Light front approach to hadrons on quantum computers

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Light Cone 2021: Physics of Hadrons on the Light Front

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Outline

1. Introduction to quantum computing
   - Recent developments
   - Variational quantum eigensolvers

2. Application to nuclear bound states: light mesons
   - Effective light-front Hamiltonian
   - Meson spectroscopy
   - Observables

3. Summary and outlook

Developments in Quantum Computing

Quantum computing has come a long way in past 40 years

- Quantum implementation of Toffoli Gate (1980)
- Deutsch-Jozsa Algorithm: First example of quantum algorithm that is exponentially faster (1992)
- Shor's Algorithm: Factoring large numbers (1994)
- Quantum Error Correction (1995)
- Transmon Qubits (2007)
- Variational Quantum Eigensolver: broad applications in quantum chemistry (2014)
- Quantum Machine Learning: Quantum classifier, Quantum Support Vector Machines, Quantum Approximation Optimization, etc (2017)

[Preskill, 2018]

Noisy Intermediate-Scale Quantum (NISQ): those devices whose qubits and quantum operations are substantially imperfect.

[Bharti, 2101.08448]

Quantum advantage: a purpose-specific computation that involves a quantum device and that can not be performed classically with a reasonable resources. [Google AI, Arute 2019]

- Sampling pseudo-random quantum circuit with 53-qubit Sycamore superconducting chip
- Gaussian boson sampling with Jiuzhang photonic quantum computer [UTSC, Zhong 2020]

Our application: **Variational Quantum Eigensolver (VQE)** and **Subspace-search VQE (SSVQE)** approaches to hadronic bound-state problems on the light front

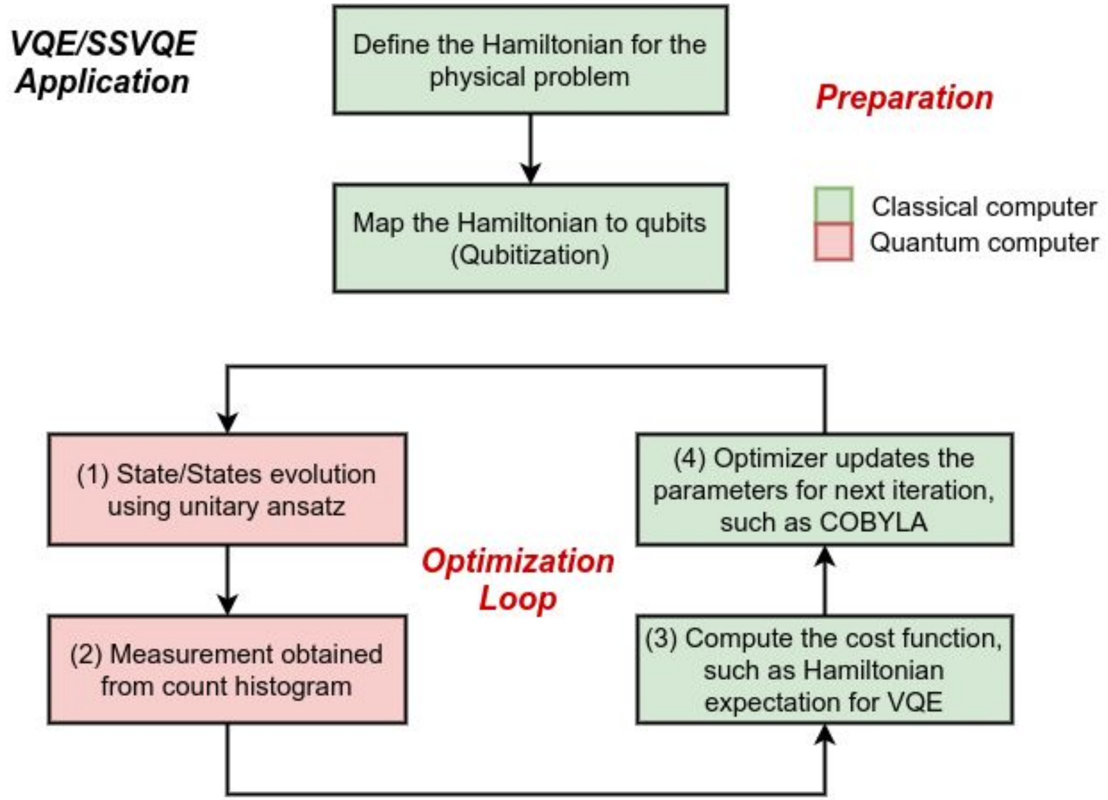
VQE is a hybrid algorithm

[Peruzzo, 1304.3061]

[Nakanishi, 1810.09434]

VQE is directly inspired by the variational principle.

The core idea is to use a **parameterized unitary ansatz** (that produces the “trial wave function”) to obtain the lowest eigenvalue via continuous optimizations.



Why are we interested in quantum computing?

- *"Nature isn't classical, dammit, and if you want to make a simulation of nature, you'd better make it quantum mechanical"* (Richard Feynman)

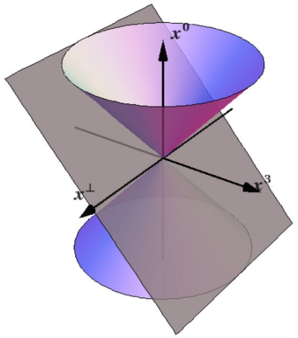
  Many problems are inherently quantum mechanical.

