Rotational symmetry in a light-front effective potential

Work done in collaboration with J.R. Hiller.

Sophia S. Chabysheva

Department of Physics
University of Idaho

December 2021
Overview

- light-front coordinates
- quenched scalar Yukawa theory
  - quenched to stabilize spectrum
- solution for single static source
  - cloud of neutrals in coherent state
  - source mass renormalized by self-energy
- solution for double source
  - overlap of clouds
  - recover Yukawa potential
  - deficit in number of neutrals
- summary
Why static source?

- develop method to consider static potential between quark and antiquark in QCD
  - mimic early lattice calculations
  - need to solve for gluon field configuration as Fock-state expansion w.r.t. fixed $q\bar{q}$

- note earlier work:
  - Burkardt and Klindworth [PRD 55, 1001 (1997)] applied a transverse lattice approach in (2+1)-dimensional QCD to the calculation of a $Q\bar{Q}$ potential which is nearly rotationally invariant
  - Rozowsky and Thorn [PRD 60, 045001 (1999)] studied force between two sources on a light front by arranging a purely transverse separation
  - Blunden, Burkardt, and Miller [PRC 61, 025206 (2000)] considered light-front models where a particle interacts with a static potential
**Light-front coordinates**

Dirac, RMP **21**, 392 (1949).

- **Time**: $x^+ = t + z$
- **Space**: $\underline{x} = (x^-, \vec{x}_\perp)$, $x^- \equiv t - z$, $\vec{x}_\perp = (x, y)$
- **Energy**: $p^- = E - p_z$
- **Momentum**: $\underline{p} = (p^+, \vec{p}_\perp)$, $p^+ \equiv E + p_z$, $\vec{p}_\perp = (p_x, p_y)$
- **Mass-shell condition**: $p^2 = m^2 \Rightarrow p^- = \frac{m^2 + p_\perp^2}{p^+}$
Static source trajectories

(a) Sources move steadily in the positive $x^-$ direction.

→ because of this consider QM of wave packets
Quenched scalar Yukawa theory

- complex scalar field $\chi$ with bare mass $m_0$ coupled to a real scalar field $\phi$ with mass $\mu$
  \[ \mathcal{L} = \partial_\mu \chi^* \partial^\mu \chi - m_0^2 |\chi|^2 + \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} \mu^2 \phi^2 - g \phi |\chi|^2. \]
- quenched form excludes pair production
  - otherwise spectrum unbounded from below
  - no mass renormalization for neutral
- light-front Hamiltonian density
  \[ \mathcal{H} = |\tilde{\partial}_\perp \chi|^2 + m_0^2 |\chi|^2 + \frac{1}{2} (\tilde{\partial}_\perp \phi)^2 + \frac{1}{2} \mu^2 \phi^2 + g \phi |\chi|^2. \]
- mode expansions for the fields
  \[
  \phi(x) = \int \frac{dp^+ d^2p_\perp}{\sqrt{16\pi^3 p^+}} \left[ a(p)e^{-ip\cdot x} + a^\dagger(p)e^{ip\cdot x} \right], \\
  \chi(x) = \int \frac{dp^+ d^2p_\perp}{\sqrt{16\pi^3 p^+}} \left[ c_+(p)e^{-ip\cdot x} + c_-^\dagger(p)e^{ip\cdot x} \right].
  \]
- nonzero commutators
  \[ [a(p), a^\dagger(p')] = \delta(p - p'), \quad [c_\pm(p), c_\pm^\dagger(p')] = \delta(p - p'). \]
Light-front Hamiltonian

\[ \mathcal{P}^- = \mathcal{P}_0^- + \mathcal{P}_{\text{int}}^- , \]
\[ \mathcal{P}_0^- = \int dp \frac{m_0^2 + \vec{p}_\perp^2}{p^+} \left[ c_+^\dagger(p) c_+(p) + c_-^\dagger(p) c_-(p) \right] \]
\[ + \int dq \frac{\mu^2 + q^2}{q^+} a^\dagger(q) a(q) \]
\[ \mathcal{P}_{\text{int}}^- = \]
\[ g \int \frac{dp dq}{\sqrt{16\pi^3 p^+ q^+ (p^+ + q^+)}} \left[ \left( c_+^\dagger(p + q) c_+(p) + c_-^\dagger(p + q) c_-(p) \right) a(q) \right. \]
\[ \left. + a^\dagger(q) \left( c_+^\dagger(p) c_+(p + q) + c_-^\dagger(p) c_-(p + q) \right) \right] \]

