# Rotational symmetry in a light-front effective potential

Work done in collaboration with J.R. Hiller.

Sophia S. Chabysheva

Department of Physics
University of Idaho

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# Overview

- light-front coordinates
- quenched scalar Yukawa theory
  - quenched to stabilize spectrum
- solution for single static source
  - cloud of neutrals in coherent state
  - source mass renormalized by self-energy
- solution for double source
  - overlap of clouds
  - recover Yukawa potential
  - deficit in number of neutrals
- summary

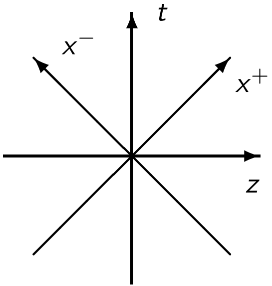
# Why static source?

- develop method to consider static potential between quark and antiquark in QCD
  - mimic early lattice calculations
  - need to solve for gluon field configuration as Fock-state expansion w.r.t. fixed $q\bar{q}$
- note earlier work:
  - Burkardt and Klindworth [PRD **55**, 1001 (1997)] applied a transverse lattice approach in (2+1)-dimensional QCD to the calculation of a $Q\bar{Q}$ potential which is nearly rotationally invariant
  - Rozowsky and Thorn [PRD **60**, 045001 (1999)] studied force between two sources on a light front by arranging a purely transverse separation
  - Blunden, Burkardt, and Miller [PRC **61**, 025206 (2000)] considered light-front models where a particle interacts with a static potential

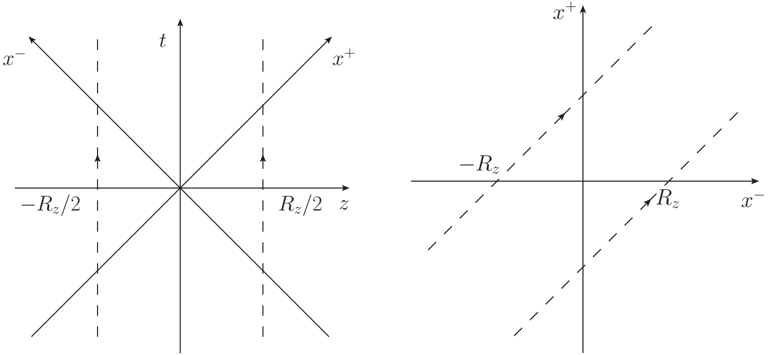
# Light-front coordinates

Dirac, RMP **21**, 392 (1949).

- *Time:* $x^+ = t + z$
- *Space:* $\underline{x} = (x^-, \vec{x}_\perp)$, $x^- \equiv t - z$, $\vec{x}_\perp = (x, y)$
- *Energy:* $p^- = E - p_z$
- *Momentum:* $\underline{p} = (p^+, \vec{p}_\perp)$, $p^+ \equiv E + p_z$, $\vec{p}_\perp = (p_x, p_y)$
- Mass-shell condition: $p^2 = m^2 \quad \Rightarrow \quad p^- = \frac{m^2 + p_\perp^2}{p^+}$



# Static source trajectories




(a) (b)

Sources move steadily in the positive $x^-$ direction.

$\rightarrow$ because of this consider QM of wave packets

# Quenched scalar Yukawa theory

- complex scalar field $\chi$ with bare mass $m_0$ coupled to a real scalar field $\phi$ with mass $\mu$
  $\mathcal{L} = \partial_\mu \chi^* \partial^\mu \chi - m_0^2 |\chi|^2 + \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}\mu^2\phi^2 - g\phi|\chi|^2.$
- quenched form excludes pair production
  - otherwise spectrum unbounded from below
  - no mass renormalization for neutral
- light-front Hamiltonian density
  $\mathcal{H} = |\vec{\partial}_\perp \chi|^2 + m_0^2|\chi|^2 + \frac{1}{2}(\vec{\partial}_\perp \phi)^2 + \frac{1}{2}\mu^2\phi^2 + g\phi|\chi|^2.$
- mode expansions for the fields
  $\phi(x) = \int \frac{dp^+ d^2p_\perp}{\sqrt{16\pi^3 p^+}} \left[a(\underline{p})e^{-ip\cdot x} + a^\dagger(\underline{p})e^{ip\cdot x}\right],$
  $\chi(x) = \int \frac{dp^+ d^2p_\perp}{\sqrt{16\pi^3 p^+}} \left[c_+(\underline{p})e^{-ip\cdot x} + c_-^\dagger(\underline{p})e^{ip\cdot x}\right].$
- nonzero commutators
  $[a(\underline{p}), a^\dagger(\underline{p}')] = \delta(\underline{p} - \underline{p}'), \quad [c_\pm(\underline{p}), c_\pm^\dagger(\underline{p}')] = \delta(\underline{p} - \underline{p}').$

