Loop calculations in the null-plane causal perturbation theory

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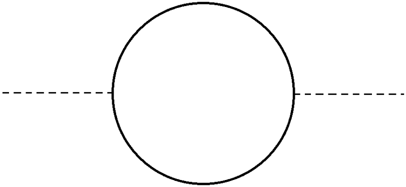
Introduction

Light-front dynamics: $\widehat{S}^F(p) = (2\pi)^{-2}\left(\frac{\not{p}+m}{p^2-m^2+i0^+} - \frac{\gamma^+}{2p_-}\right)$.



Feynman's rules:

$$\int d^4q \mathrm{Tr}\left[\widehat{S}^F(q)\widehat{S}^F(p-q)\right]$$

Spurious poles need regularization $\rightarrow$ Prescriptions.
Non-local interaction terms in the Hamiltonian.

$S(g)$-operator:

$$S(g) = 1 + \sum_{n=1}^{+\infty} \frac{1}{n!} \int dX T_n(X) g(X)$$

$$T_n(X) \equiv T_n(x_1; \cdots; x_n)\ ,\ g(X) \equiv g(x_1) \cdots g(x_n)\ ,\ dX \equiv d^4x_1 \cdots d^4x_n$$

Axiom of causality: Let $g_1, g_2 \in \mathscr{S}(\mathbb{R}^4)$ be two switching functions. If their supports satisfy $\text{supp}(g_1) < \text{supp}(g_2)$, then:

$$S(g_1 + g_2) = S(g_2)S(g_1)\ .$$

If $\text{supp}(g_1) \sim \text{supp}(g_2)$, then the causal decomposition remains valid, and $S(g_1)$ and $S(g_2)$ commute.

Perturbatively:

$$T_n(X) = T_m(X_2)T_{n-m}(X_1) \quad ; \quad X_1 < X_2$$

$$[T_n(X); T_m(Y)] = 0 \quad ; \quad X \sim Y$$

How to chronologically order?

1) Product of distributions by Heaviside's functions is ill-defined:

$$\widehat{\Theta\delta}(p) = (2\pi)^{-1/2} \int dq \widehat{\Theta}(q)\hat{\delta}(p-q) = i(2\pi)^{-3/2} \int \frac{dq}{q + i0^+}$$

2) Use causality axiom[3,4]:

$$T(X_1 \cup X_2) - T(X_2)T(X_1) = \begin{cases} 0 & ; \quad X_1 < X_2 \text{ (or } X_1 \sim X_2) \\ [T(X_1); T(X_2)] & ; \quad X_1 > X_2 \end{cases}$$

This is an advanced distribution with respect to X_1. Generalizing:

$$A_n(Y; x_n) := \sum_{r=1}^{n} (-1)^{r-1} \sum_{\substack{X_1, \cdots, X_r \neq \emptyset \\ X_1 \cup \cdots \cup X_r = Y \cup \{x_n\} \\ X_j \cap X_k = \emptyset, \forall j \neq k \\ x_n \in X_r}} T_{n_1}(X_1) \cdots T_{n_r}(X_r)$$

$$R_n(Y; x_n) := \sum_{r=1}^{n} (-1)^{r-1} \sum_{\substack{X_1, \cdots, X_r \neq \emptyset \\ X_1 \cup \cdots \cup X_r = Y \cup \{x_n\} \\ X_j \cap X_k = \emptyset, \forall j \neq k \\ x_n \in X_1}} T_{n_1}(X_1) \cdots T_{n_r}(X_r)$$

[3] Epstein and Glaser. Ann. Inst. H. Poincaré A **19**: 211-295 (1973)

[4] Scharf. *Finite Quantum Electrodynamics. The Causal Approach*. 3rd edition. Dover (2014).

