} dw\,\frac{1}{w-w_+}\,\frac{1}{w-w_-}\,\frac{1}{w-w_0}$$



3 poles in complex w plane:

$$w_\pm = \pm\left(t + \frac{1+x-i\epsilon}{1\pm\alpha}\right)$$

$$w_0 = \frac{x+\gamma-i\epsilon}{\alpha}$$

Result:

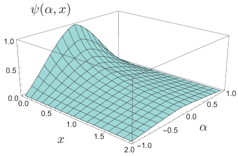
$$\psi(\alpha,x) = \frac{\mathcal{N}}{m^2}\,\frac{1}{x+A}\,\frac{1-|\alpha|}{x+A+(1-|\alpha|)\,B}\,, \qquad A = 1+(1-\alpha^2)\,t, \quad B = \gamma - 1 - t$$

Abbreviate:

$$t = -\frac{M^2}{4m^2} \in [-1, 0]$$

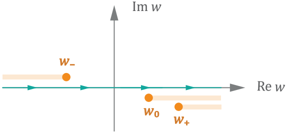
$$w = \frac{M}{2m}\,q^- \in \mathbb{R}$$

$$x = \frac{q_\perp^2}{m^2} > 0$$



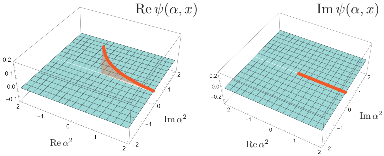
- vanishes at endpoints $\alpha = \pm 1$
- falls off like $1/x^2$
- support only for $-1 < \alpha < 1$ (not an analytic function!)

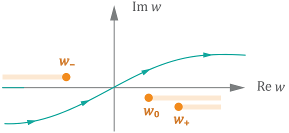
Example: monopole



$$\int_{-\infty}^{\infty} dw$$

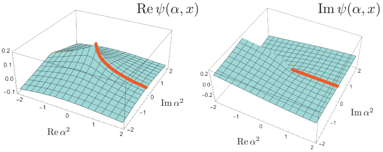
⇒ support only for $-1 < \alpha < 1$,
not an analytic function,
x, t ∈ ℝ





$$\int_{-\infty\,(1+i\epsilon)}^{\infty\,(1+i\epsilon)} dw$$

⇒ analytic function in α^2 for any x, t ∈ ℂ,
for $-1 < \alpha < 1$ result is the same



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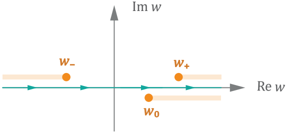

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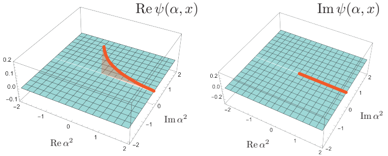

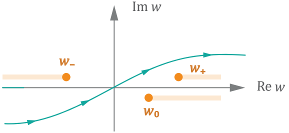

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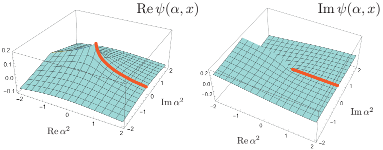

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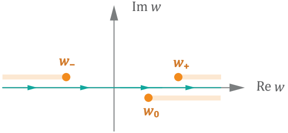

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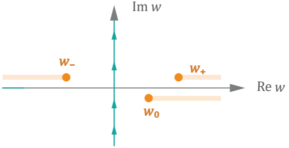
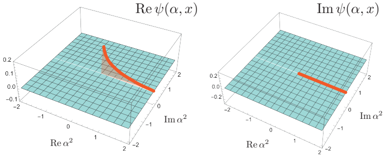
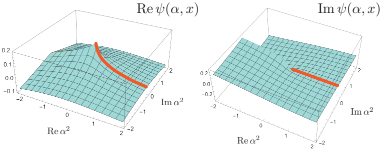


$$\int_{-\infty\,(1+i\epsilon)}^{\infty\,(1+i\epsilon)} dw \longrightarrow \int_{-i\infty}^{+i\infty} dw$$

$\Rightarrow$ support only for $-1 < \alpha < 1$,
not an analytic function,
x, t $\in \mathbb{R}$