# Light-front Hamiltonian

- $\mathcal{P}^- = \mathcal{P}_0^- + \mathcal{P}_{\rm int}^-,$
  $$\mathcal{P}_0^- = \int d\underline{p}\frac{m_0^2+\vec{p}_\perp^2}{p^+}\left[c_+^\dagger(\underline{p})c_+(\underline{p}) + c_-^\dagger(\underline{p})c_-(\underline{p})\right] + \int d\underline{q}\frac{\mu^2+\vec{q}_\perp^2}{q^+}a^\dagger(\underline{q})a(\underline{q})$$
  $$\mathcal{P}_{\rm int}^- = g\int \frac{d\underline{p}d\underline{q}}{\sqrt{16\pi^3p^+q^+(p^++q^+)}}\left[\left(c_+^\dagger(\underline{p}+\underline{q})c_+(\underline{p}) + c_-^\dagger(\underline{p}+\underline{q})c_-(\underline{p})\right)a(\underline{q}) + a^\dagger(\underline{q})\left(c_+^\dagger(\underline{p})c_+(\underline{p}+\underline{q}) + c_-^\dagger(\underline{p})c_-(\underline{p}+\underline{q})\right)\right]$$
  - this includes only emission and absorption
- light-front momentum operator
  $\mathcal{P}^+ = \int d\underline{q}\, q^+a^\dagger(\underline{q})a(\underline{q}) + \int d\underline{p}\, p^+[c_+^\dagger(\underline{p})c_+(\underline{p}) + c_-^\dagger(\underline{p})c_-(\underline{p})].$
- ordinary energy operator $\mathcal{E} = \frac{1}{2}(\mathcal{P}^- + \mathcal{P}^+)$.
  - key role, because (ordinary) momentum $\vec{p}$ is not conserved when static sources are present.

# Single source at $\pm\vec{R}/2$

- $|F^{\pm}\rangle = \int d\underline{p}\sqrt{p^{+}}F^{\pm}(\underline{p})c_{\pm}^{\dagger}(\underline{p})|0\rangle,$
  - with the momentum-space envelope function $F^{\pm}$ peaked at $\underline{p} = (m, \vec{0}_{\perp})$
  - Fourier transform yields the spatial probability which is peaked at $\pm\vec{R}/2$
    $\psi^{\pm}(\underline{x}) = \int \frac{d\underline{p}}{\sqrt{16\pi^3}} F^{\pm}(\underline{p})e^{-i\underline{p}\cdot\underline{x}}$
- the expectation value of the current for the static sources is $\langle F^{\pm}|:|\chi^2|:|F^{\pm}\rangle|_{x^+=0} = |\psi^{\pm}(\underline{x})|^2$
- we require that this become a delta function when the spatial packet is infinitesimally narrow
  $\rightarrow N^2\delta(x^{-} \pm R_z)\delta(\vec{x}_{\perp} \mp \vec{R}_{\perp}/2)$
  - with $N^2 = \int d\underline{x}|\psi^{\pm}(\underline{x})|^2 = \int d\underline{p}|F^{\pm}(\underline{p})|^2$
  - the $p^{+}$ integral is extended to $-\infty$ for $F^{\pm}$ peaked at $p^{+} = m$

# Expectation value of free $\mathcal{P}^-$

- For a single source, the expectation value for the free part of the LF energy for the complex scalar field is given by $\int d\underline{x}\langle F^{\pm}| : |\vec{\partial}_\perp \chi|^2 + m_0^2|\chi|^2 : |F^{\pm}\rangle = m_0^2/2m$
- recall that $\mathcal{E} = (\mathcal{P}^- + \mathcal{P}^+)/2$
  The expectation value for the LF longitudinal momentum is $m$, because the wave packet is sharply peaked at $p^+ = m$
- $\Rightarrow \langle F^{\pm}|\mathcal{E}|F^{\pm}\rangle = m_0^2/2m + m/2$
  - which reduces to $m$ for $g = 0 \ \rightarrow m_0 = m$
- the normalization is determined by
  $1 = \langle F^{\pm}|F^{\pm}\rangle = \int d\underline{p}\, p^+|F^{\pm}(\underline{p})|^2 = m\int d\underline{p}|F^{\pm}(\underline{p})|^2 = mN^2$