Separate the n-point transition distribution:

$$\begin{aligned} A_n(Y; x_n) &= T_n(Y \cup \{x_n\}) + A'_n(Y; x_n) \\ R_n(Y; x_n) &= T_n(Y \cup \{x_n\}) + R'_n(Y; x_n) \end{aligned}$$

Causal distribution:

$$\begin{aligned} D_n(Y; x_n) &:= R_n(Y; x_n) - A_n(Y; x_n) \\ &= R'_n(Y; x_n) - A'_n(Y; x_n) \end{aligned}$$

Recover the n-point transition distribution by splitting D_n:

$$\begin{aligned} T_n(Y \cup \{x_n\}) &= A_n(Y; x_n) - A'_n(Y; x_n) \\ &= R_n(Y; x_n) - R'_n(Y; x_n) \end{aligned}$$

$\Rightarrow$ Inductive procedure for finding T_n by knowing T_m, $m \leq n-1$.
How to split the causal distribution?

$$D_n(x_1;\cdots;x_n) = \sum_k d_n^k(x_1;\cdots;x_n) \ :C_k\left(u^A\right):$$

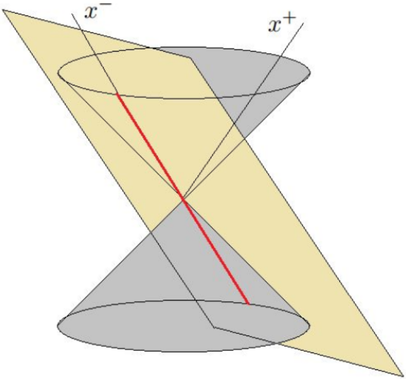
Numerical distribution:

$$d(x) := d_n^k(x_1 - x_n;\cdots;x_{n-1} - x_n;0)$$

$$d = r - a$$

$$\text{supp}(r) \subseteq \Gamma^+_{n-1}(0) \quad , \quad \text{supp}(a) \subseteq \Gamma^-_{n-1}(0)$$

Splitting according to the x^+ time requires to know the behaviour of the causal distribution at the x^- axis[5,6].



[5] Acevedo, Pimentel and Soto. Proc. Sci. **LC2019**: 021 (2020).
[6] Acevedo and Pimentel. *Null-plane causal perturbation theory*. Submitted.

Def.: Let $d \in \mathscr{S}'(\mathbb{R}^m)$ be a distribution, and let ρ be a continuous positive function. If the limit

$$\lim_{s \to 0^+} \rho(s) s^{3m/4} d\left(sx^+; sx^\perp; x^-\right) = d_-(x)$$

exists in $\mathscr{S}'(\mathbb{R}^m)$ and is non-null, then d_- is called the **quasi-asymptotics** of d at the x^- axis, with regard to the function ρ.

$\Rightarrow$ ρ is an automodel function:

$$\lim_{s \to 0^+} \frac{\rho(as)}{\rho(s)} = a^\alpha \quad , \quad \rho(s) = s^\alpha \rho_0(s)$$

ρ_0: slowly varying function.

Def.: If the quasi-asymptotics of the distribution $d \in \mathscr{S}'(\mathbb{R}^m)$ at the x^- axis is obtained with the automodel function $\rho(s) = s^{\omega_-} \rho_0(s)$, then the number ω_- is the **singular order at the x^- axis** of the distribution d.

Steps for the causal distribution splitting:

1) Replace Heaviside's function by a continuous, monotonous non-decreasing function

$$\chi(t) = \begin{cases} 0 & ; \quad t \leq 0 \\ < 1 & ; \quad 0 < t < 1 \\ 1 & ; \quad t \geq 1 \end{cases}$$

2) Multiply the causal distribution by a function of compact support in the x^- coordinate:

$$d_{\mathfrak{b}}(x) := d(x)\mathfrak{b}(x^-) \quad , \quad \mathfrak{b} \in \mathscr{D}\left(\mathbb{R}^{m/4}\right) \quad , \quad \mathfrak{b}(x^-) \in [0;1]$$

3) Define the auxiliary retarded and auxiliary advanced distributions:

$$r_{\mathfrak{b}}(x) := \lim_{s \to 0^+} \chi\left(\frac{x^+}{s}\right) d_{\mathfrak{b}}(x) \quad , \quad a_{\mathfrak{b}}(x) := -\lim_{s \to 0^+} \chi\left(-\frac{x^+}{s}\right) d_{\mathfrak{b}}(x)$$

4) Recover the retarded and advanced distributions by the limit $\mathfrak{b} \to 1$:

$$r(x) := \lim_{\mathfrak{b} \to 1} r_{\mathfrak{b}}(x) \quad , \quad a(x) := \lim_{\mathfrak{b} \to 1} a_{\mathfrak{b}}(x)$$