- this includes only emission and absorption

- light-front momentum operator
\[ \mathcal{P}^+ = \int dq \ q^+ a^\dagger(q) a(q) + \int dp \ p^+ [c_+^\dagger(p) c_+(p) + c_-^\dagger(p) c_-(p)] . \]

- ordinary energy operator \( \mathcal{E} = \frac{1}{2} (\mathcal{P}^- + \mathcal{P}^+) \).

- key role, because (ordinary) momentum \( \vec{p} \) is not conserved when static sources are present.
Single source at $\pm \vec{R}/2$

\[ |F^\pm \rangle = \int dp \sqrt{p^+} F^\pm (p) c^\dagger_\pm (p) |0\rangle, \]

- with the momentum-space envelope function $F^\pm$ peaked at $p = (m, \vec{0}_\perp)$
- Fourier transform yields the spatial probability which is peaked at $\pm \vec{R}/2$
  \[ \psi^\pm (\vec{x}) = \int \frac{dp}{\sqrt{16\pi^3}} F^\pm (p) e^{-i\vec{p} \cdot \vec{x}} \]

- the expectation value of the current for the static sources is
  \[ \langle F^\pm |: |\chi^2 : |F^\pm \rangle|_{x^+ = 0} = |\psi^\pm (\vec{x})|^2 \]
- we require that this become a delta function when the spatial packet is infinitesimally narrow
  \[ \rightarrow N^2 \delta (x^- \pm R_z) \delta (\vec{x}_\perp \mp \vec{R}_\perp/2) \]
- with $N^2 = \int d\vec{x} |\psi^\pm (\vec{x})|^2 = \int dp |F^\pm (p)|^2$
- the $p^+$ integral is extended to $-\infty$ for $F^\pm$ peaked at $p^+ = m$
For a single source, the expectation value for the free part of
the LF energy for the complex scalar field is given by
\[ \int dx \langle F^\pm | : \bar{\partial}_\perp \chi |^2 + m_0^2 |\chi|^2 : |F^\pm \rangle = \frac{m_0^2}{2m} \]

recall that \( E = (P^- + P^+)/2 \)
The expectation value for the LF longitudinal momentum is
\( m \), because the wave packet is sharply peaked at \( p^+ = m \)

\[ \langle F^\pm | E | F^\pm \rangle = \frac{m_0^2}{2m} + m/2 \]
which reduces to \( m \) for \( g = 0 \) \( \rightarrow m_0 = m \)

the normalization is determined by
\[ 1 = \langle F^\pm | F^\pm \rangle = \int dp p^+ |F^\pm (p)|^2 = m \int dp |F^\pm (p)|^2 = mN^2 \]
Single-source eigenvalue problem

- define the coherent state

$$|G^\pm\rangle = \sqrt{Z^\pm} e^{\int dq G^\pm(q) a(q)} |0\rangle$$

and the total (product) state

$$|G^\pm F^\pm\rangle = |G^\pm\rangle |F^\pm\rangle$$

- this is the charged source dressed by infinite number of neutral scalars

- $\vec{p}$ momentum is not conserved, and the coherent state includes all possible momenta

- $Z^\pm = e^{-\int dq |G^\pm(p)|^2}$

- the key property for this coherent state is

$$a(q)|G^\pm F^\pm\rangle = G^\pm(q)|G^\pm F^\pm\rangle$$

- we require that the coherent state is an eigenstate of the (ordinary) energy

$$\mathcal{E}|G^\pm F^\pm\rangle = E^\pm |G^\pm F^\pm\rangle$$
Projection

- Project onto $ \langle F^\pm \rangle$ and use the delta functions in $\langle F^\pm \rangle : \chi^2 : \langle F^\pm \rangle$ to do remaining $\times$ integrals.