# Single-source eigenvalue problem

- define the coherent state
  $|G^{\pm}\rangle = \sqrt{Z^{\pm}} e^{\int d\underline{q} G^{\pm}(\underline{q}) a^{\dagger}(\underline{q})} |0\rangle$
  and the total (product) state $|G^{\pm}F^{\pm}\rangle = |G^{\pm}\rangle|F^{\pm}\rangle$
  - this is the charged source dressed by infinite number of neutral scalars
  - $\vec{p}$ momentum is not conserved, and the coherent state includes all possible momenta
  - $Z^{\pm} = e^{-\int d\underline{q} |G^{\pm}(\underline{p})|^2}$
- the key property for this coherent state is
  $a(\underline{q})|G^{\pm}F^{\pm}\rangle = G^{\pm}(\underline{q})|G^{\pm}F^{\pm}\rangle$
- we require that the coherent state is an eigenstate of the (ordinary) energy
  $\mathcal{E}|G^{\pm}F^{\pm}\rangle = E^{\pm}|G^{\pm}F^{\pm}\rangle$

# Projection

- project onto $\langle F^{\pm}|$ and use the delta functions in $\langle F^{\pm}|:|\chi^2|:|F^{\pm}\rangle$ to do remaining $\underline{x}$ integrals
- this leaves an equation for the structure of the coherent state

$$\left[\frac{m_0^2}{2m}+\frac{1}{2}m\right]|G^{\pm}\rangle+\frac{1}{2}\int d\underline{q}\left[\frac{q_\perp^2+\mu^2}{q^+}+q^+\right]a^\dagger(\underline{q})G^{\pm}(\underline{q})|G^{\pm}\rangle$$
$$+\frac{g}{2m}\int\frac{d\underline{q}}{\sqrt{16\pi^3 q^+}}\left\{e^{\pm iq^+R_z/2\pm i\vec{q}_\perp\cdot\vec{R}_\perp/2}G^{\pm}(\underline{q})\right.$$
$$\left.+e^{\mp iq^+R_z/2\mp i\vec{q}_\perp\cdot\vec{R}_\perp/2}a^\dagger(\underline{q})\right\}|G^{\pm}\rangle=E^{\pm}|G^{\pm}\rangle.$$

- terms that contain $a^\dagger$ must cancel, to be consistent with the absence of $a^\dagger$ on the right
- $E^{\pm}$ determined by remaining terms

# Single-source solution

- the cancellation of terms yields
  $G^{\pm}(\underline{q}) = -\frac{g}{m}\sqrt{\frac{q^+}{16\pi^3}}\frac{e^{\mp iq^+R_z/2 \mp i\vec{q}_\perp\cdot\vec{R}_\perp/2}}{(q^+)^2+q_\perp^2+\mu^2}$
- collecting the remaining terms gives
  $E^{\pm} = \frac{m_0^2}{2m} + \frac{1}{2}m + \frac{g}{2m}\int\frac{d\underline{q}}{\sqrt{16\pi^3 q^+}}e^{\pm iq^+R_z/2 \pm i\vec{q}_\perp\cdot\vec{R}_\perp/2}G^{\pm}(\underline{q})$
  $= \frac{m_0^2}{2m} + \frac{1}{2}m - \frac{1}{2}\left(\frac{g}{m}\right)^2\mu I(\Lambda)$
  - $I(\Lambda) \equiv \int\frac{d\underline{q}}{16\pi^3\mu}\frac{\theta(\Lambda^2-(q^+)^2-q_\perp^2)}{(q^+)^2+q_\perp^2+\mu^2}$
- with the renormalization $m_0^2 = m^2 + g^2\frac{\mu}{m}I(\Lambda)$,
  we can arrange $E^{\pm} = m$
- the average number of neutral scalars in this single-source state is
  $\langle n\rangle_{\pm} \equiv \int d\underline{q}\langle G_1^{\pm}F^{\pm}|a^\dagger(\underline{q})a(\underline{q})|G_1^{\pm}F^{\pm}\rangle = \int d\underline{q}|G_1^{\pm}(\underline{q})|^2 = \infty$