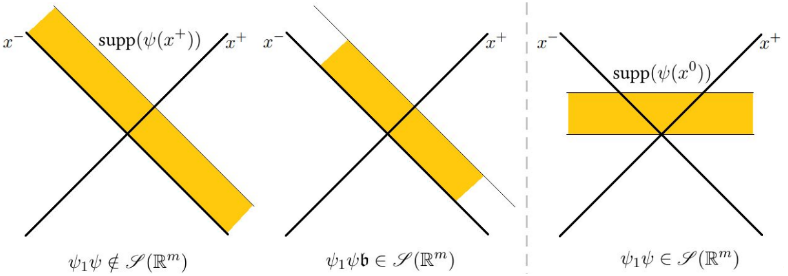
Cauchy's criterion: The limit defining $r_{\mathfrak{b}}(x)$ exists if and only if:

$$\forall k \geq 1 \;:\quad \lim_{s\to 0^+} \psi\left(\frac{x^+}{s}\right) d_{\mathfrak{b}}(x) = 0 \quad ; \quad \psi(t) := \chi(kt) - \chi(t)$$

Applying to a test function $\varphi \in \mathscr{S}(\mathbb{R}^m)$ and introducing $\psi_1 \in \mathscr{C}^{+\infty}(\mathbb{R}^m)$ equal to 1 in a neighbourhood of $\Gamma^+_{n-1}(0) \cup \Gamma^-_{n-1}(0)$ and equal to 0 outside it:

$$\lim_{s\to 0^+} \left\langle \psi\left(\frac{x^+}{s}\right) d_{\mathfrak{b}}(x); \varphi(x) \right\rangle = \lim_{s\to 0^+} \frac{\langle \varphi_- d_-; \psi\psi_1\mathfrak{b}\rangle}{s^{\omega_-}} = 0$$

Here: $\varphi_-(x^-) := \varphi(0; 0^\perp; x^-)$.
This is satisfied for $\omega_- < 0$, but could be unsatisfied for $\omega_- \geq 0$!



For $\omega_- < 0$:

Retarded distribution is well-defined. In momentum space:

$$\hat{r}_{\mathfrak{b}}(p) = \frac{i}{2\pi} \int \frac{\hat{d}_{\mathfrak{b}}\left((p_{1+} - k; \boldsymbol{p}_1); \cdots ; (p_{n-1+} - k; \boldsymbol{p}_{n-1})\right)}{k + i0^+} dk$$

$$\equiv \frac{i}{2\pi} \int \frac{\hat{d}_{\mathfrak{b}}(p_+ - k; \boldsymbol{p})}{k + i0^+} dk$$

Here:

$$\hat{d}_{\mathfrak{b}}(p) = (2\pi)^{-m/8} \int \hat{d}(p_+; p_\perp; q)\hat{\mathfrak{b}}(p_- - q) d^{m/4}q$$

The limit $\mathfrak{b}(x^-) \to 1$ is taken as: $\hat{\mathfrak{b}}(p_-) \to (2\pi)^{m/8}\delta(p_-)$.

For $\omega_- \geq 0$:

Consider $\varphi(x) = \left(x^+\right)^{b_1}\left(x^\perp\right)^{b_2}\widetilde{\varphi}(x)$, $\widetilde{\varphi}(x) \in \mathscr{S}\left(\mathbb{R}^m\right)$, $b_1 + b_2 = |b|$.

$$\lim_{s\to 0^+}\frac{\langle \varphi_- d_-; \psi\psi_1\mathfrak{b}\rangle}{s^{\omega_-}} \to s^{|b|-\omega_-}\left\langle \widetilde{\varphi}_- d_-; \left(x^+\right)^{b_1}\left(x^\perp\right)^{b_2}\psi\psi_1\mathfrak{b}\right\rangle$$

The retarded distribution is well-defined on $\mathscr{S}(\mathbb{R}^m)$ if it acts after the projection into the restricted space:

$$(W\varphi)(x) := \varphi(x) - w\left(x^+; x^\perp\right)\sum_{|b|=0}^{\lfloor\omega_-\rfloor}\frac{\left(x^{+,\alpha}\right)^b}{b!}D^b_{+,\alpha}\varphi\left(0; 0^\perp; x^-\right)$$