$\Rightarrow$ analytic function in α^2 for any x, t $\in \mathbb{C}$,
for $-1 < \alpha < 1$ result is the same





Example: monopole

After integrating over w,
poles in w become **branch cuts** in x:

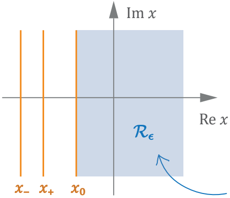
$$w_\pm = \pm\left(t + \frac{1+x}{1\pm\alpha}\right) \qquad w_0 = \frac{x+\gamma}{\alpha}$$

$$\Rightarrow$$

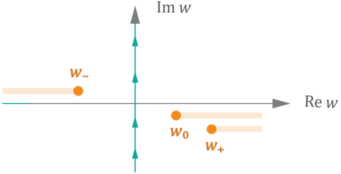
$$x_\pm = (1\pm\alpha)(\pm w - t) - 1 \qquad x_0 = \alpha w - \gamma$$

$w = -i\infty \dots i\infty$

These separate different **regions**
in complex x plane:



As long as we stay
inside this region,
we will get correct
result for the LFWF



$$\int_{-i\infty}^{+i\infty} dw$$

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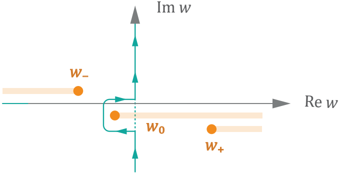

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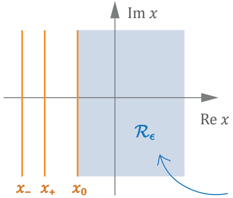


As long as we stay inside this region, we will get correct result for the LFWF

→ Outside this region, the poles in w have moved to the wrong side of the **Euclidean** integration contour (would need to deform it as well)

Euclidean variables

Now, go to **Euclidean metric:**

$$k_E = \begin{bmatrix} \boldsymbol{k} \\ k_4 \end{bmatrix} = \begin{bmatrix} \boldsymbol{k} \\ ik^0 \end{bmatrix} \qquad k_3 = \frac{k^+ - k^-}{2}, \qquad k_4 = i\,\frac{k^+ + k^-}{2} \qquad k_E \cdot p_E = -k \cdot p = \boldsymbol{k}_\perp \cdot \boldsymbol{p}_\perp - \tfrac{1}{2}(k^- p^+ + k^+ p^-)$$

Drop index E.
BSWF depends on two four-vectors k, P:

$$\Psi(q, P) = \Psi(x, \omega, t, \alpha) \qquad q = k + \alpha\,\frac{P}{2}$$

Three Lorentz invariants:

$$x = \frac{k^2}{m^2} = \frac{\boldsymbol{k}_\perp^2}{m^2}$$

q^-

$$\omega = \hat{k} \cdot \hat{P} = -\frac{w + \alpha t}{2\sqrt{xt}}$$

$$t = \frac{P^2}{4m^2} = -\frac{M^2}{4m^2}$$

⇒ **LFWF & PDA:**

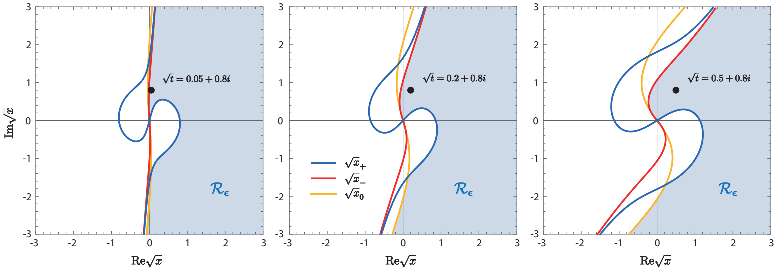
$$\psi(\alpha, x, t) = \frac{\mathcal{N} m^2}{i\pi}\, 2\sqrt{xt} \int_{-\infty}^{\infty} d\omega\, \Psi(x, \omega, t, \alpha)$$

$$\phi(\alpha) = \frac{m^2}{(4\pi)^2 f} \int_0^{\infty} dx\, \psi(\alpha, x, t)$$