- Vast amount of encoded information in a many-qubit state: the total state space of n-qubit goes with 2^n
- High scalability in quantum applications (compact encoding)
- Many-body problems and quantum computing are similar by nature
- Rapid progress in quantum hardware (~100 qubit capability)

Effective model of light mesons

Basis Light-front Quantization (BLFQ) [Vary, 0905.1411]

- Light-front dynamics
  simple dispersion relation
- Hamiltonian approach
  eigenvalues => mass spectrum
  eigenfunctions => calculating observables
- Basis function approach
  exploit symmetry; discretize the Hamiltonian



front form

In the application of the light mesons within the valence $|q\bar{q}\rangle$ Fock sector

[Qian, 2005.13806]

$$H_{\mathrm{eff},\gamma_5} = \overbrace{\underbrace{\frac{\mathbf{k}_\perp^2 + m_q^2}{x} + \frac{\mathbf{k}_\perp^2 + m_{\bar{q}}^2}{1-x}}_{\text{LF kinetic energy}} + \underbrace{\kappa^4 x(1-x)\mathbf{r}_\perp^2 - \frac{\kappa^4}{(m_q+m_{\bar{q}})^2}\frac{\partial}{\partial x}\left(x(1-x)\frac{\partial}{\partial x}\right)}_{\text{confinement}}}^{H_{\mathrm{eff}}} + V_g + H_{\gamma_5}$$

one-gluon exchange

m_q ($m_{\bar{q}}$) is the mass of the quark/antiquark, κ is the confining strength

V_g is the one-gluon exchange, H_{γ_5} is the pseudoscalar contact interaction

[Li, 1704.06968]

Effective model of light mesons

Transverse and longitudinal basis functions are truncated separately with $N_{\max}$ (basis energy scale) and $L_{\max}$ (longitudinal resolution):

$$\psi_{s\bar{s}}^{m_j}(\boldsymbol{k}_\perp, x) = \sum_{nml} \tilde{\psi}_{s\bar{s}}^{m_j}(n,m,l)\phi_{nm}\left(\frac{\boldsymbol{k}_\perp}{\sqrt{x(1-x)}}\right)\chi_l(x)$$

2D harmonic oscillator *Jacobi polynomial*

$$2n + |m| + 1 \leq N_{\max} \qquad l \leq L_{\max}$$

In this first work, we limit basis size to $(N_{\max}, L_{\max}) = (1,1)$ and $(N_{\max}, L_{\max}) = (4,1)$

	N_f	$\alpha_\mathrm{s}(0)$	κ (MeV)	m_q (MeV)	$N_{\max}$	$L_{\max}$	Matrix dimension
$H_\mathrm{eff}^{(1,1)}$	3	0.89	560 ± 10	300 ± 10	1	1	4 by 4
$H_\mathrm{eff}^{(4,1)}$					4	1	16 by 16

The quark mass m_q and confining strength κ are slightly modified by fitting $\rho(770)$ mass at $N_{\max} = L_{\max} = 1$, compared with original paper. [Qian, 2005.13806]

Effective model of light mesons

Example of $N_{\max} = L_{\max} = 1$ (smallest Hamiltonian matrix) where matrix element corresponds to $(n,\ m,\ l,\ s,\ \bar{s})$ basis state,

$$H_{\text{eff}}^{(1,1)} = \begin{pmatrix} 568487 & 0 & 25428 & 0 \\ 0 & 1700976 & 0 & -15767 \\ 25428 & 0 & 568487 & 0 \\ 0 & -15767 & 0 & 1700976 \end{pmatrix}$$

(All units in MeV^2)

	n	m	l	s	$\bar{s}$	Direct encoding	Compact encoding
①	0	0	0	1/2	-1/2	$\|0001\rangle$	$\|00\rangle$
②	0	0	0	-1/2	1/2	$\|0010\rangle$	$\|01\rangle$
③	0	0	1	1/2	-1/2	$\|0100\rangle$	$\|10\rangle$
④	0	0	1	-1/2	1/2	$\|1000\rangle$	$\|11\rangle$

From second quantization, the Hamiltonian can be written in terms of creation and annihilation operators,

$$\hat{H} = \hat{H}_1 + \hat{H}_2 + \cdots = \sum_{ij} h_{ij} \hat{a}_i^\dagger \hat{a}_j + \frac{1}{4} \sum_{ijkl} h_{ijkl} \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k \hat{a}_l + \cdots$$

We focus only on the single-body interactions and identify h_{ij} as the Hamiltonian matrix elements.

Encoding onto qubits

Setting up rules to map from physics problem to quantum computing problem

Suppose H of dimension $(N, N) = (2^n, 2^n) \;\rightarrow H_q = \sum_\alpha c_\alpha P_\alpha$

[Jordan and Wigner (1928)]
[Seeley, 1208.5986]

1. **Direct encoding**: Jordan-Wigner (JW) encoding, basically map directly from fermions to spin orbitals using the 2 x 2 Pauli spin matrices $\sigma_k \in \{I_k, X_k, Y_k, Z_k\}$

$$\hat{a}_j^\dagger = \bigotimes_{i=1}^{j-1} Z_i \otimes \frac{X_j - iY_j}{2}$$

$$\hat{a}_j = \bigotimes_{i=1}^{j-1} Z_i \otimes \frac{X_j + iY_j}{2}$$

$$O(N)$$

$$\begin{aligned} H^{(1,1)}_{\text{direct}} &= 2269462\,\text{IIII} - 284243\,(\text{ZIII} + \text{IIZI}) \\ &- 850488\,(\text{IZII} + \text{IIIZ}) + 12714\,(\text{XZXI} + \text{YZYI}) \\ &- 7883\,(\text{IXZX} + \text{IYZY}), \end{aligned}$$

[Kreshchuk, 2002.04016]

2. **Compact encoding**: utilize orthogonal basis formed by Pauli strings $P_\alpha = \otimes_{k=1}^n \sigma_k$ under trace, one can further reduce the N-by-N Hamiltonian (Hilbert-Schmidt decomposition)

$$H_q = \frac{1}{N} \sum_{\alpha=1}^{N^2} \text{Tr}(P_\alpha H) \cdot P_\alpha$$

$$O(\log N) = O(n)$$

$$\begin{aligned} H^{(1,1)}_{\text{compact}} &= 1134731\,\text{II} - 566245\,\text{IZ} \\ &+ 4831\,\text{XI} + 20598\,\text{XZ} \end{aligned}$$

$$\text{Tr}(P_j P_k) = 2^n \delta_{j,k} = N\delta_{j,k}$$

Variational ansatz $\hat{U}(\vec{\theta})$

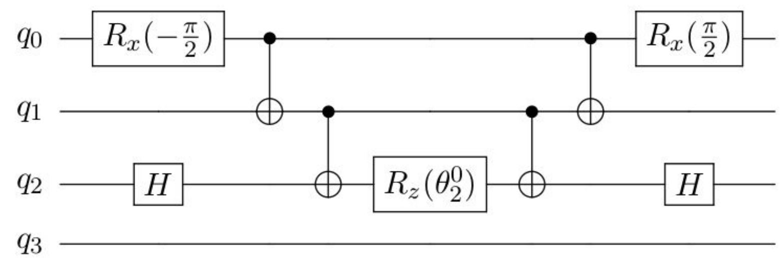
Variational ansatz is an educated guess of the unitary circuit with parameters to be optimized in each iteration.