# Double-source solution

- can show that $|G^+F^+\rangle|G^-F^-\rangle$ is an eigenstate of $\mathcal{E}$ with eigenenergy $E = \frac{m_0^2}{m} + m - \left(\frac{g}{m}\right)^2 \mu I(\Lambda) - \frac{1}{2}\left(\frac{g}{m}\right)^2 Y(R)$
  - $Y(R) \equiv \int \frac{d\underline{q}}{16\pi^3} \frac{e^{iq^+ R_z + i\vec{q}_\perp \cdot \vec{R}_\perp} + e^{-iq^+ R_z - i\vec{q}_\perp \cdot \vec{R}_\perp}}{(q^+)^2 + q_\perp^2 + \mu^2}$
- the bare mass is renormalized as before $m_0^2 = m^2 + g^2 \frac{\mu}{m} I(\Lambda)$
- the integrals in $Y(R)$ can be done, to obtain $Y(R) = \frac{1}{4\pi^2 R} \frac{\pi}{2} e^{-\mu R}$
- this leaves an energy of $E = 2m - \left(\frac{g}{2m}\right)^2 \frac{e^{-\mu R}}{4\pi R}$
  - $E$ is equal to the total mass plus the (correct) attractive, rotationally symmetric Yukawa potential for scalar exchange between scalars
  - note that unlike Yukawa theory for fermions, $g$ has units of mass

# Evaluation of $Y(R)$

- $Y(R) \equiv \int_{q^+>0} \frac{dq}{16\pi^3} \frac{e^{iq^+ R_z + i\vec{q}_\perp \cdot \vec{R}_\perp} + e^{-iq^+ R_z - i\vec{q}_\perp \cdot \vec{R}_\perp}}{(q^+)^2 + q_\perp^2 + \mu^2} = \frac{1}{2}\int$
  - replacing $q^+$ by $-q^+$ and $\vec{q}_\perp = -\vec{q}_\perp$ yields equal integral
  - can then write as $\frac{1}{2}$ of integral with no restriction on $q^+$
- evaluate in spherical coordinates $\vec{q} = (q_x, q_y, q^+) = (q, \theta, \phi)$, relative to an axis parallel to $\vec{R}$
- the $\phi$ integral is trivial
  $Y(R) = \frac{1}{16\pi^2} \int_0^\infty q^2 dq \int_{-1}^{1} d\cos\theta \frac{e^{iqR\cos\theta} + e^{-iqR\cos\theta}}{q^2+\mu^2}$
- the $\cos\theta$ integral reduces this to
  $Y(R) = \frac{1}{16\pi^2} \int_0^\infty \frac{q^2 dq}{q^2+\mu^2} \left[ \frac{e^{iqR} - e^{-iqR}}{iqR} + \frac{e^{-iqR} - e^{iqR}}{-iqR} \right]$
  $= \frac{1}{4\pi^2 R} \int_0^\infty \frac{q^2 dq}{q^2+\mu^2} \sin(qR) = \frac{1}{4\pi^2 R} \frac{\pi}{2} e^{-\mu R}$

# Differential number of neutrals

- change in number of neutral scalars induced by the proximity of two sources
  $\langle\delta n\rangle \equiv \int d\underline{q}\langle a^\dagger(\underline{q})a(\underline{q})\rangle - \langle n\rangle_+ - \langle n\rangle_-$
- only the interference terms contribute
  $\langle\delta n\rangle = \int d\underline{q}\left[G^{+*}(\underline{q})G^-(\underline{q}) + G^{-*}(\underline{q})G^+(\underline{q})\right]$
- can show this reduces to
  $\langle\delta n\rangle = -\frac{1}{16\pi^2}\left(\frac{g}{m}\right)^2\left[e^{\mu R}\mathrm{Ei}(-\mu R) + e^{-\mu R}\mathrm{Ei}(\mu R)\right]$
  - $\mathrm{Ei}$ is the exponential integral function
- for large separations $R$, this simplifies to
  $\langle\delta n\rangle = -\frac{1}{8\pi^2}\left(\frac{g}{m}\right)^2\frac{1}{(\mu R)^2} + \mathcal{O}(\frac{1}{(\mu R)^3})$
- correctly goes to zero as the separation becomes infinite
- the interaction results from the interference between overlapping clouds of neutrals

# Remarks

- effective potential arises from the overlap between clouds of neutral scalars that dress the charged sources.
  - essentially an interference term in the expectation value of the energy
- the calculation is nonperturbative, even though the Yukawa potential is order $g^2$
  - the eigensolution is obtained to all orders in $g$ as a coherent state of neutral scalars
  - static sources remove the constraint of momentum conservation
- Next $\rightarrow$ Yukawa theory with 2 fermions as sources
  - static in position but dynamic w.r.t. spin

# Summary

- have obtained the correct Yukawa potential for scalar exchange between static sources
  - rotationally symmetric
  - no fine tuning
- comes from a nonperturbative calculation of the overlap between coherent states dressing the static sources
- must consider ordinary energy and have sources static w.r.t ordinary time
  - neither $P^-$ nor $P^+$ is conserved
- changing coordinate systems does not change the physics