- Can be integrated **numerically** (check for monopole: same result)
- BSWF is **Lorentz-invariant**, can be calculated in any frame (also rest frame)
- But now the **branch cuts** in complex x plane will look different, need to stay inside $\mathcal{R}_\epsilon$

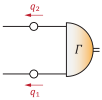
Euclidean variables

Branch cuts in complex $\sqrt{x}$ plane:



Propagator poles

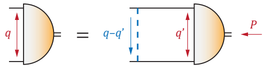
Pole in BS amplitude



- Correct result for LFWF inside $\mathcal{R}_\epsilon$
- Physical region 0 < M < 2m is imaginary axis: $\sqrt{t} = \frac{iM}{2m}$
  Not possible ⇒ must calculate for complex $\sqrt{t}$
- If cuts cross real $\sqrt{x}$ axis, must deform contour to calculate PDA

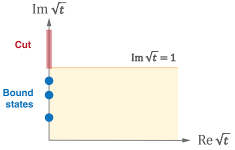
Bethe-Salpeter equation

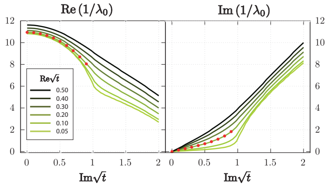
Ultimately, we want to compute BSWF dynamically from its **Bethe-Salpeter equation.**
Here: massive Wick-Cutkosky model (scalar ladder exchange)
Wick 1954, Cutkosky 1954, Nakanishi 1969



$$\Gamma(q,P) = \int \frac{d^4q'}{(2\pi)^4}\, K(q,q',P)\, G_0(q',P)\, \Gamma(q',P)$$

- Two parameters: overall coupling strength **c**, mass ratio $\beta = \mu/m$
- Compute eigenvalue spectrum for complex $\sqrt{t} = \frac{iM}{2m}$, bound states if $\frac{1}{\lambda(t)} \stackrel{!}{=} c$





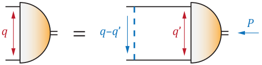
branch point
at threshold

Lines: direct result using
contour deformations

Red dots: Nakanishi method

Virtual state if c too small,
tachyon if c too large
GE, Duarte, Peña, Stadler, PRD 100 (2019)

Singularities in BSE

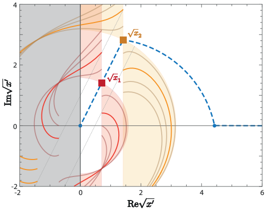


$$\Gamma(\color{red}{x},\omega,t,\alpha) = \int_0^\infty \color{red}{dx'} \int_{-1}^{1} d\omega' \int_{-1}^{1} dy\, K(\color{red}{x},\color{red}{x'},\Omega)\, G_0(\color{red}{x'},\omega',t,\alpha)\, \Gamma(\color{red}{x'},\omega',t,\alpha)$$

$$\Omega = \omega\omega' + y\sqrt{1-\omega^2}\sqrt{1-\omega'^2}$$

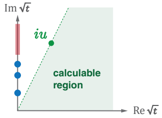
- BSE must be solved for $\sqrt{x} \in \mathcal{R}_\epsilon$
- BSE is **integral equation** ⇒ must be solved along path $\sqrt{x}$ that coincides with **integration path** $\sqrt{x'}$, must lie inside $\mathcal{R}_\epsilon$ ⇒ need contour deformations

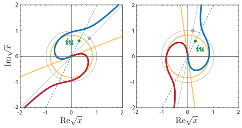
- **Kernel** has pole ⇒ after integrating over y and ω', becomes branch cut in $\sqrt{x'}$, automatically avoided if Re $\sqrt{x'}$ and $|\sqrt{x'}|$ increase along integration path



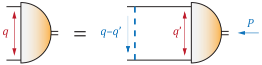
- **Propagators** have poles ⇒ branch cuts from before, automatically avoided if integration path connects $\sqrt{x'} = 0 \dots (1+|\alpha|)\sqrt{t}$ and goes back to real axis

- **BS amplitude** may dynamically generate singularities at $q^2/m^2 = -u^2$ ⇒ avoided as long as arg $\sqrt{t}$ < arg(iu)