Barkoutsos [1805.04340]
Romero [1701.02691]

1. **Unitary coupled cluster (UCC)** ansatz
- motivated by coupled cluster methods

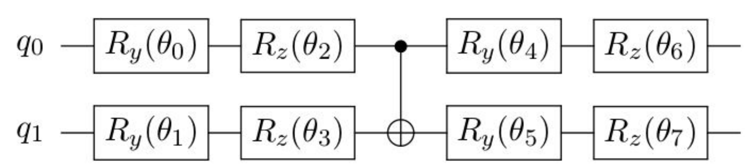
$$\hat{U}(\vec{\theta}) = e^{\hat{T}(\vec{\theta})-\hat{T}^\dagger(\vec{\theta})}, \quad \hat{T}(\vec{\theta}) = \hat{T}_1(\vec{\theta}) = \sum_{\substack{r\in \text{occ}\\ p\in \text{virt}}} \theta_p^r \hat{a}_p^\dagger \hat{a}_r$$

$$\hat{U}(\vec{\theta}) = e^{i\sum_\alpha c_\alpha P_\alpha}$$



UCC(Single) circuit for $e^{i\,\theta_3^0\,\hat{a}_3^\dagger\,\hat{a}_0}$

2. **Hardware efficient (HE)** ansatz [Kandala, 1704.05018]
- heuristic ansatz
- consists of alternating single-qubit rotations and entangling blocks (repetition layers)
- proven to work for general problems



EfficientSU2 ansatz with 1 repetition layer

[Qiskit 0.32.1 library]

Optimization algorithms

❖ **Variational Quantum Eigensolver (VQE)** algorithm [Peruzzo, 1304.3061]

VQE finds the ground state

Initial state $|\psi\rangle$ → Unitary evolution $|\psi(\vec{\theta})\rangle = \hat{U}(\vec{\theta})\,|\psi\rangle$ → Measure & compute cost function $\langle\psi(\vec{\theta})|\,\hat{H}\,|\psi(\vec{\theta})\rangle$

❖ **Subspace-search VQE (SSVQE)** algorithm [Nakanishi, 1810.09434]

In particular, **Weighted SSVQE** for finding up to *k*-th excited states

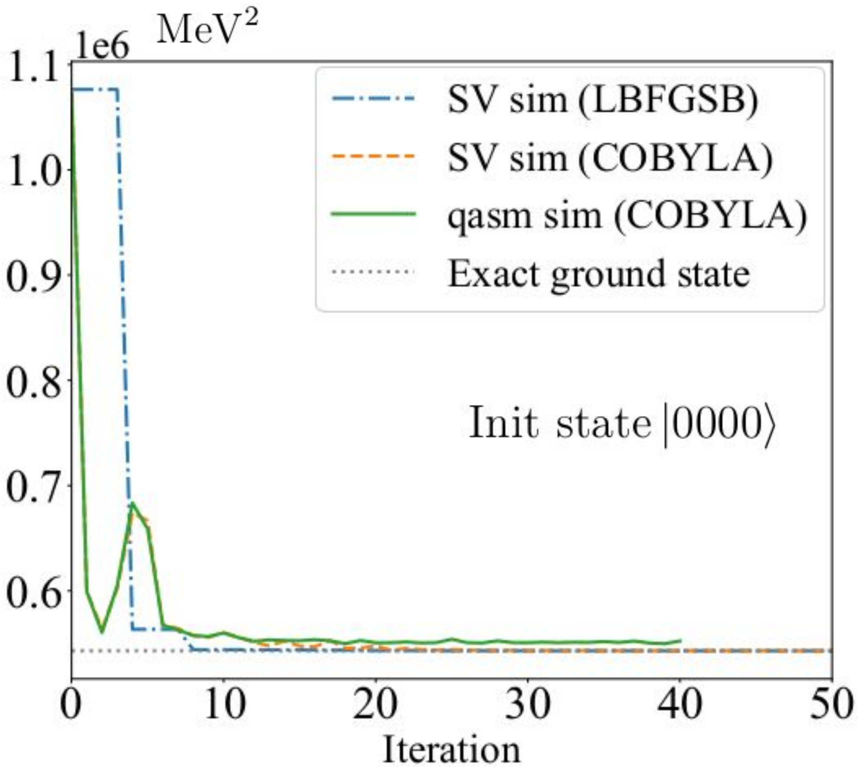
Set of orthogonal states $\{|\psi_i\rangle\}_{i=0}^{k}$ → Evolution for each state $|\psi_i(\vec{\theta})\rangle = \hat{U}(\vec{\theta})\,|\psi_i\rangle$ → Measure & compute cost function $C_{\vec{\omega}}(\vec{\theta}) = \vec{\omega}\cdot\vec{E}$

$$\vec{E} = (E_0, E_1, \cdots, E_k) \qquad E_i = \langle\psi_i(\vec{\theta})|\hat{H}|\psi_i(\vec{\theta})\rangle$$

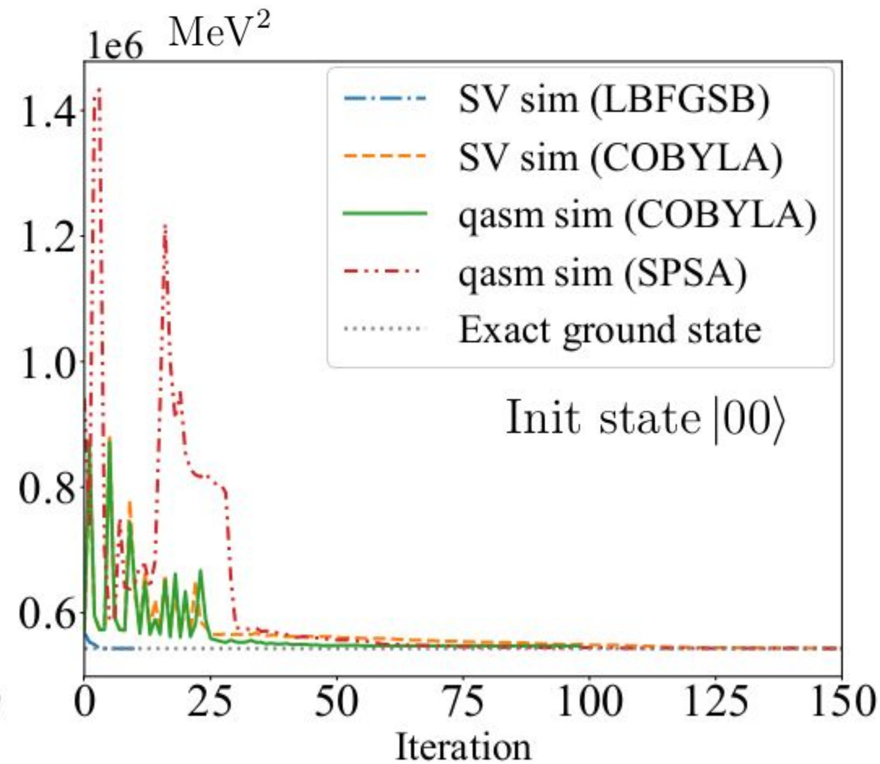
$\vec{\omega}$ is a strictly decreasing weight vector prioritizing lower-lying states

