A perturbative expansion for bound states

Online talk at LC2021, 29 November 2021

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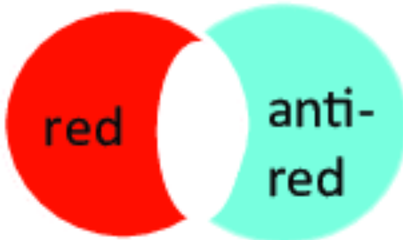
Hadrons and atoms have unexpected similarities

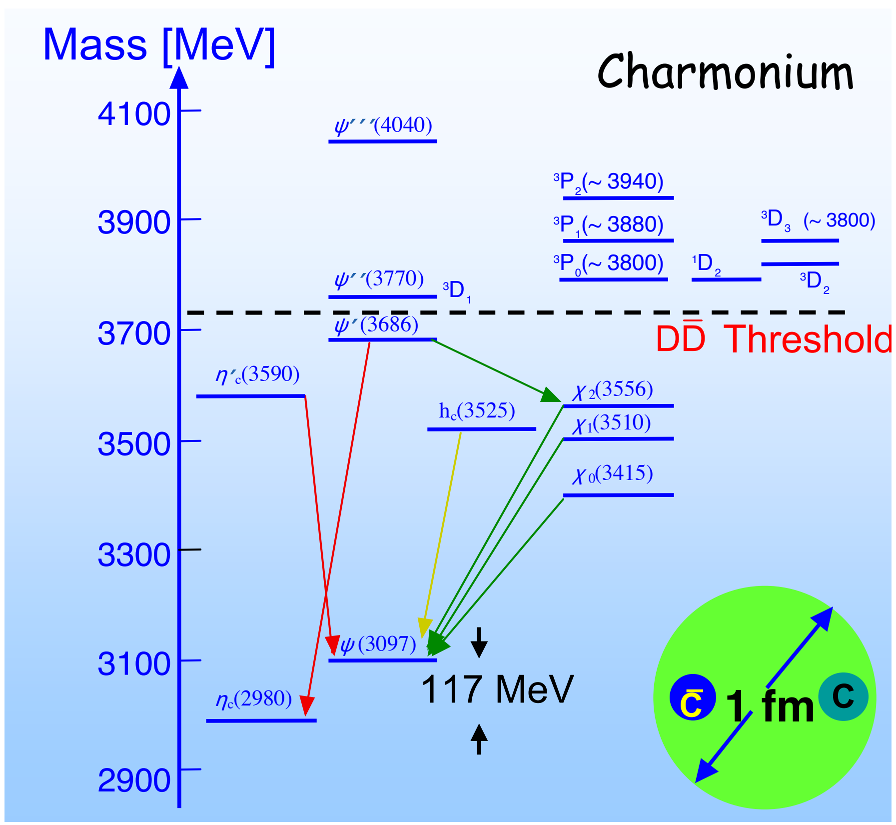
Can the first-principles bound state methods of QED be applied to QCD?

This is a real possibility!