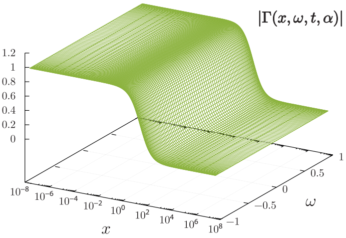


Bethe-Salpeter amplitude



$$\Gamma(\color{red}{x}\color{black},\omega,t,\alpha) = \int_0^\infty \color{red}{dx'} \color{black}\int_{-1}^{1} d\omega' \int_{-1}^{1} dy\, K(\color{red}{x},\color{red}{x'}\color{black},\Omega)\, G_0(\color{red}{x'}\color{black},\omega',t,\alpha)\,\Gamma(\color{red}{x'}\color{black},\omega',t,\alpha)$$

$$\Omega = \omega\omega' + y\sqrt{1-\omega^2}\sqrt{1-\omega'^2}$$



$\alpha = 0.6$,
$\sqrt{t} = 0.2 + 0.2i$

- BS amplitude falls off like 1/x
- only weak dependence on ω and α
  $\Rightarrow \alpha$ dependence in LFWF comes from propagators:

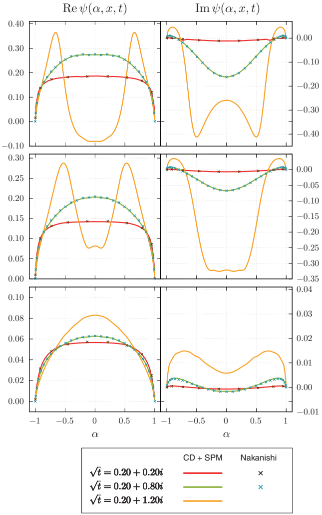
$$\psi(\alpha,x,t) \;\propto \int_{-\infty}^{\infty} d\omega\, G_0(x,\omega,t,\alpha)\,\Gamma(x,\omega,t,\alpha)$$

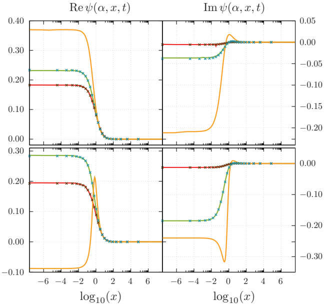
- LFWF needs Γ for $\omega \in (-\infty,\infty)$, but
  BSE solution only known for $\omega \in [-1,1]$

  Use **Schlessinger point method** (SPM)
  for analytic continuation
  Schlessinger, Phys. Rev. 167 (1968)

$$f(\omega) = \cfrac{c_1}{1 + \cfrac{c_2\,(\omega-\omega_1)}{1 + \cfrac{c_3\,(\omega-\omega_2)}{1 + \cfrac{c_4\,(\omega-\omega_3)}{\dots}}}}$$

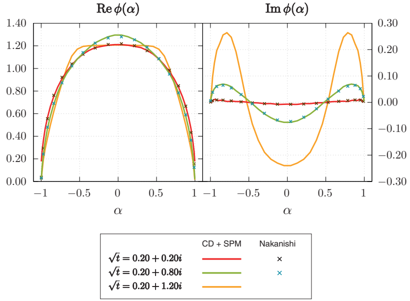
Light-front wave function





- Results agree with Nakanishi method
- LFWF vanishes at endpoints $\alpha = \pm 1$
- No expansion in moments involved, plain numerical result
- Also works above threshold (unphysical, no poles on 1st sheet, no resonances either)

Distribution amplitude



$$\phi(\alpha) = \frac{m^2}{(4\pi)^2 f} \int_0^\infty dx\, \psi(\alpha, x, t)$$

integrate along
previous contour

- Results agree with Nakanishi method
- PDA vanishes at endpoints $\alpha = \pm 1$
- No expansion in moments involved, plain numerical result
- Also works above threshold (unphysical, no poles on 1st sheet, no resonances either)

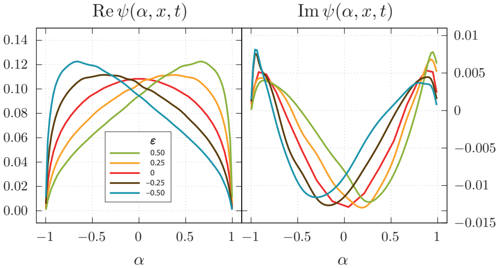
Unequal masses

Straightforward to implement:

$$G_0(q,P) = \frac{1}{q_1^2+m_1^2}\,\frac{1}{q_2^2+m_2^2} \qquad\qquad m_1 = m\,(1+\varepsilon)\,, \quad m_2 = m\,(1-\varepsilon)$$