Retarded distribution normalized at q:

$$\hat{r}_{\mathfrak{b},q}(p) = \frac{i}{2\pi}\int\frac{dk}{k+i0^+}\left[\hat{d}_{\mathfrak{b}}(p_+ - k; \boldsymbol{p}) - \sum_{|c|=0}^{\lfloor\omega_-\rfloor}\frac{1}{c!}(p_{+,\alpha} - q_{+,\alpha})^c D^c_{+,\alpha}\hat{d}_{\mathfrak{b}}(q_+ - k; q_\perp; p_-)\right]$$
$$+\sum_{b=0}^{\lfloor\omega_-\rfloor}\widehat{C}_b(p_-)p_+^{b_1}p_\alpha^{b_2} \quad , \quad |b| = b_1 + b_2$$

Yukawa's model:

$$T_1(x) = \mathfrak{g}\ :\overline{\psi}(x)\gamma^5\psi(x):\ \varphi(x)$$

Second order causal distribution:

$$D_2(x_1; x_2) = [T_1(x_1); T_1(x_2)]$$

For fermion's self-energy:

$$D_2^{(\mathrm{FSE})}(x_1; x_2) = :\overline{\psi}(x_1)d(y)\psi(x_2): - :\overline{\psi}(x_2)d(-y)\psi(x_1): \quad , \quad y := x_1 - x_2$$

$$d(y) = -\mathfrak{g}^2\gamma^5 \left(S_+(y)D_+(y) - S_-(y)D_-(y)\right)\gamma^5$$

In momentum space:

$$\widehat{D}_\pm(p) = \pm\frac{i}{2\pi}\Theta\left(\pm p_-\right)\delta(p^2 - m^2) \quad , \quad \widehat{S}_\pm(p) = (\not{p} + m)\widehat{D}_\pm(p)$$

Then:

$$\hat{d}(p) = \frac{\mathfrak{g}^2}{4(2\pi)^3}\left\{m_1 p^2 - \frac{\not{p}}{2}\left[p^2 + (m_1^2 - m_2^2)\right]\right\}\hat{d}_1(p)$$

$$\hat{d}_1(p) = \mathrm{sgn}(p_-)\Theta\left[p^2 - (m_1 + m_2)^2\right]\frac{1}{p^2}\sqrt{1 - \frac{2(m_1^2 + m_2^2)}{p^2} + \frac{(m_1^2 - m_2^2)^2}{p^4}}$$

[7] Acevedo and Pimentel. Phys. Rev. D **103**: 076022 (2021).

[8] Acevedo, Beltrán, Pimentel and Soto. Eur. Phys. J. Plus **136**: 895 (2021).

Fermion's self-energy in Yukawa's model

Splitting *formulae* $\Rightarrow$ It suffices to split d_1, $\omega_-[d_1] = -1 < 0$.

$$\hat{r}_1(p) = \frac{i}{2\pi} \int \frac{dk}{k + i0^+} \hat{d}_1(p_+ - k; p_\perp; p_-)$$
$$= \frac{i}{2\pi} J + \frac{1}{2p^2} \text{sgn}(p_-) \Theta\left[p^2 - (m_1 + m_2)^2\right] \sqrt{1 - \frac{2(m_1^2 + m_2^2)}{p^2} + \frac{(m_1^2 - m_2^2)^2}{p^4}}$$

$$J = \mathscr{P} \int\limits_{(m_1+m_2)^2}^{+\infty} \frac{ds}{s^2(p^2 - s)} \sqrt{[s - (m_1 + m_2)^2]\,[s - (m_1 - m_2)^2]}$$

Multiplying by the factored polynomial to obtain r, then subtracting r':

$$\widehat{\Sigma}(p) = \frac{\mathfrak{g}^2}{4(2\pi)^4} \left(m_1 p^2 - \frac{\not{p}}{2} \left[p^2 + (m_1^2 - m_2^2)\right] \right)$$
$$\times \left\{ J(p) - i\pi\Theta\left[p^2 - (m_1 + m_2)^2\right] \frac{1}{p^2} \sqrt{1 - \frac{2(m_1^2 + m_2^2)}{p^2} + \frac{(m_1^2 - m_2^2)^2}{p^4}} \right\}$$

Fermion's self-energy in Yukawa's model

J can be analytically calculated by using third Euler's substitution:

$$s = \frac{(m_1+m_2)^2 - (m_1-m_2)^2 x^2}{1-x^2} \qquad 0 < x < 1$$

For $p^2 < (m_1-m_2)^2 \;\vee\; p^2 > (m_1+m_2)^2$:

$$J = \frac{p^2-(m_1-m_2)^2}{p^4}\left[\frac{b^2-a^2}{a^2-1} + b\log\left(\left|\frac{1+b}{1-b}\right|\right) - \frac{a^2+b^2}{2a}\log\left(\frac{a+1}{a-1}\right)\right]$$

