Light-front dynamic analysis of the transition form factors in 1 + 1 dimensional scalar field model

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LFD and hadron physics

- Light-Front Dynamics (LFD) is an essential theoretical tool for the JLab/EIC physics, e.g. GPDs, TMDs, 3D femtography of the hadrons, etc.
- It has the remarkable features of the maximum number (7) of kinematic operators among the 10 Poincare generators, boost invariant wave function, and simpler vacuum property.
- The advantage of LFD is maximized in 1+1 dimensional models due to the absence of transverse rotation operators which are dynamical in LFD.
Drell-Yan-West vs other frames

- For the study of exclusive processes, the Drell-Yan-West frame \( (q^+ = 0) \) is well-established, but it cannot be taken in this case of 1+1 dimensions, as it results in \( q^2 = 0 \).
- As one must use a \( q^+ \neq 0 \) frame, one must include both the valence and non-valence graphs for the calculation of the form factors.
- Choosing a \( q^+ \neq 0 \) frame, one can also directly access not only the space-like \( (q^2 < 0) \), but also the time-like \( (q^2 > 0) \) momentum regions.
Transition Form Factor in 1+1-dimensional simple scalar model

- We give a clear example demonstrating direct access to the time-like region of photon momenta without resorting to analytic continuation.
- We use the exactly solvable scalar field model of Sawicki and Mankiewicz.
- Even though the model is not very realistic, it serves as an initial trial for the more complicated 't Hooft model or the phenomenological light-front quark model.
The covariant Bethe-Salpeter (BS) model

- The meson wave function is a product of two free single particle propagators
- Dirac delta function for overall momentum conservation
- And a constant vertex function

\[ \Gamma^{\mu\nu} = \frac{p - k (S)}{p (S)} \frac{q (\gamma^*)}{k (S)} + \frac{p - k (S)}{p (S)} \frac{q - k (S)}{k (S)} \frac{q (\gamma^*)}{p - q (\gamma^*)} + \frac{p - k (S)}{p (S)} \frac{q (\gamma^*)}{k (S)} \frac{q (\gamma^*)}{p - q (\gamma^*)} \]

\[ \Gamma^{\mu\nu} = F(q^2, q'^2) (g^{\mu\nu} q \cdot q' - q'^\mu q') \]
Valence and non-valence contributions: how to define them

\[ p - k (S) \Rightarrow q (\gamma^*) \]

\[ p - k - q (S) \]

\[ p(S) \]

\[ k(S) \]

\[ p - q (\gamma^*) \]

(a)

\[ 1 \]

\[ 1 - x \]

\[ 1 - x - \alpha \]

\[ 1 - \alpha \]

(b)

(c)

\[ f_i^{++} = \frac{\Gamma_i^{++}}{g^{++}q \cdot q' - q'^+q^+} \]

\[ f_i^{+-} = \frac{\Gamma_i^{+-}}{g^{+-}q \cdot q' - q'^+q^-} \]

where \( i \) represents \( D(b), D(c), C(b), C(c), \) or \( S \)
Problem with this definition is that \( f_{i}^{++} \neq f_{i}^{+-} \)

This is because each individual \( \Gamma_{i}^{\mu \nu} \) may not be gauge invariant by themselves, thus there may be a gauge-non-invariant component \( \tilde{g}_{i} \) in addition to the gauge-invariant part \( \tilde{f}_{i} \), i.e., although the total current must satisfy gauge invariance \( \Gamma^{\mu \nu} = F(q^{2}, q'^{2}) (g^{\mu \nu} q \cdot q' - q'^{\mu} q^{\nu}) \) and there is only one form factor \( F(q^{2}, q'^{2}) \), each individual LFTTO contributions may be of the form \( \Gamma_{i}^{\mu \nu} = \tilde{f}_{i} (g^{\mu \nu} q \cdot q' - q'^{\mu} q^{\nu}) + \tilde{g}_{i} (q^{\mu} q'^{\nu}) \).

i.e., in our ++ and +- currents example,

\[
\begin{pmatrix}
\Gamma_{i}^{++} \\
\Gamma_{i}^{+-}
\end{pmatrix} = \begin{pmatrix}
g^{++} q \cdot q' - q'^{+} q^{+} & q^{+} q'^{+} \\
g^{+-} q \cdot q' - q'^{+} q^{-} & q^{+} q'^{-}
\end{pmatrix} \cdot \begin{pmatrix}
\tilde{f}_{i} \\
\tilde{g}_{i}
\end{pmatrix}
\]
Inverting this equation, we get
\[
\begin{pmatrix}
\tilde{f}_i \\
\tilde{g}_i
\end{pmatrix} = \begin{pmatrix}
-\frac{1}{2q^+q'^+} & \frac{1}{2q^+q'^-} \\
\frac{1}{2q^+q'^+} & -\frac{1}{2q^+q'^-}
\end{pmatrix} \cdot \begin{pmatrix}
\Gamma^{++}_i \\
\Gamma^{+-}_i
\end{pmatrix}.
\]
Recall that
\[
f_{i}^{++} = \frac{\Gamma^{++}_i}{g^{++}q \cdot q' - q'^+q^+} = -\frac{\Gamma^{++}_i}{q'^+q^+},
\]
and
\[
f_{i}^{+-} = \frac{\Gamma^{+-}_i}{g^{+-}q \cdot q' - q'^+q^-} = \frac{\Gamma^{+-}_i}{q^+q'^-}.
\]
So, in fact,
\[
\tilde{f}_i = \frac{f_{i}^{++} + f_{i}^{+-}}{2},
\]
and
\[
\tilde{g}_i = \frac{-f_{i}^{++} + f_{i}^{+-}}{2}.
\]
Of course, we have