Non-relativistic bound states

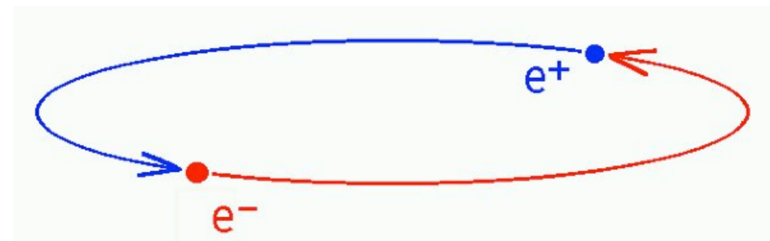
QCD: $b\bar{b}$, $c\bar{c}$ quarkonia

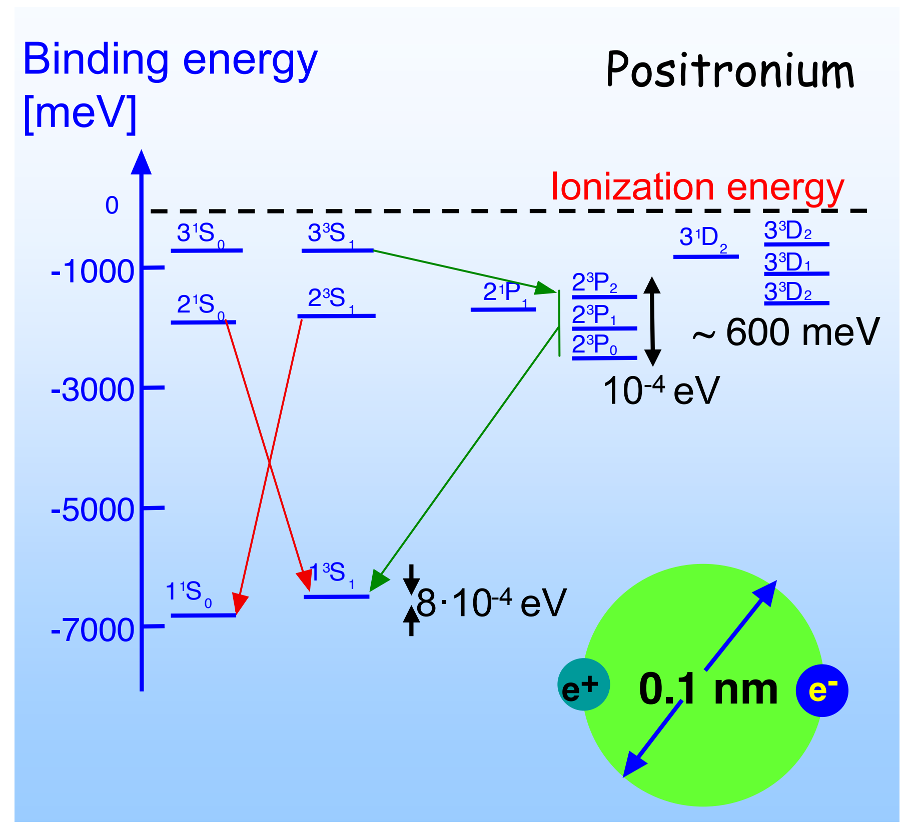




$$V(r) = V'r - \frac{4}{3}\frac{\alpha_s}{r}$$

QED: e^+e^- atoms





$$V(r) = -\frac{\alpha}{r}$$

Light quarks: relativistic bound states

Valence Fock states govern quantum numbers and decays,
even for highly relativistic constituents.