For example: $C_{\vec{\omega}}(\vec{\theta}) = E_0 + 0.5E_1 + 0.25E_2, \quad \vec{\omega} = (1, 0.5, 0.25)$

Results: VQE ground state $(N_{\max}, L_{\max}) = (1, 1)$



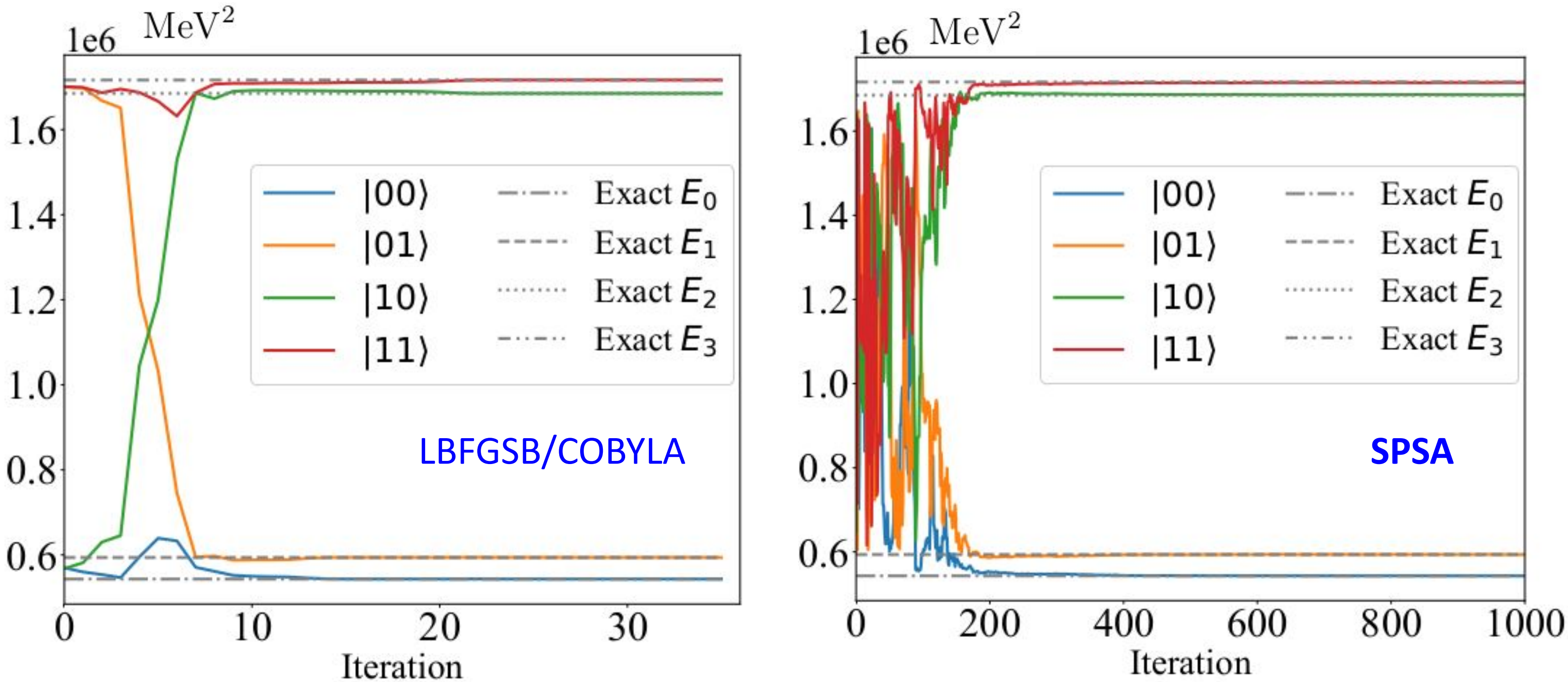
(a) Direct encoding with VQE



(b) Compact encoding with VQE

- All quantum simulated results agree with exact ground state energy.
- Ansatz: (a) UCCS ansatz, 50 gates (b) HE ansatz, 9 gates (1 repetition layer)
- Two IBM simulators are used: statevector (SV) simulator (exact simulation) and qasm simulator (8192 shots per measurement) [Qiskit 0.32.1 library]