\[ \sum_i \tilde{f}_i = \sum_i f_i^{++} = \sum_i f_i^{+-} = F(q^2, q'^2) \]

and

\[ \sum_i \tilde{g}_i = 0 \]

as it must be.
The amplitude $\Gamma^{\mu\nu}$ is calculated as such, following the Feynman rules for the scalar field theory.

$$\Gamma^{\mu\nu} = \Gamma^D_{\mu\nu} + \Gamma^C_{\mu\nu} + \Gamma^S_{\mu\nu}$$

$$= ie^2 g_s \int \frac{d^2 k}{(2\pi)^2} \times$$

$$\left\{ \frac{(2p - 2k - q)^\mu (p - 2k - q)^\nu}{((p - k - q)^2 - m^2) ((p - k)^2 - m^2) (k^2 - m^2)} + \frac{(q - 2k)^\mu (p - 2k + q)^\nu}{((p - k)^2 - m^2) (k^2 - m^2) ((q - k)^2 - m^2)} + \frac{-2g^{\mu\nu}}{((p - k)^2 - m^2) (k^2 - m^2)} \right\},$$

where the coupling constant of the simple scalar model $g_s$ is fixed from the normalization condition. For simplicity, we take all the intermediate scalar particles’ mass to be $m$ and their charge to be $e$, but it can be easily generalized to unequal mass/charge cases. The initial scalar meson has mass $M$. 
For the manifestly covariant calculation, we use Feynman parametrization method and obtain

\[
F(q^2, q'^2) = \frac{e^2 g_s}{4\pi} \int_0^1 dx \int_0^{1-x} dy (1 - 2y) \left( \frac{1}{\Delta_1^2} + \frac{1}{\Delta_2^2} \right),
\]

where

\[
\Delta_1 = x(x-1)q^2 + 2x(x+y-1)q\cdot q' + (x+y)(x+y-1)q'^2 + m^2,
\]

and

\[
\Delta_2 = x(x-1)q'^2 + 2x(x+y-1)q\cdot q' + (x+y)(x+y-1)q^2 + m^2.
\]

Doing the x and y integrations, it ends up being

\[
F(q^2, q'^2) = \frac{e^2 g_s}{4\pi} \frac{(2 - \omega - \gamma' - \gamma) \frac{\sqrt{\omega}}{\sqrt{1-\omega}} \tan^{-1} \left( \frac{\sqrt{\omega}}{\sqrt{1-\omega}} \right) + (\gamma - \gamma' - \omega) \frac{\sqrt{1-\gamma'}}{\sqrt{\gamma'}} \tan^{-1} \left( \frac{\sqrt{\gamma'}}{\sqrt{1-\gamma'}} \right) + (\gamma' - \gamma - \omega) \frac{\sqrt{1-\gamma}}{\sqrt{\gamma}} \tan^{-1} \left( \frac{\sqrt{\gamma}}{\sqrt{1-\gamma}} \right)}{m^4 \left[ 4\omega\gamma'\gamma + \omega^2 + (\gamma' - \gamma)^2 - 2\omega(\gamma' + \gamma) \right]},
\]

where \( \gamma = \frac{q^2}{4m^2} \), \( \gamma' = \frac{q'^2}{4m^2} \), and \( \omega = \frac{M^2}{4m^2} \).
For the LFD calculation, we define the light-front momentum fraction parameter of the emitted photon with respect to the initial scalar meson $\alpha = q^+/p^+$

So that the 2-momenta of the external particles are

$$p = (p^+, p^-) = \left( p^+, \frac{M^2}{2p^+} \right)$$

$$q = (q^+, q^-) = \left( \alpha p^+, \frac{M^2}{2p^+} - \frac{q'^2}{2(1 - \alpha)p^+} \right)$$

$$q' = p - q = (q'^+, q'^-) = \left( (1 - \alpha)p^+, \frac{q'^2}{2(1 - \alpha)p^+} \right),$$

where $\alpha$ satisfies

$$\alpha^2 M^2 - \alpha M^2 + \alpha q'^2 - \alpha q^2 + q^2 = 0,$$

for which we get the solutions

$$\alpha_\pm = \frac{(M^2 - q'^2 + q^2) \pm \sqrt{(-M^2 + q'^2 - q^2)^2 - 4M^2q^2}}{2M^2}.$$ 

We ensure $q^+ > 0$ by taking the $\alpha_+$ root in our calculation
For the LFD calculation, we pick the ++ component of the current and the +− one to show their difference.