Valence quantum numbers

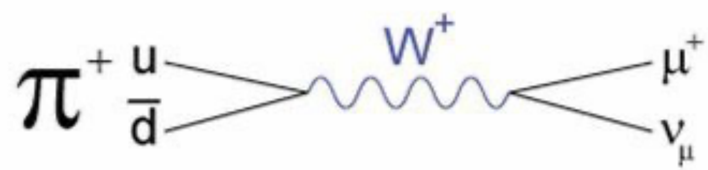
$n^{2s+1}\ell_J$	J^{PC}	$I=1$ $u\bar{d},\ \bar{u}d,$ $\frac{1}{\sqrt{2}}(d\bar{d}-u\bar{u})$	$I=\frac{1}{2}$ $u\bar{s},\ d\bar{s};$ $\bar{d}s,\ \bar{u}s$	$I=0$ f'	$I=0$ f	$\theta_{\rm quad}$ [°]	$\theta_{\rm lin}$ [°]
1^1S_0	0^{-+}	π	K	η	$\eta'(958)$	-11.3	-24.5
1^3S_1	1^{--}	$\rho(770)$	$K^*(892)$	$\phi(1020)$	$\omega(782)$	39.2	36.5
1^1P_1	1^{+-}	$b_1(1235)$	$K_{1B}{}^{\dagger}$	$h_1(1415)$	$h_1(1170)$		
1^3P_0	0^{++}	$a_0(1450)$	$K_0^*(1430)$	$f_0(1710)$	$f_0(1370)$		
1^3P_1	1^{++}	$a_1(1260)$	$K_{1A}{}^{\dagger}$	$f_1(1420)$	$f_1(1285)$		
1^3P_2	2^{++}	$a_2(1320)$	$K_2^*(1430)$	$f_2'(1525)$	$f_2(1270)$	29.6	28.0
1^1D_2	2^{-+}	$\pi_2(1670)$	$K_2(1770)^{\dagger}$	$\eta_2(1870)$	$\eta_2(1645)$		
1^3D_1	1^{--}	$\rho(1700)$	$K^*(1680)^{\ddagger}$		$\omega(1650)$		
1^3D_2	2^{--}		$K_2(1820)^{\dagger}$				
1^3D_3	3^{--}	$\rho_3(1690)$	$K_3^*(1780)$	$\phi_3(1850)$	$\omega_3(1670)$	31.8	30.8
1^3F_4	4^{++}	$a_4(1970)$	$K_4^*(2045)$	$f_4(2300)$	$f_4(2050)$		
1^3G_5	5^{--}	$\rho_5(2350)$	$K_5^*(2380)$				
2^1S_0	0^{-+}	$\pi(1300)$	$K(1460)$	$\eta(1475)$	$\eta(1295)$		
2^3S_1	1^{--}	$\rho(1450)$	$K^*(1410)^{\ddagger}$	$\phi(1680)$	$\omega(1420)$		
2^3P_1	1^{++}	$a_1(1640)$					
2^3P_2	2^{++}	$a_2(1700)$	$K_2^*(1980)$	$f_2(1950)$	$f_2(1640)$		

Current quark Fock states

Mesons have a sizeable current $q\bar{q}$ Fock component

E.g., pion decay

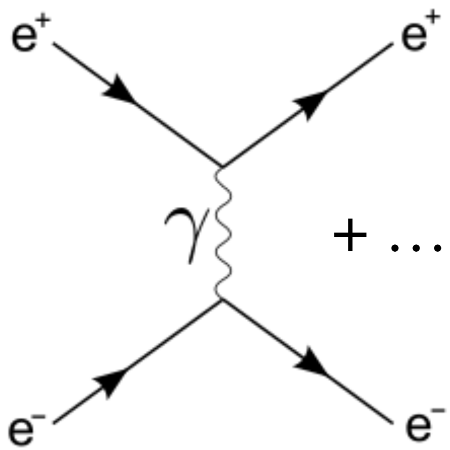
Stan Brodsky



What prevents the strong color field from creating abundant $q\bar{q}$, g constituents?

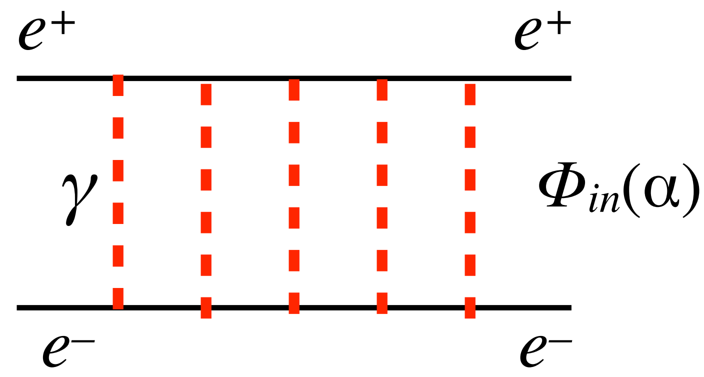
Perturbative expansion: Scattering vs. bound states

$e^+e^- \to e^+e^-$



Scattering amplitudes are expanded around free states

Positronium



Atoms are expanded around an initial bound state

Schrödinger wave functions for atoms $\Phi_{in}(\alpha)$ are exponential in α

Their power corrections $\Phi_{in}(\alpha)(1 + c_1\alpha + c_2\alpha^2 \ldots)$ depend on $\Phi_{in}(\alpha)$

The perturbative expansion of bound states is not unique, it depends on the choice of initial state.