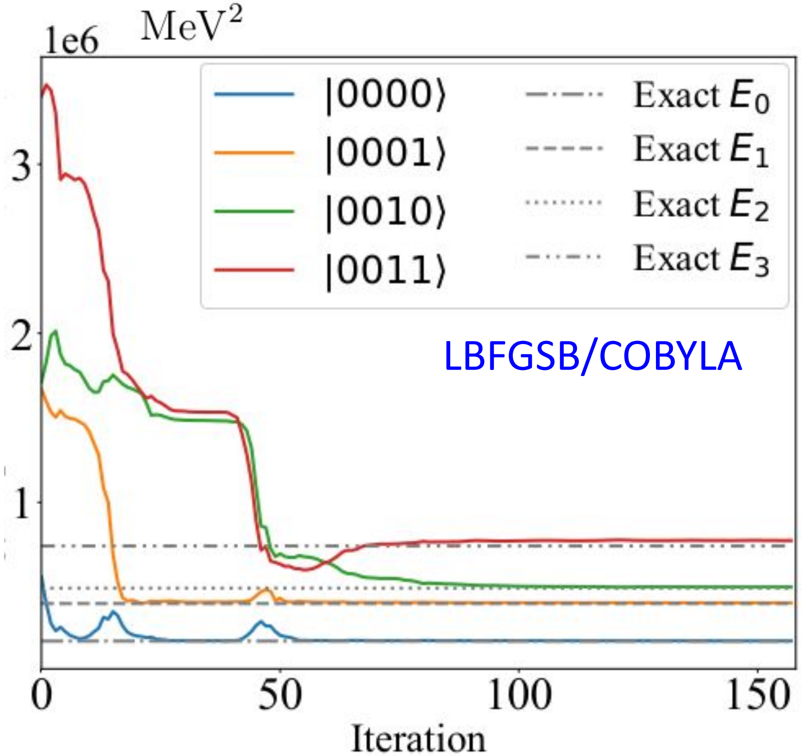
Results: SSVQE spectrum $(N_{\max}, L_{\max}) = (1, 1)$



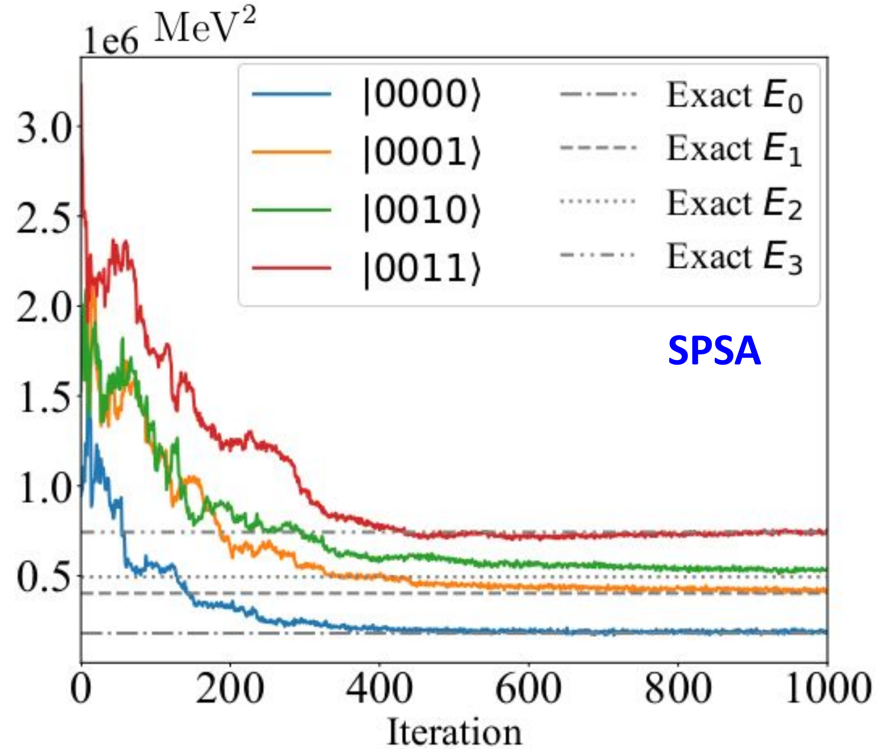
(a) SSVQE with statevector simulator (b) SSVQE with qasm simulator

- Both SV simulator (noise free) and qasm simulator (sampling noise) agree with exact classical spectrum.
- Both use HE ansatz (2 repetition layers, 12 params)
- Cost function: $1.0 \cdot E_{|00\rangle} + 0.5 \cdot E_{|01\rangle} + 0.25 \cdot E_{|10\rangle} + 0.125 \cdot E_{|11\rangle}$
- Note: states emerge in accordance with the specified order of the cost function

Results: SSVQE spectrum $(N_{\max}, L_{\max}) = (4, 1)$



(a) SSVQE with statevector simulator



(b) SSVQE with qasm simulator

- Low-lying spectrum, only look for lowest 4 states instead of 16
- Both SV simulator and qasm simulator do NOT completely agree with exact classical spectrum.
- Both use HE ansatz (6 repetition layer, 53 params)

Results: decay constants

Decay constants are defined as vacuum-to-hadron matrix element of the quark current operator. In BLFQ basis, we write the decay constants as:

$$f_{\mathrm{P,V}} = 2\sqrt{2N_c}\int_0^1 \frac{\mathrm{d}x}{2\sqrt{x(1-x)}}\int \frac{\mathrm{d}^2\boldsymbol{k}_\perp}{(2\pi)^3}\psi^{(m_j=0)}_{\uparrow\downarrow\mp\downarrow\uparrow}(x,\boldsymbol{k}_\perp)$$

$$\equiv \frac{\kappa\sqrt{N_c}}{\pi}\sum_{nl}(-1)^n C_l(m_q,\kappa)\Big(\tilde{\psi}^{(m_j=0)}_{\uparrow\downarrow}(n,0,l) \mp \tilde{\psi}^{(m_j=0)}_{\downarrow\uparrow}(n,0,l)\Big)$$

[Li, 1704.06968]

In the VQE/SSVQE, we basically encode the basis representation of the LFWF on the qubits and evaluation of observables such as decay constants are straightforward:

$$f_{\mathrm{P,V}} \propto \;|\langle \nu_{\mathrm{P,V}}|\psi(\vec{\theta})\rangle\,| = \sqrt{\langle\psi(\vec{\theta})|(|\nu_{P,V}\rangle\langle\nu_{P,V}|)|\psi(\vec{\theta})\rangle}$$

For example, in $(N_{\max}, L_{\max}) = (1,1)$

$$\nu_{\mathrm{P}}^{(1,1)} = (1,-1,0,0)$$
$$\nu_{\mathrm{V}}^{(1,1)} = (1,1,0,0)$$

→

$$|\nu^{(1,1)}_{P,V}\rangle\langle\nu^{(1,1)}_{P,V}|_q = 0.5\,(II \mp IX + ZI \pm ZX)$$

Results: decay constants

Summary of decay constants for the lowest two states (π and ρ mesons). Experimental decay constants are around 130 MeV and 216 MeV, respectively.