For $(m_1-m_2)^2 < p^2 < (m_1+m_2)^2$:

$$J = \frac{p^2-(m_1-m_2)^2}{p^4}\,\frac{(b^2-a^2)^2}{2a(a^2+b^2)^2} \times\left[(b^2-a^2)\log\left(\frac{a+1}{a-1}\right) + 4ab\tan^{-1}\left(\frac{1}{b}\right) - \frac{2a(a^2+b^2)}{a^2-1}\right]$$

In these expressions:

$$a = \frac{m_1+m_2}{m_1-m_2} > 1 \qquad b = \sqrt{\left|\frac{p^2-(m_1+m_2)^2}{p^2-(m_1-m_2)^2}\right|}$$

Vacuum polarization in QED[9]

QED:

$$T_1(x) = ie \ :\overline{\psi}(x)\gamma^a\psi(x):\ A_a(x)$$

Causal distribution of second order vacuum polarization:

$$D_2^{(\mathrm{VP})}(x_1;x_2) = :A_a(x_1)d^{ab}(y)A_b(x_2):$$

$$d^{ab}(y) = P^{ab}(y) - P^{ba}(-y) \quad , \quad P^{ab}(y) = e^2\mathrm{Tr}\left[\gamma^a S_+(y)\gamma^b S_-(-y)\right]$$

In momentum space:

$$\hat{d}^{ab}(k) = \frac{e^2}{3(2\pi)^3}\left(k^2 g^{ab} - k^a k^b\right)\hat{d}_1(k)$$

$$\hat{d}_1(k) = \left(1 + \frac{2m^2}{k^2}\right)\mathrm{sgn}(k_-)\Theta(k^2 - 4m^2)\sqrt{1 - \frac{4m^2}{k^2}}$$

Singular order: $\omega_-\left[d_1\right] = 0 \geq 0$.

$$\hat{r}_1(k) = \frac{i}{2\pi}\int\frac{dq}{q + i0^+}\left\{\hat{d}_1(k_+ - q; k_\perp; k_-) - \hat{d}_1(-q; 0_\perp; k_-)\right\}$$

$$= \frac{i}{2\pi}k^2 E(k) + \frac{1}{2}\mathrm{sgn}(k_-)\Theta(k^2 - 4m^2)\left(1 + \frac{2m^2}{k^2}\right)\sqrt{1 - \frac{4m^2}{k^2}}$$

[9] Acevedo and Pimentel. *Quantum electrodynamics in the null-plane causal perturbation theory I*. In preparation.

Vacuum polarization in QED

Using parameter ξ defined by $\dfrac{k^2}{m^2} = -\dfrac{(1-\xi)^2}{\xi}$:

$$E(k) = \mathscr{P}\int_{4m^2}^{+\infty} \frac{(s+2m^2)\sqrt{1-\dfrac{4m^2}{s}}}{s^2(k^2-s)}\,ds$$

$$= \frac{m^2}{k^4}\left[\frac{1+\xi}{1-\xi}\left(\xi - 4 + \frac{1}{\xi}\right)\log(\xi) + \frac{5}{3}\left(\xi + \frac{1}{\xi}\right) - \frac{22}{3}\right]$$

Multiplying by the factored polynomial to obtain r, then subtracting r':

$$\widehat{\Pi}^{ab}(k) = (2\pi)^{-4}\left(\frac{k^a k^b}{k^2} - g^{ab}\right)\widehat{\Pi}(k)$$

$$\widehat{\Pi}(k) = \frac{e^2 m^2}{3}\left\{\left[\frac{1+\xi}{1-\xi}\left(\xi - 4 + \frac{1}{\xi}\right)\log(\xi) + \frac{5}{3}\left(\xi + \frac{1}{\xi}\right) - \frac{22}{3}\right]\right.$$
$$\left. - i\pi\Theta(k^2 - 4m^2)(k^2 + 2m^2)\sqrt{1 - \frac{4m^2}{k^2}}\right\}$$

Conclusions

- Null-plane CPT avoids the necessity of using Feynman's propagators in loop calculations.
- No spurious poles appear, no prescriptions are needed.
- The equivalence with instant dynamics can be established in a very clear way.

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