Caswell & Lepage (1975)

Instantaneous (Δt = 0) interactions

A $q\bar{q}$ Fock state is bound by an instantaneous interaction.

Cf. the $V(r) = -\alpha/r$ potential of the NR Schrödinger equation.

Theories with a local action generally do not have instantaneous potentials.

Gauge theories are an exception:
Although their action is local, the gauge may be fixed non-locally

The lack of $\partial_0 A^0$ and $\nabla\cdot\boldsymbol{A}$ in $\mathcal{L}_{QED}$ means that A^0 and $\boldsymbol{A}_L$ do not propagate

Feynman gauge fixing: $\mathcal{L}_{GF} = (\partial_\mu A^\mu)^2$ adds the missing terms
$\Rightarrow$ All gauge fields propagate, explicit Poincaré invariance

Instantaneous gauge interactions for

$\nabla\cdot\boldsymbol{A}(t,\boldsymbol{x}) = 0$ (Coulomb gauge)

$A^0(t,\boldsymbol{x}) = 0$ (Temporal gauge)

Canonical quantization

Conjugate field π_α

$$\pi_\alpha(t,\boldsymbol{x}) = \frac{\partial\mathcal{L}(\varphi,\partial\varphi)}{\partial[\partial_0\varphi_\alpha(t,\boldsymbol{x})]}$$

Commutation relations

$$[\varphi_\alpha(t,\boldsymbol{x}),\pi_\beta(t,\boldsymbol{y})]_\pm = i\delta_{\alpha\beta}\delta^3(\boldsymbol{x}-\boldsymbol{y})$$

A^0 has no conjugate field, due to the absence of $\partial_0 A^0$ in $\mathcal{L}_{QED}$.

Not a problem in temporal gauge: $A^0(t,\boldsymbol{x}) = 0$.

Choose temporal gauge.

$A^0(t,\boldsymbol{x}) = 0$ is preserved under time-independent gauge transformations, which are generated by

$$\frac{\delta\mathcal{S}_{QED}}{\delta A^0(x)} = \partial_i E^i(x) - e\psi^\dagger\psi(x)$$

Willemsen (1978)

Physical states are required to satisfy the constraint:

$$\frac{\delta\mathcal{S}_{QED}}{\delta A^0(x)}|phys\rangle = 0$$

This determines $\nabla\cdot\boldsymbol{E}_L$ in terms of the charge distribution, and ensures that the states are invariant under t-independent gauge transformations.

The classical, instantaneous field E_L

$$\frac{\delta \mathcal{S}_{QED}}{\delta A^0(x)}\,|phys\rangle = 0$$ is not an operator relation, it is a constraint on $|phys\rangle$

$$\frac{\delta \mathcal{S}_{QED}}{\delta A^0(x)}\,|0\rangle = 0$$ implies $\boldsymbol{E}_L = 0$ in the vacuum.

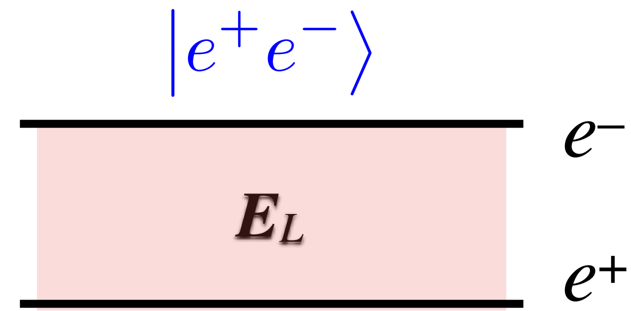
No particles are created.

In temporal gauge the electric field $\boldsymbol{E}_L$ is classical, not an operator.