[Zyla, 2020]

$N_{\max}$	$L_{\max}$	Decay constants	Exact result (MeV)	SV sim (MeV)	qasm sim (MeV)
1	1	f_π	178.18	178.18	178.17 ± 1.97
		f_ρ	178.18	178.18	178.17 ± 1.97
4	1	f_π	193.71	193.32	194.28 ± 15.49
		f_ρ	231.00	232.93	225.72 ± 13.44

Uncertainty (sampling error) in qasm simulator results from measurements of 8192 shots.

Summary and outlook

We presented two promising quantum computing approaches, VQE and SSVQE, to find meson spectroscopy and observables (including decay constants) on the light front for the light meson systems.

Basis light-front quantization approach (BLFQ) works very well with the VQE and SSVQE approaches.

Future plans:

- Optimize the VQE and SSVQE programs to solve the original Hamiltonian and compute on real quantum backends.
  $N_{\max} = 8,\ L_{\max} = 1$ => (32, 32) Hamiltonian => 5 qubits
  $N_{\max} = 8,\ L_{\max} = 3$ => (64, 64) Hamiltonian => 6 qubits

Logarithmic scaling
(compact encoding)

- Calculations on transition matrix element of an operator using SSVQE approach. Take advantage of **quantum superposition!**

$$A = \langle \psi_i(\vec{\theta})|\hat{A}|\psi_j(\vec{\theta})\rangle = \langle \psi_i|U^\dagger(\vec{\theta})\hat{A}U(\vec{\theta})|\psi_j\rangle$$

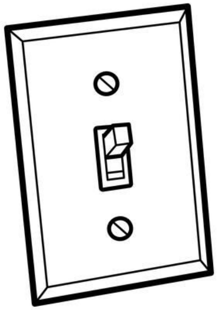
$$\mathrm{Re}(A) = \langle \psi_{ij}^+|U^\dagger(\vec{\theta})\hat{A}U(\vec{\theta})|\psi_{ij}^+\rangle - \frac{1}{2}(\langle \psi_i|U^\dagger(\vec{\theta})\hat{A}U(\vec{\theta})|\psi_i\rangle + \langle \psi_j|U^\dagger(\vec{\theta})\hat{A}U(\vec{\theta})|\psi_j\rangle)$$

$$|\psi_{ij}^+\rangle = \frac{1}{\sqrt{2}}(|\psi_i\rangle + |\psi_j\rangle)$$

Potential application:
- EM radiative transition

Backup slides:
Introduction to Quantum Computing

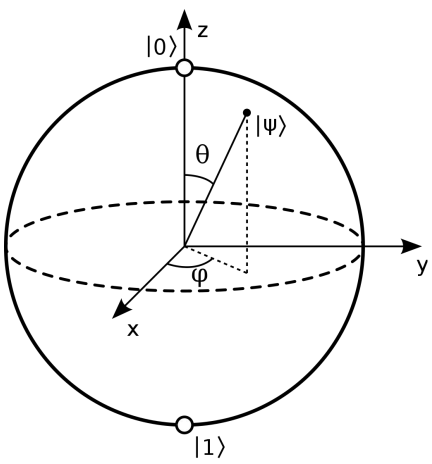
❖ Classical computers:
- classical bit: 0, 1
- implementation: electric voltage Low, High
- classical gates: AND, OR, NOT, Bitwise logic gates
- Deterministic nature



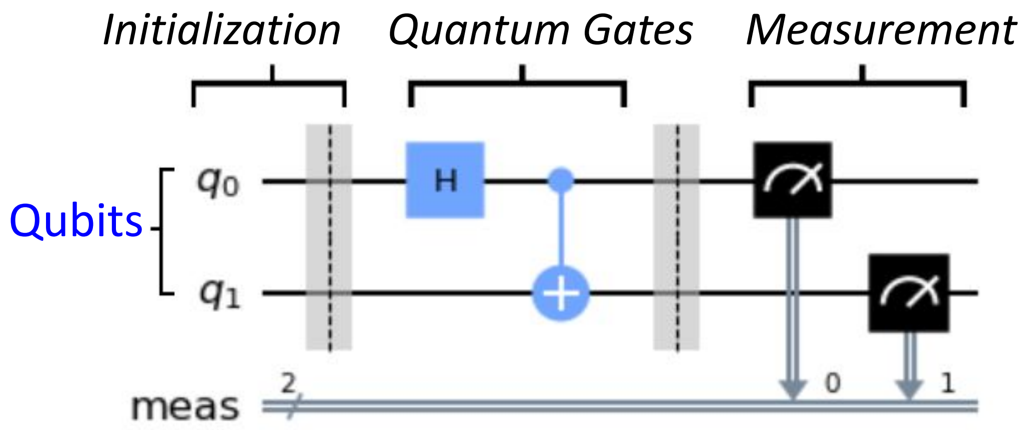
❖ Quantum computers:
- quantum bit (qubit): $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- implementation: two-level quantum systems

$$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle, \ |\alpha|^2 + |\beta|^2 = 1$$

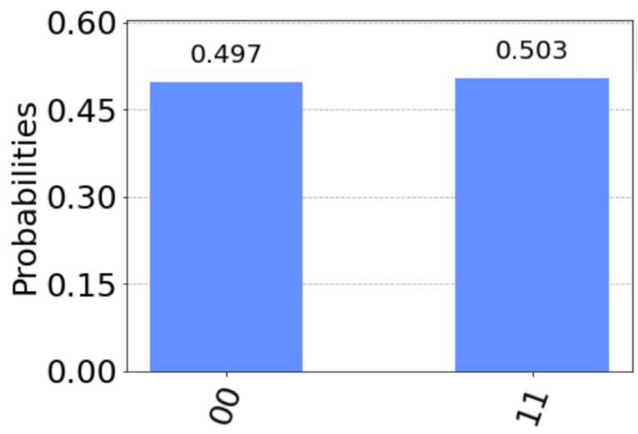
- quantum gates: unitary operators
- States only collapse when measured
- <u>superposition</u>
- Can theoretically solve problems classical computers cannot solve



Backup slides:
Quantum circuit crash course



Quantum circuit



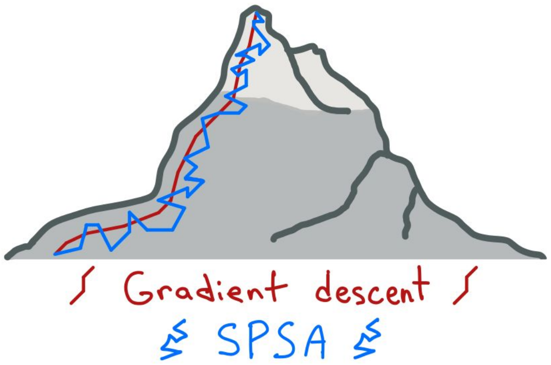
Outcome

- Each horizontal line represents the evolution of a qubit (from left to right)
- Each measurement collapses the wave function and we obtain either 0 or 1 in the computation basis
- In practice, measurements are statistical outcomes by running the same circuit repeatedly for thousands of times (or shots)

Backup slides: Optimizers

Optimizers (on classical computers) aim to minimize the loss functions and updates the parameter for next iteration.

