Exploring the Dyson-Schwinger equation in Minkowski space

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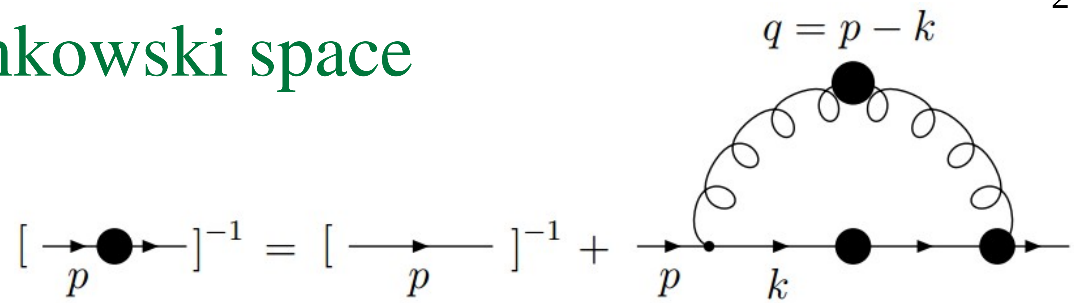
DSE Formulation in Minkowski space



Lattice gauge simulations and perturbation theory calculations are formulated in Euclidean space; furthermore system descriptions originally formulated in Minkowski space has to deal with many difficulties due to its singular behavior. So, why Minkowski?

- Possibility of calculation of dynamical observables, such as nucleon form factors and structure observables defined on the light front, such as the parton distributions, that cannot be obtained trivially from Euclidean solutions;

- Electromagnetic form factors: integral contains singularities different from those appearing in BSE, and whose positions depend on the momentum transfer $\Longrightarrow$ Invalidation of Wick rotation (results for FF are only approximated).

- QCD at finite density: Lattice sign problem.
- ...

Nakanishi Integral Representation (NIR)

"Parametric representation for any Feynman diagram for any interacting bosons with a denominator carrying the overall analytical behavior in Minkowski space."

- Generalization of the Källén-Lehmman integral representation of two point functions, for n-point functions. †N. Nakanishi, Phys. Rev. 130, 1230 (1963) and Prog.Theor.Phys.Suppl. 43, 1 (1969).

- Wick rotation is the exact analytical continuation of the Minkowski space Nakanishi representation: Possibility of the exploration in the complex plane.

Fermion Dyson Schwinger Equation (Rainbow-Ladder)

⟹ Dressed fermion propagator: $S_f\left(k\right) = \dfrac{1}{\not{k} - m_B + \not{k}A_f\left(k^2\right) - B_f\left(k^2\right) + i\epsilon}$

$$A_F\left(k^2\right) = \int_0^\infty d\gamma \frac{\rho_A\left(\gamma\right)}{k^2 - \gamma + i\epsilon} \quad ; \qquad B_f\left(k^2\right) = \int_0^\infty d\gamma \frac{\rho_B\left(\gamma\right)}{k^2 - \gamma + i\epsilon}$$

⟹ Integral representation of the fermion propagator:

$$S_f = R\frac{\not{k} + \overline{m}_0}{k^2 - \overline{m}_0^2 + i\epsilon} + \not{k}\int_0^\infty d\gamma \frac{\rho_v\left(\gamma\right)}{k^2 - \gamma + i\epsilon} + \int_0^\infty d\gamma \frac{\rho_s\left(\gamma\right)}{k^2 - \gamma + i\epsilon}$$

Fermion Dyson Schwinger Equation (Rainbow-Ladder)

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Vector and scalar Self-Energy densities

$\Longrightarrow$ Integral representation of the fermion propagator:

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Vector and scalar spectral densities

$$\not{k} A(k^2) - B(k^2) = i g^2 \int \frac{d^4 q}{(2\pi)^4} \frac{\gamma_\mu\, S(k-q) \gamma_\nu}{q^2 - m_\sigma^2 + i\epsilon} \left[g^{\mu\nu} - \frac{(1-\xi)\, q^\mu q^\nu}{q^2 - \xi m_\sigma^2 + i\epsilon} \right]$$
$$- i\, g^2 \int \frac{d^4 q}{(2\pi)^4} \frac{\gamma_\mu\, S(k-q) \gamma_\nu}{q^2 - \Lambda^2 + i\epsilon} \left[g^{\mu\nu} - \frac{(1-\xi) q^\mu q^\nu}{q^2 - \xi \Lambda^2 + i\epsilon} \right]$$