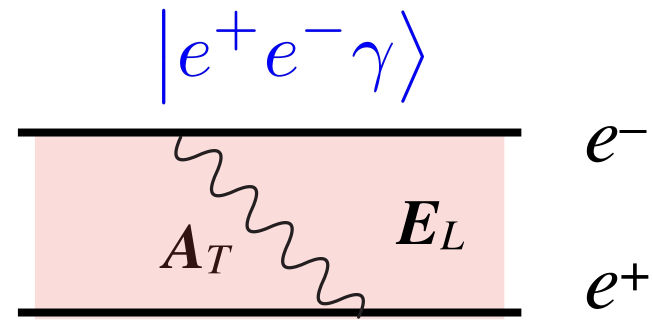
$\boldsymbol{E}_L$ can bind e^+e^- Fock states strongly, without pair creation.

Fock state expansion for Positronium in $A^0=0$ gauge

A perturbative expansion in α can start from the $|e^+e^-\rangle$ Fock state, bound by its classical field $\boldsymbol{E}_L$:



Higher order corrections include states with transverse photons and e^+e^- pairs, as determined by $H_{QED}\,|e^+e^-\rangle$



Each Fock component of the bound state includes its particular instantaneous $\boldsymbol{E}_L$ field.

This Fock expansion is valid in any frame, and is formally exact at $O(\alpha^\infty)$.

Temporal gauge in QCD: $A_a^0 = 0$

The instantaneous gauge constraint determines $\boldsymbol{E}_{L,a}$ for all hadron Fock states:

$$\partial_i E^i_{L,a}(\boldsymbol{x})\,|phys\rangle = g\big[-f_{abc}A^i_b E^i_c + \psi^\dagger T^a \psi(\boldsymbol{x})\big]\,|phys\rangle$$

In QED we impose the boundary condition: $\boldsymbol{E}_L(\boldsymbol{x}) \to 0$ for $|\boldsymbol{x}| \to \infty$

In QCD $\boldsymbol{E}_{L,a}(\boldsymbol{x}) \equiv 0$ for (globally) color singlet Fock states.

Each color component of the Fock state has $\boldsymbol{E}_{L,a}(\boldsymbol{x}) \neq 0$ $\Rightarrow$

$$E^i_{L,a}(\boldsymbol{x})\,|phys\rangle = -\partial^x_i \int d\boldsymbol{y}\Big[\kappa\,\boldsymbol{x}\cdot\boldsymbol{y} + \frac{g}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\Big]\mathcal{E}_a(\boldsymbol{y})\,|phys\rangle$$

where $\quad \mathcal{E}_a(\boldsymbol{y}) = -f_{abc}A^i_b E^i_c(\boldsymbol{y}) + \psi^\dagger T^a \psi(\boldsymbol{y})$

The homogeneous solution $\propto \kappa$ is the only one that is compatible with invariance under space translations and rotations

Including the κ ≠ 0 homogeneous solution for $E^i_{L,a}$

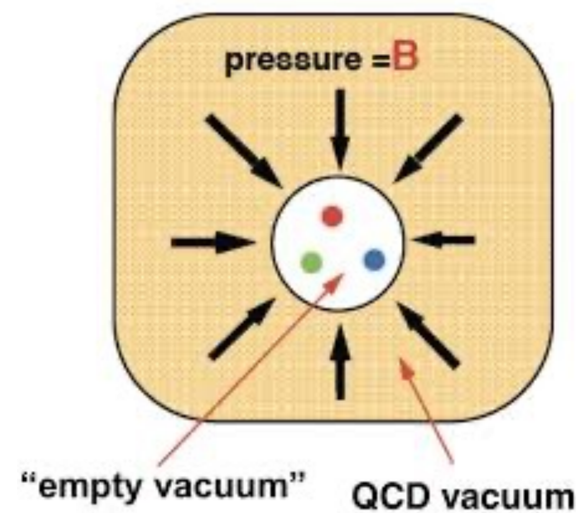
$$E^i_{L,a}(\boldsymbol{x})\,|phys\rangle = -\partial_i^x \int d\boldsymbol{y}\Big[\kappa\,\boldsymbol{x}\cdot\boldsymbol{y} + \frac{g}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\Big]\mathcal{E}_a(\boldsymbol{y})\,|phys\rangle$$

where $\mathcal{E}_a(\boldsymbol{y}) = -f_{abc}A^i_b E^i_c(\boldsymbol{y}) + \psi^\dagger T^a \psi(\boldsymbol{y})$