- **COBYLA**: constrained optimization, derivative-free
- **LBFGSB**: quasi-Newton method, derivative-based
- **SPSA**: Simultaneous Perturbation Stochastic Approximation; can handle measurement uncertainty [diagram from Pennylane]
- **QNSPSA**: Quantum Natural SPSA; improve SPSA by sampling natural gradients; significant speed up



More optimizers at
scipy.optimize.minimize
qiskit.algorithms.optimizers

Backup slides:
Parton distribution functions

We can also use quantum simulators to measure parton distribution functions (PDFs), which is the probability of finding a particle with longitudinal momentum fraction x under some factorization scale related to experimental conditions,

[Li, 1704.06968]

$$q(x;\mu) = \frac{1}{x(1-x)} \sum_{s\bar{s}} \int \frac{d^2\boldsymbol{k}_\perp}{2(2\pi)^3} |\psi_{s\bar{s}}^{(m_j=0)}(x,\boldsymbol{k}_\perp)|^2$$

$$\equiv \frac{1}{4\pi} \sum_{s\bar{s}} \sum_{nm} \sum_{l\bar{l}} \tilde{\psi}_{s\bar{s}}^{*(m_j=0)}(n,m,\bar{l}) \tilde{\psi}_{s\bar{s}}^{(m_j=0)}(n,m,l) \chi_l(x) \chi_{\bar{l}}(x)$$

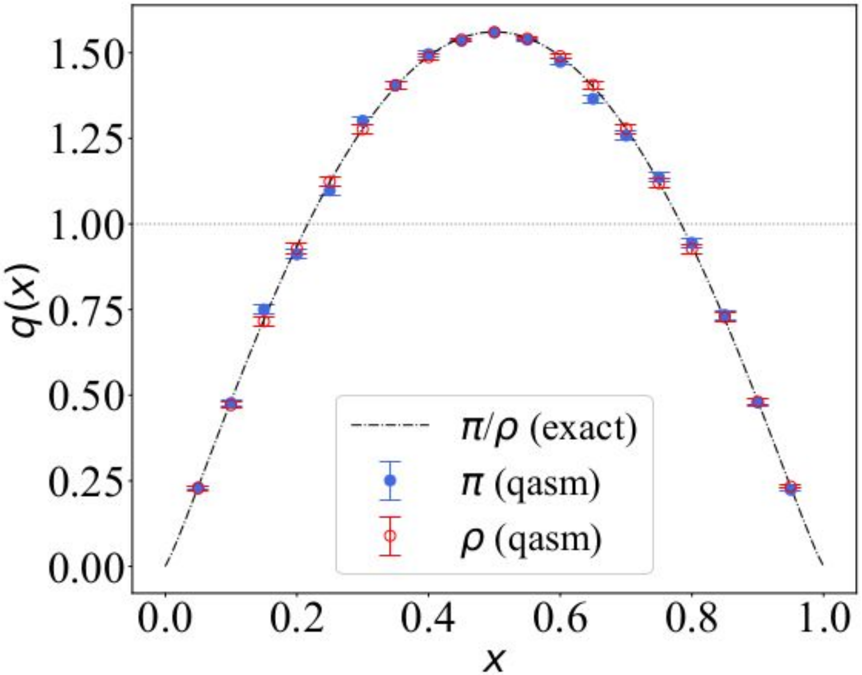
Using projection operators, we can compute the PDF on quantum computers as well. Note that the PDF operator on qubits will be dependent on x.

$$q(x) = \sum_{s\bar{s}} \sum_{nm} \sum_{l\bar{l}} \langle \psi(\vec{\theta}) | \hat{O}_{\mathrm{pdf}}(x) | \psi(\vec{\theta}) \rangle$$

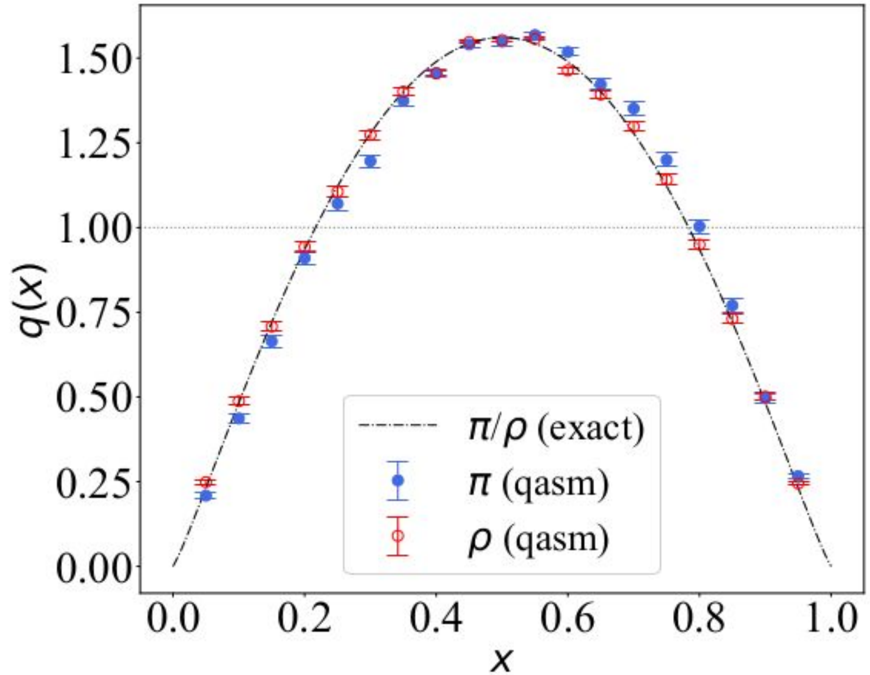
Qubitized operators: $\hat{O}_{\mathrm{pdf}}(x) = \hat{U}_{\mathrm{p}}(s,\bar{s},n,m,\bar{l})^\dagger \hat{U}_{\mathrm{p}}(s,\bar{s},n,m,l) \chi_l(x) \chi_{\bar{l}}(x)/4\pi$

$$\hat{O}^{(1,1)}_{\mathrm{pdf}}(0.5)_q = 1.30\, II - 1.29\, IX - 0.18\, IZ, \qquad \hat{O}^{(1,1)}_{\mathrm{pdf}}(0.25)_q = 0.78\left(II + IZ\right)$$