Results depend on mass difference ε,
contour deformation works in same way

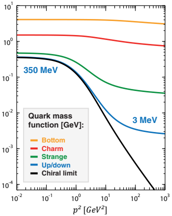
LFWF is no longer symmetric in α:



Complex conjugate poles

Typical situation for quark and gluon propagators in QCD within truncations

Maris, Roberts, PRC 56 (1997),
GE, Williams, Sanchis-Alepuz, Alkofer, Fischer, PPNP 91 (2016),
Windisch, PRC 95 (2017),
Fischer, Huber, PRD 102 (2020),
...

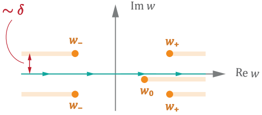


Straightforward to implement in scalar model, contour deformations work in same way

$$D(q^2) = \frac{1}{2}\left(\frac{1}{q^2+m^2\,(1+i\delta)} + \frac{1}{q^2+m^2\,(1-i\delta)}\right)$$
$$= \frac{q^2+m^2}{(q^2+m^2)^2+m^4\,\delta^2}$$

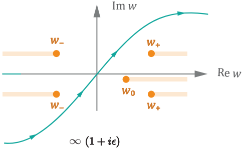
Careful when doing residue calculus in Minkowski space:

see also Tiburzi, Detmold, Miller, PRD 68 (2003)



$$\int_{-\infty}^{\infty} dw$$

does not give correct limit for $\delta \to 0$



$$\int_{-\infty\,(1+i\epsilon)}^{\infty\,(1+i\epsilon)} dw$$

correct limit for $\delta \to 0$, proper analytic continuation

Complex conjugate poles

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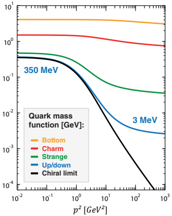

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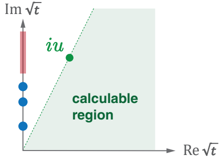
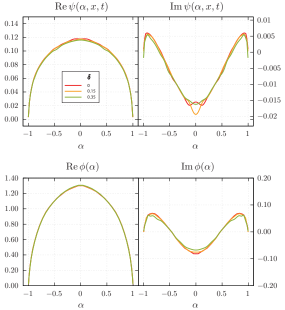
$$= \frac{q^2+m^2}{(q^2+m^2)^2+m^4\,\delta^2}$$

One does not even need to know pole positions $q^2_{1,2}/m^2 = -u^2$ $\Rightarrow$ avoided as long as arg $\sqrt{t}$ < arg(iu)



works for general singularities in n-point functions!

LFWF and PDA:



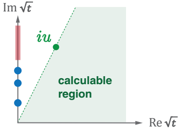
Summary

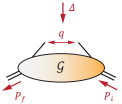
- We explored new method to compute LFWF & PDA using contour deformations & analytic continuations
  GE, Ferreira, Stadler, in preparation

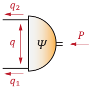
- Method is fast and efficient

- Can integrate **numerically** without explicit knowledge of singularities in the integrand (as long as confined to certain region)

- Proof of concept for scalar model and Bethe-Salpeter wave function, but should work in same way for hadron-to-hadron correlators







$\mathcal{G}(z,P,\Delta) = \langle P_f |\, \mathsf{T}\, \Phi(z)\, \mathcal{O}\, \Phi(0)\, | P_i \rangle$ $\qquad$ $\Psi(z,P) = \langle 0 | \mathsf{T}\, \Phi(z)\, \Phi(0) | P \rangle$

Thank you!

	$\mathcal{G}(q,P,\Delta=0)$	$\mathcal{G}(q,P,\Delta)$	$\Psi(q,P)$
$\int dq^-$	TMD	GTMD	LFWF
$\int d^2\boldsymbol{q}_\perp \int dq^-$	PDF	GPD	PDA