- This leaves an equation for the structure of the coherent state:

$$\left[ \frac{m_0^2}{2m} + \frac{1}{2}m \right] |G^\pm\rangle + \frac{1}{2} \int dq \left[ \frac{q_1^2 + \mu^2}{q^+} + q^+ \right] a^\dagger(q) G^\pm(q) |G^\pm\rangle$$

$$+ \frac{g}{2m} \int dq \frac{dq}{\sqrt{16\pi^3 q^+}} \left\{ e^{\pm iq^+ R_z/2} \mp i q_\perp \cdot \vec{R}_\perp/2 G^\pm(q) \right.$$  

$$\left. + e^{\mp iq^+ R_z/2} \mp i q_\perp \cdot \vec{R}_\perp/2 a^\dagger(q) \right\} |G^\pm\rangle = E^\pm |G^\pm\rangle.$$

- Terms that contain $a^\dagger$ must cancel, to be consistent with the absence of $a^\dagger$ on the right.

- $E^\pm$ determined by remaining terms.
Single-source solution

the cancellation of terms yields
\[ G^{\pm}(q) = -\frac{g}{m} \sqrt{\frac{q^+}{16\pi^3}} e^{\mp i q^+ R_z / 2 \mp i q_\perp \cdot \vec{R}_\perp / 2} \]

collecting the remaining terms gives
\[ E^{\pm} = \frac{m_0^2}{2m} + \frac{1}{2} m + \frac{g}{2m} \int \frac{dq}{\sqrt{16\pi^3 q^+}} e^{\pm i q^+ R_z / 2 \pm i q_\perp \cdot \vec{R}_\perp / 2} G^{\pm}(q) \]
\[ = \frac{m_0^2}{2m} + \frac{1}{2} m - \frac{1}{2} \left( \frac{g}{m} \right)^2 \mu I(\Lambda) \]
\[ I(\Lambda) \equiv \int \frac{dq}{16\pi^3 \mu} \frac{\theta(\Lambda^2 - (q^+)^2 - q_\perp^2)}{(q^+)^2 + q_\perp^2 + \mu^2} \]

with the renormalization \( m_0^2 = m^2 + g^2 \frac{\mu}{m} I(\Lambda) \),
we can arrange \( E^{\pm} = m \)

the average number of neutral scalars in this single-source state is
\[ \langle n \rangle^{\pm} \equiv \int dq \langle G_1^{\pm} F^{\pm} | a^\dagger(q) a(q) | G_1^{\pm} F^{\pm} \rangle = \int dq \left| G_1^{\pm}(q) \right|^2 = \infty \]
can show that $|G^+ F^+\rangle |G^- F^-\rangle$ is an eigenstate of $\mathcal{E}$ with eigenenergy $E = \frac{m_0^2}{m} + m - (\frac{g}{m})^2 \mu I(\Lambda) - \frac{1}{2} (\frac{g}{m})^2 Y(R)$

\[ Y(R) \equiv \int \frac{dq}{16\pi^3} \frac{e^{iq^+ R_z + i\vec{q} \cdot \vec{R}} + e^{-iq^+ R_z - i\vec{q} \cdot \vec{R}}}{(q^+)^2 + q^2 + \mu^2} \]

the bare mass is renormalized as before

\[ m_0^2 = m^2 + g^2 \frac{\mu}{m} I(\Lambda) \]

the integrals in $Y(R)$ can be done, to obtain $Y(R) = \frac{1}{4\pi^2 R} \frac{\pi}{2} e^{-\mu R}$