Backup slides:
Parton distribution functions



(a) PDFs at $N_{\max} = L_{\max} = 1$



(b) PDFs at $N_{\max} = 4, L_{\max} = 1$

- In both basis truncations, the PDFs for lowest two states are comparable due to the lack of longitudinal excitations. Note: SV and exact results agree with each other (not shown).
- For $(N_{\max}, L_{\max}) = (1, 1)$, the qasm agree with the exact calculations.
- For $(N_{\max}, L_{\max}) = (4, 1)$, the PDF is rescaled due to lack of normality of the PDF (or LFWF on qubits) as the SSVQE evolution is sensitive to sample noise at this cutoff.

Backup slides: Detailed Statistics of VQE results

Simulator	Encoding	Optimizer	Ground state energy (MeV^2)	Iterations
SV	Direct	LBFGSB	543059 (0%)	60
	Direct	COBYLA	543059 (0%)	90
	Compact	LBFGSB	543059 (0%)	11
	Compact	COBYLA	543059 (0%)	344
qasm	Direct	COBYLA	552344 ± 996 (1.53%)	41
	Direct	SPSA	545767 ± 152 (0.47%)	1051
	Compact	COBYLA	547405 ± 211 (0.76%)	99
	Compact	SPSA	543065 ± 6 (0%)	1551
Exact solution	-	-	543059	-

Backup slides:
Detailed Statistics of SSVQE results

$N_{\max}$	$L_{\max}$	State	Exact energy (MeV^2)	SV sim (MeV^2)	qasm sim (MeV^2)
1	1	$\|00\rangle$	543059	543059 (0%)	543059 ± 0 (0%)
		$\|01\rangle$	593915	593915 (0%)	593915 ± 0 (0%)
		$\|10\rangle$	1686541	1685210 (0.08%)	1686541 ± 70 (0%)
		$\|11\rangle$	1715577	1716743 (0.07%)	1715577 ± 66 (0%)
4	1	$\|0000\rangle$	180012	180802 (0.44%)	189263 ± 6511 (1.08%)
		$\|0001\rangle$	402071	405796 (0.93%)	419139 ± 6324 (1.73%)
		$\|0010\rangle$	493293	499376 (1.23%)	532381 ± 7008 (5.21%)
		$\|0011\rangle$	742530	774189 (4.26%)	745422 ± 6503 (2.88%)

Backup slides: Hamiltonian on qubits

$$(N_{\max}, L_{\max}) = (4,1)$$

$$\begin{aligned}
H^{(4,1)}_{\text{compact}} = {} & 1868696\, IIII - 623614\, IIIZ + 518799\, IIXI + 44344\, IIXZ \\
& - 531599\, IIZI + 11950\, IIZZ + 29183\, IYIY - 21316\, IYXY \\
& + 28874\, IYYI + 22502\, IYYX - 1474\, IYYZ + 6301\, IYZY \\
& + 1762\, XXII + 7092\, XXIZ - 310\, XXXI - 4214\, XXXZ \\
& + 653\, XXZI + 3207\, XXZZ + 77283\, XZII - 61720\, XZIX \\
& + 4548\, XZIZ - 38263\, XZXI + 33154\, XZXX - 3510\, XZXZ \\
& + 844\, XZYY + 19387\, XZZI - 15666\, XZZX + 2304\, XZZZ \\
& + 29183\, YIIY - 21316\, YIXY - 28874\, YIYI + 22502\, YIYX \\
& + 1474\, YIYZ + 6301\, YIZY + 1762\, YYII + 7092\, YYIZ \\
& - 310\, YYXI - 4214\, YYXZ + 653\, YYZI + 3207\, YYZZ \\
& - 77283\, ZXII - 61720\, ZXIX - 4548\, ZXIZ + 38263\, ZXXI \\
& + 33154\, ZXXX + 3510\, ZXXZ + 844\, ZXYY - 19387\, ZXZI \\
& - 15666\, ZXZX - 2304\, ZXZZ + 215302\, ZZII - 34396\, ZZIZ \\
& + 70683\, ZZXI + 19390\, ZZXZ - 12936\, ZZZI - 11024\, ZZZZ,
\end{aligned}$$

Backup slides: Basis mapping for $(N_{\max}, L_{\max}) = (4, 1)$

	n	m	l	s	$\bar{s}$	Compact encoding
①	0	0	0	1/2	-1/2	$\vert 0000\rangle$
②	0	0	0	-1/2	1/2	$\vert 0001\rangle$
③	0	0	1	1/2	-1/2	$\vert 0010\rangle$
④	0	0	1	-1/2	1/2	$\vert 0011\rangle$
⑤	0	1	0	-1/2	-1/2	$\vert 0100\rangle$
⑥	0	1	1	-1/2	-1/2	$\vert 0101\rangle$
⑦	0	-1	0	1/2	1/2	$\vert 0110\rangle$
⑧	0	-1	1	1/2	1/2	$\vert 0111\rangle$
⑨	1	0	0	1/2	-1/2	$\vert 1000\rangle$
⑩	1	0	0	-1/2	1/2	$\vert 1001\rangle$
⑪	1	0	1	1/2	-1/2	$\vert 1010\rangle$
⑫	1	0	1	-1/2	1/2	$\vert 1011\rangle$
⑬	1	1	0	-1/2	-1/2	$\vert 1100\rangle$
⑭	1	1	1	-1/2	-1/2	$\vert 1101\rangle$
⑮	1	-1	0	1/2	1/2	$\vert 1110\rangle$
⑯	1	-1	1	1/2	1/2	$\vert 1111\rangle$