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Vector and scalar spectral densities

Gauge fixing

$$\not{k} A(k^2) - B(k^2) = i g^2 \int \frac{d^4 q}{(2\pi)^4} \frac{\gamma_\mu S(k-q)\gamma_\nu}{q^2 - m_\sigma^2 + i\epsilon} \left[g^{\mu\nu} - \frac{(1-\xi)\, q^\mu q^\nu}{q^2 - \xi m_\sigma^2 + i\epsilon} \right]$$
$$- i g^2 \int \frac{d^4 q}{(2\pi)^4} \frac{\gamma_\mu S(k-q)\gamma_\nu}{q^2 - \Lambda^2 + i\epsilon} \left[g^{\mu\nu} - \frac{(1-\xi) q^\mu q^\nu}{q^2 - \xi \Lambda^2 + i\epsilon} \right]$$

Pauli-Villars regulator

Fermion Dyson Schwinger Equation

- Parameters: $\alpha = \dfrac{g^2}{4\pi}$, Λ , m_σ , $\overline{m}_0$.

- Self energy densities: $\rho_A(\gamma)$ = - Im[A(γ)]/π and $\rho_B(\gamma)$ = – Im[B(γ)]/π.

- Solutions of DSE obtained writing the trivial relation $S_f^{-1} S_f = 1$ in a suitable form:

$$\frac{R}{k^2 - \overline{m}_0^2 + i\epsilon} + \int_0^\infty d\gamma \frac{\rho_v(\gamma)}{k^2 - \gamma + i\epsilon} = \frac{1 + A_f(k^2)}{k^2(1 + A_f(k^2))^2 - (m_B + B_f(k^2))^2 + i\epsilon}$$

$$\frac{R\,\overline{m}_0}{k^2 - \overline{m}_0^2 + i\epsilon} + \int_0^\infty d\gamma \frac{\rho_s(\gamma)}{k^2 - \gamma + i\epsilon} = \frac{m_B + B_f(k^2)}{k^2(1 + A_f(k^2))^2 - (m_B + B_f(k^2))^2 + i\epsilon}$$

Fermion DSE solution

$$\rho_A\left(p^2\right) = R\left[\mathcal{K}_A^{\xi=1}\left(p^2,\overline{m}_0^2;m_\sigma^2\right) + \frac{1}{m_\sigma^2}\mathcal{K}_A^{\xi}\left(p^2,\overline{m}_0^2;m_\sigma^2\right)\right] + \int_{s_\xi^{\text{thres}}}^{\infty} ds\rho_v\left(s\right)\left[\mathcal{K}_A^{\xi=1}\left(p^2,s;m_\sigma^2\right) + \frac{1}{m_\sigma^2}\mathcal{K}_A^{\xi}\left(p^2,s;m_\sigma^2\right)\right] - \left[m_\sigma \to \Lambda\right]$$

$$\rho_B\left(p^2\right) = R\overline{m}_0\left[\mathcal{K}_B^{\xi=1}\left(p^2,\overline{m}_0^2;m_\sigma^2\right) + \frac{1}{m_\sigma^2}\mathcal{K}_B^{\xi}\left(p^2,\overline{m}_0^2;m_\sigma^2\right)\right] + \int_{s_\xi^{\text{thres}}}^{\infty} ds\rho_v\left(s\right)\left[\mathcal{K}_B^{\xi=1}\left(p^2,s;m_\sigma^2\right) + \frac{1}{m_\sigma^2}\mathcal{K}_B^{\xi}\left(p^2,s;m_\sigma^2\right)\right] - \left[m_\sigma \to \Lambda\right]$$