$\kappa \neq \kappa(\boldsymbol{x},\boldsymbol{y})$: this is a homogeneous solution of $\partial_i \boldsymbol{E}^i(\boldsymbol{x}) = 0$

The linear dependence on $\boldsymbol{x}$ makes $\boldsymbol{E}_L$ independent of $\boldsymbol{x}$, as required by translation invariance: The field energy density is spatially constant.

The field energy ∝ volume of space is irrelevant only if it is universal. This relates the normalisation $\varkappa$ of all Fock components, leaving an overall scale Λ as the single parameter.



"Bag model without a bag"

The potential energy $\mathcal{H}_V = \frac{1}{2}\int d\boldsymbol{x} \sum_a \boldsymbol{E}_L^a \cdot \boldsymbol{E}_L^a$

$$\mathcal{H}_V = \int d\boldsymbol{y} d\boldsymbol{z} \Big\{ \boldsymbol{y}\cdot\boldsymbol{z} \Big[\tfrac{1}{2}\kappa^2 \int d\boldsymbol{x} + g\kappa \Big] + \tfrac{1}{2}\frac{\alpha_s}{|\boldsymbol{y}-\boldsymbol{z}|} \Big\} \mathcal{E}_a(\boldsymbol{y})\mathcal{E}_a(\boldsymbol{z})$$

Recall: $\mathcal{E}_a(\boldsymbol{y}) = -f_{abc}A_b^i E_c^i(\boldsymbol{y}) + \psi^\dagger T^a \psi(\boldsymbol{y})$

Gives translation invariant potentials for (globally) color singlet states

Meson:

$$|q(\boldsymbol{x}_1)\bar{q}(\boldsymbol{x}_2)\rangle \equiv \sum_A \bar{\psi}^A(\boldsymbol{x}_1)\,\psi^A(\boldsymbol{x}_2)\,|0\rangle \qquad \mathcal{H}_V\,|q\bar{q}\rangle = V_{q\bar{q}}\,|q\bar{q}\rangle$$

$$V_{q\bar{q}}(\boldsymbol{x}_1,\boldsymbol{x}_2) = \Lambda^2|\boldsymbol{x}_1-\boldsymbol{x}_2| - C_F\frac{\alpha_s}{|\boldsymbol{x}_1-\boldsymbol{x}_2|}$$

Cornell potential

This potential is valid also for relativistic $q\bar{q}$ Fock states,
in any frame

Baryon Fock state potential

Baryon: $|q(\boldsymbol{x}_1)q(\boldsymbol{x}_2)q(\boldsymbol{x}_3)\rangle \equiv \sum_{A,B,C} \epsilon_{ABC}\psi_A^\dagger(\boldsymbol{x}_1)\,\psi_B^\dagger(\boldsymbol{x}_2)\,\psi_C^\dagger(\boldsymbol{x}_3)\,|0\rangle$

$$V_{qqq}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3) = \Lambda^2 d_{qqq}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3) - \frac{2}{3}\alpha_s\Big(\frac{1}{|\boldsymbol{x}_1-\boldsymbol{x}_2|}+\frac{1}{|\boldsymbol{x}_2-\boldsymbol{x}_3|}+\frac{1}{|\boldsymbol{x}_3-\boldsymbol{x}_1|}\Big)$$

$$d_{qqq}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3) \equiv \frac{1}{\sqrt{2}}\sqrt{(\boldsymbol{x}_1-\boldsymbol{x}_2)^2+(\boldsymbol{x}_2-\boldsymbol{x}_3)^2+(\boldsymbol{x}_3-\boldsymbol{x}_1)^2}$$