this leaves an energy of $E = 2m - (\frac{g}{2m})^2 \frac{e^{-\mu R}}{4\pi R}$

$E$ is equal to the total mass plus the (correct) attractive, rotationally symmetric Yukawa potential for scalar exchange between scalars

note that unlike Yukawa theory for fermions, $g$ has units of mass
Evaluation of $Y(R)$

\[ Y(R) \equiv \int_{q^+ > 0} \frac{dq}{16\pi^3} \frac{e^{iq^+ R_z + i\vec{q}_\perp \cdot \vec{R}_\perp} + e^{-iq^+ R_z - i\vec{q}_\perp \cdot \vec{R}_\perp}}{(q^+)^2 + q^2_\perp + \mu^2} = \frac{1}{2} \int \]

Replacing $q^+$ by $-q^+$ and $\vec{q}_\perp = -\vec{q}_\perp$ yields equal integral.

Can then write as $\frac{1}{2}$ of integral with no restriction on $q^+$.

Evaluate in spherical coordinates $\vec{q} = (q_x, q_y, q^+) = (q, \theta, \phi)$, relative to an axis parallel to $\vec{R}$.

The $\phi$ integral is trivial.

\[ Y(R) = \frac{1}{16\pi^2} \int_{0}^{\infty} q^2 dq \int_{-1}^{1} d\cos \theta \frac{e^{iqR \cos \theta} + e^{-iqR \cos \theta}}{q^2 + \mu^2} \]

The $\cos \theta$ integral reduces this to

\[ Y(R) = \frac{1}{16\pi^2} \int_{0}^{\infty} \frac{q^2 dq}{q^2 + \mu^2} \left[ \frac{e^{iqR} - e^{-iqR}}{iqR} + \frac{e^{-iqR} - e^{iqR}}{-iqR} \right] \]

\[ = \frac{1}{4\pi^2 R} \int_{0}^{\infty} \frac{q^2 dq}{q^2 + \mu^2} \sin(qR) = \frac{1}{4\pi^2 R} \frac{\pi}{2} e^{-\mu R} \]
Differential number of neutrals

- change in number of neutral scalars induced by the proximity of two sources
  \[ \langle \delta n \rangle \equiv \int dq \langle a^\dagger(q) a(q) \rangle - \langle n \rangle_+ - \langle n \rangle_- \]

- only the interference terms contribute
  \[ \langle \delta n \rangle = \int dq \left[ G^{++}(q) G^{--}(q) + G^{--}(q) G^{++}(q) \right] \]

- can show this reduces to
  \[ \langle \delta n \rangle = - \frac{1}{16\pi^2} \left( \frac{g}{m} \right)^2 \left[ e^{\mu R} Ei(-\mu R) + e^{-\mu R} Ei(\mu R) \right] \]
  - \( Ei \) is the exponential integral function

- for large separations \( R \), this simplifies to
  \[ \langle \delta n \rangle = - \frac{1}{8\pi^2} \left( \frac{g}{m} \right)^2 \frac{1}{(\mu R)^2} + \mathcal{O}\left( \frac{1}{(\mu R)^3} \right) \]

- correctly goes to zero as the separation becomes infinite

- the interaction results from the interference between overlapping clouds of neutrals
effective potential arises from the overlap between clouds of neutral scalars that dress the charged sources.
- essentially an interference term in the expectation value of the energy

the calculation is nonperturbative, even though the Yukawa potential is order $g^2$
- the eigensolution is obtained to all orders in $g$ as a coherent state of neutral scalars
- static sources remove the constraint of momentum conservation

Next $\rightarrow$ Yukawa theory with 2 fermions as sources
- static in position but dynamic w.r.t. spin
have obtained the correct Yukawa potential for scalar exchange between static sources
- rotationally symmetric
- no fine tuning

comes from a nonperturbative calculation of the overlap between coherent states dressing the static sources

must consider ordinary energy and have sources static w.r.t ordinary time
- neither $P^-$ nor $P^+$ is conserved

changing coordinate systems does not change the physics