$$s_\xi^{\text{thres}} = \left(\overline{m}_0 + \xi m_\sigma\right)^2$$

$$f_A(p^2) = 1 + P\int_{s^{\text{thres}}}^{\infty} ds\,\frac{\rho_A(s)}{p^2 - s}$$

$$f_B(p^2) = m_B + P\int_{s^{\text{thres}}}^{\infty} ds\,\frac{\rho_B(s)}{p^2 - s}$$

$$d(p^2) = \left[p^2 f_A^2(p^2) - \pi^2\, p^2\, \rho_A^2(p^2) - f_B^2(p^2) + \pi^2\, \rho_B^2(p^2)\right]^2 + 4\pi^2\left[p^2\rho_A(p^2)\, f_A(p^2) - \rho_B(p^2)\, f_B(p^2)\right]^2$$

Connection Formulas

$$\rho_v\left(p^2\right) = -2\frac{f_A\left(p^2\right)}{d\left(p^2\right)}\left[p^2\rho_A\left(p^2\right)f_A\left(p^2\right) - \rho_B\left(p^2\right)f_B\left(p^2\right)\right] + \frac{\rho_A\left(p^2\right)}{d\left(p^2\right)}\left[p^2 f_A^2\left(p^2\right) - \pi^2 p^2\rho_A^2\left(p^2\right) - f_B^2\left(p^2\right) + \pi^2\rho_B^2\left(p^2\right)\right]$$

$$\rho_s\left(p^2\right) = -2\frac{f_B\left(p^2\right)}{d\left(p^2\right)}\left[p^2\rho_A\left(p^2\right)f_A\left(p^2\right) - \rho_B\left(p^2\right)f_B\left(p^2\right)\right] + \frac{\rho_B\left(p^2\right)}{d\left(p^2\right)}\left[p^2 f_A^2\left(p^2\right) - \pi^2 p^2\rho_A^2\left(p^2\right) - f_B^2\left(p^2\right) + \pi^2\rho_B^2\left(p^2\right)\right]$$