When two of the quarks coincide the potential reduces to the $q\bar{q}$ potential:

$$V_{qqq}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_2) = \Lambda^2|\boldsymbol{x}_1-\boldsymbol{x}_2| - \frac{4}{3}\frac{\alpha_s}{|\boldsymbol{x}_1-\boldsymbol{x}_2|} = V_{q\bar{q}}(\boldsymbol{x}_1,\boldsymbol{x}_2)$$

Analogous potentials are obtained for any quark and gluon Fock state, such as $q\bar{q}g$ and gg.

Summary

The similarities of hadrons and atoms are unlikely to be “accidental”

Need to consider the principles of QED bound states

Temporal gauge ($A^0 = 0$) is advantageous for equal-time bound states

The gauge constraint determines the classical, instantaneous $\boldsymbol{E}_L$ field for each Fock component

Perturbative expansion, starting from non-perturbative valence Fock states

A homogeneous solution of the gauge constraint gives confinement in QCD

Many features of hadrons thus obtained look promising & intrigiuing

PH 2109.06257

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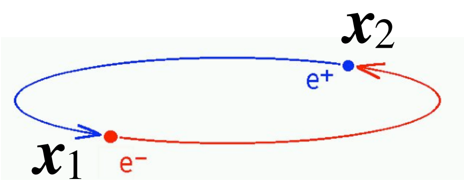
Back-up slides

There is a difference between QED and QCD

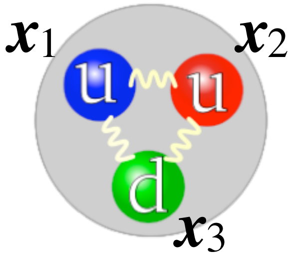
Global gauge invariance allows a classical gauge field for neutral atoms, but not a color octet gluon field for color singlet hadrons.

Positronium (QED)



$$\boldsymbol{E}_L(\boldsymbol{x}) = -\frac{e}{4\pi}\boldsymbol{\nabla}_x\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_1|} - \frac{1}{|\boldsymbol{x}-\boldsymbol{x}_2|}\right)$$

Proton (QCD)



$$\boldsymbol{E}_L^a(\boldsymbol{x}) = 0 \quad \text{for all } \boldsymbol{x}$$

However:
The classical gluon field is non-vanishing for each color component C of the state

$$\boldsymbol{E}_L^a(\boldsymbol{x}, C) \neq 0$$

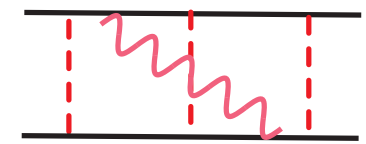
The blue quark feels the color field generated by the red and green quarks.

An external observer sees no field:
The gluon field generated by a color singlet state vanishes.

$$\sum_C \boldsymbol{E}_L^a(\boldsymbol{x}, C) = 0$$

The qg$\bar{\text{q}}$ potential

A q$\bar{\text{q}}$ state, after the emission of a transverse gluon:



$$|q(\boldsymbol{x}_1)g(\boldsymbol{x}_g)\bar{q}(\boldsymbol{x}_2)\rangle \equiv \sum_{A,B,b} \bar{\psi}_A(\boldsymbol{x}_1)\, A_b^j(\boldsymbol{x}_g) T_{AB}^b \psi_B(\boldsymbol{x}_2)\, |0\rangle$$

$$V_{qgq}^{(0)}(\boldsymbol{x}_1, \boldsymbol{x}_g, \boldsymbol{x}_2) = \frac{\Lambda^2}{\sqrt{C_F}}\, d_{qgq}(\boldsymbol{x}_1, \boldsymbol{x}_g, \boldsymbol{x}_2) \qquad \text{(universal } \Lambda)$$