We use the Cauchy integration method to do the energy integration enclosing the poles, then there is only the kinematic momentum fraction integration left over.

We get

\[
f^{++}_{D(b)} = \frac{e^2 g_s}{4\pi} \int_0^{1-\alpha} dx \left( 2 - 2x - \alpha \right) (1 - 2x - \alpha) \left[ \alpha(\alpha-1)(1-x-\alpha)(1-x)x \left( \frac{m^2}{x} + \frac{m^2}{1-x-\alpha} - \frac{q'^2}{1-\alpha} \right) \left( \frac{m^2}{x} + \frac{m^2}{1-x-M^2} \right) \right]^{-1},
\]

\[
f^{++}_{D(c)} = \frac{e^2 g_s}{4\pi} \int_{1-\alpha}^{1} dx \left( 2 - 2x - \alpha \right) (1 - 2x - \alpha) \left[ \alpha(\alpha-1)(1-x-\alpha)(1-x)x \left( \frac{m^2}{1-x-\alpha} - \frac{m^2}{1-x} + M^2 - \frac{q'^2}{1-\alpha} \right) \left( \frac{m^2}{1-x} + \frac{m^2}{x-M^2} \right) \right]^{-1},
\]

\[
f^{++}_{C(b)} = f^{++}_{D(c)}, \quad \text{and} \quad f^{++}_{C(c)} = f^{++}_{D(b)}.
\]

The x integration can be computed analytically to confirm that

\[
f^{++}_{D(b)} + f^{++}_{D(c)} + f^{++}_{C(b)} + f^{++}_{C(c)} = F_{cov}.
\]

Similarly, one can also calculate the +− component of the current.
Numerical Results
The 3D plot of the real part of the form factor for the case of $m = 0.25 \text{ GeV}$, $M = 0.14 \text{ GeV}$, normalized to $F(q^2 = 0, q'^2 = 0) = 1$, with $-2 \text{GeV}^2 < q^2 < 2 \text{GeV}^2$ and $-2 \text{GeV}^2 < q'^2 < 2 \text{GeV}^2$. 
The 3D plot of the imaginary part of the form factor for the case of $m = 0.25 \text{ GeV}$, $M = 0.14 \text{ GeV}$, normalized to $F(q^2 = 0, q'^2 = 0) = 1$, with $-2\text{GeV}^2 < q^2 < 2\text{GeV}^2$ and $-2\text{GeV}^2 < q'^2 < 2\text{GeV}^2$. 
The numerical results of the form factor for the case of $m = 0.25 \text{ GeV}$, $M = 0.14 \text{ GeV}$, and $q'^2 = -0.1 \text{ GeV}^2$, normalized to $F(q^2 = 0, q'^2 = 0) = 1$, from the manifestly covariant calculation as well as the light-front one, and their agreements with the Dispersion Relation.
The numerical results of individual $x^+$-ordered contributions to the form factor for the case of $m = 0.25 \text{ GeV}$, $M = 0.14 \text{ GeV}$, and $q'^2 = -0.1 \text{ GeV}^2$, normalized to $F(q^2 = 0, q'^2 = 0) = 1$, by picking the $++$ component of the current. Note that $f_{D(b)}^{++} = f_{C(b)}^{++}$ and $f_{D(c)}^{++} = f_{C(c)}^{++}$. 
The numerical results of individual $x^+$-ordered contributions to the form factor for the case of $m = 0.25$ GeV, $M = 0.14$ GeV, and $q'^2 = -0.1$ GeV$^2$, normalized to $F(q^2 = 0, q'^2 = 0) = 1$, by picking the $+-$ component of the current. Note that $f_{D(b)}^{-+} = f_{C(c)}^{+-}$ and $f_{D(c)}^{+-} = f_{C(b)}^{+-}$. 
The numerical results of the gauge-invariant part of the individual $x^+$-ordered contributions to the form factor for the case of $m = 0.25 \text{ GeV}$, $M = 0.14 \text{ GeV}$, and $q'^2 = -0.1 \text{ GeV}^2$, normalized to $F(q^2 = 0, q'^2 = 0) = 1$, taking the tilde definition. Note that $\tilde{f}_{D(b)} = \tilde{f}_{C(c)}$ and $\tilde{f}_{D(c)} = \tilde{f}_{C(b)}$. 
The numerical results of the gauge-non-invariant part of the individual \( x^+ \)-ordered contributions to the form factor for the case of \( m = 0.25 \text{ GeV}, M = 0.14 \text{ GeV}, \) and \( q^2 = -0.1 \text{ GeV}^2 \) by taking the tilde definition. Imaginary parts are all zero here.

Note that \( \tilde{g}_{D(b)} = \tilde{g}_{C(c)} \) and \( \tilde{g}_{D(c)} = \tilde{g}_{C(b)} \).
Thank you!