$$d_{qgq}(\boldsymbol{x}_1, \boldsymbol{x}_g, \boldsymbol{x}_2) \equiv \sqrt{\tfrac{1}{4}(N - 2/N)(\boldsymbol{x}_1 - \boldsymbol{x}_2)^2 + N(\boldsymbol{x}_g - \tfrac{1}{2}\boldsymbol{x}_1 - \tfrac{1}{2}\boldsymbol{x}_2)^2}$$

$$V_{qgq}^{(1)}(\boldsymbol{x}_1, \boldsymbol{x}_g, \boldsymbol{x}_2) = \tfrac{1}{2}\,\alpha_s \Big[\frac{1}{N}\, \frac{1}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|} - N\Big(\frac{1}{|\boldsymbol{x}_1 - \boldsymbol{x}_g|} + \frac{1}{|\boldsymbol{x}_2 - \boldsymbol{x}_g|}\Big)\Big]$$

When q and g coincide:

$$V_{qgq}^{(0)}(\boldsymbol{x}_1 = \boldsymbol{x}_g, \boldsymbol{x}_2) = \Lambda^2 |\boldsymbol{x}_1 - \boldsymbol{x}_2| = V_{q\bar{q}}^{(0)}$$

$$V_{qgq}^{(1)}(\boldsymbol{x}_1 = \boldsymbol{x}_g, \boldsymbol{x}_2) = V_{q\bar{q}}^{(1)}$$

The gg potential

A "glueball" component: $|g(\boldsymbol{x}_1)g(\boldsymbol{x}_2)\rangle \equiv \sum_a A_a^i(\boldsymbol{x}_1)\, A_a^j(\boldsymbol{x}_2)\,|0\rangle$

has the potential $V_{gg} = \sqrt{\frac{N}{C_F}}\,\Lambda^2\,|\boldsymbol{x}_1 - \boldsymbol{x}_2| - N\,\frac{\alpha_s}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|}$

This agrees with the qgq̄ potential where the quarks coincide:

$$V_{gg}(\boldsymbol{x}, \boldsymbol{x}_g) = V_{qg\bar{q}}(\boldsymbol{x}, \boldsymbol{x}_g, \boldsymbol{x})$$

It is straightforward to work out the instantaneous potential for any Fock state.

$\mathcal{O}\left(\alpha_s^0\right)$ light $q\bar{q}$ bound states

An $\mathcal{O}\left(\alpha_s^0\right)$ meson state with $\boldsymbol{P} = 0$ and wave function Φ:

$$|M\rangle = \sum_{A,B;\alpha,\beta} \int d\boldsymbol{x}_1 d\boldsymbol{x}_2\, \bar{\psi}_\alpha^A(t=0,\boldsymbol{x}_1)\delta^{AB}\Phi_{\alpha\beta}(\boldsymbol{x}_1-\boldsymbol{x}_2)\psi_\beta^B(t=0,\boldsymbol{x}_2)\,|0\rangle$$

The bound state condition $H\,|M\rangle = M\,|M\rangle$ gives, at $\mathcal{O}\left(\alpha_s^0\right)$

$$\left[i\gamma^0\boldsymbol{\gamma}\cdot\overrightarrow{\boldsymbol{\nabla}} + m\gamma^0\right]\Phi(\boldsymbol{x}) + \Phi(\boldsymbol{x})\left[i\gamma^0\boldsymbol{\gamma}\cdot\overleftarrow{\boldsymbol{\nabla}} - m\gamma^0\right] = \left[M - V(|\boldsymbol{x}|)\right]\Phi(\boldsymbol{x})$$

where $\boldsymbol{x} \equiv \boldsymbol{x}_1 - \boldsymbol{x}_2$ and $V(|\,\boldsymbol{x}\,|) = V'|\,\boldsymbol{x}\,| = \Lambda^2|\,\boldsymbol{x}\,|$.

In the non-relativistic limit ($m \gg \Lambda$) this reduces to the Schrödinger equation.

If we add the instantaneous gluon exchange potential:

$\Rightarrow$ The quarkonium phenomenology with the Cornell potential.