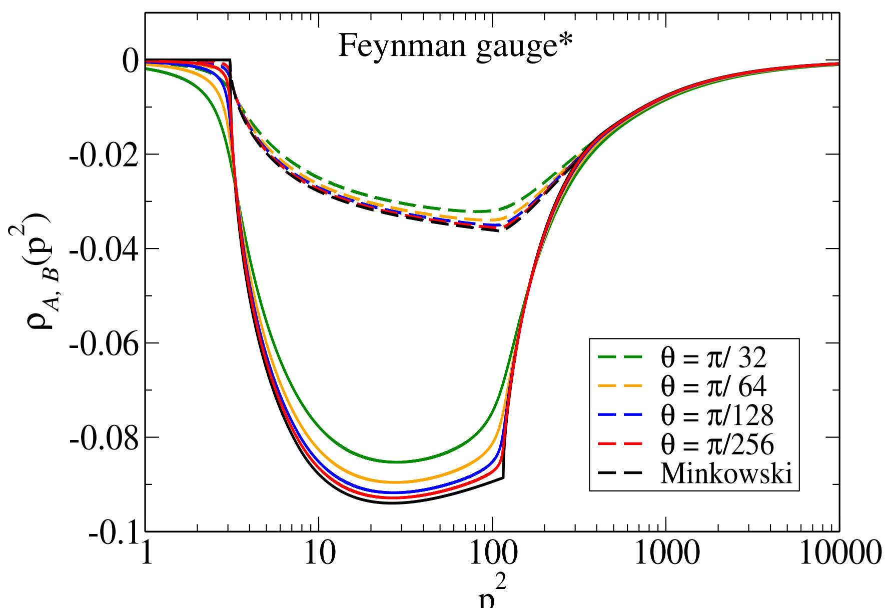
Comparison with Un-Wick rotated results

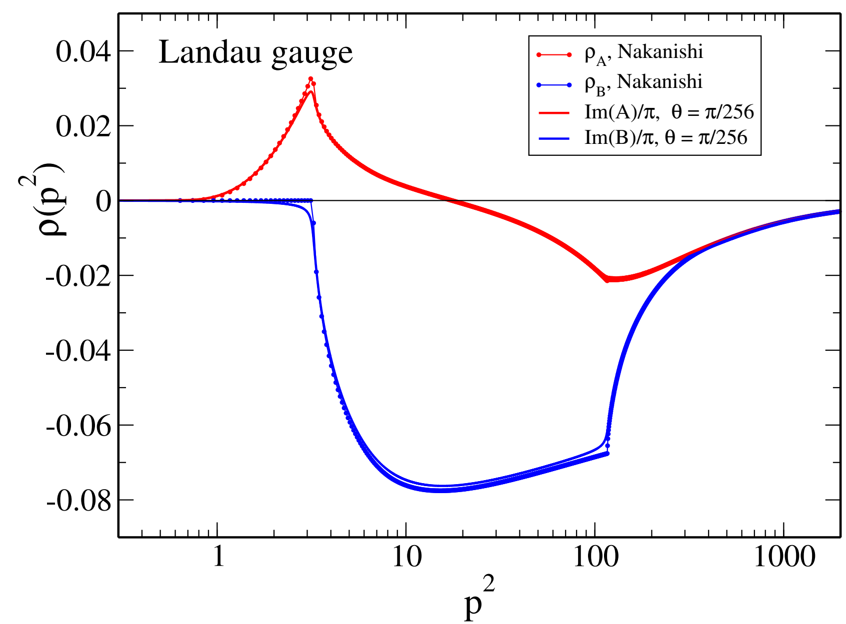
- From Euclidean space formulation, in increments of δ: $$p_0 =\!\rightarrow \exp(-i\delta)p_0$$ $$k_0 =\!\rightarrow \exp(-i\delta)k_0$$

A. Castro et al., Journal of Physics: Conf. Series 1291 012006 (2019): Un-Wick comparison withrotation in the ladder bosonic BSE

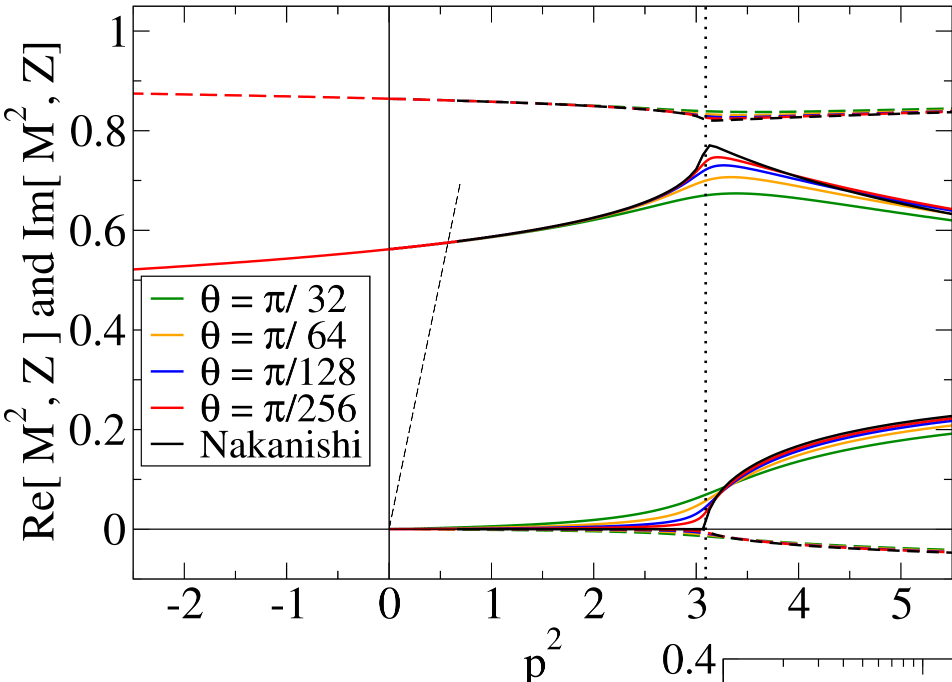
- Minkowski space: δ = π/2, or in a more conveninent notation Θ = π/2 − δ.

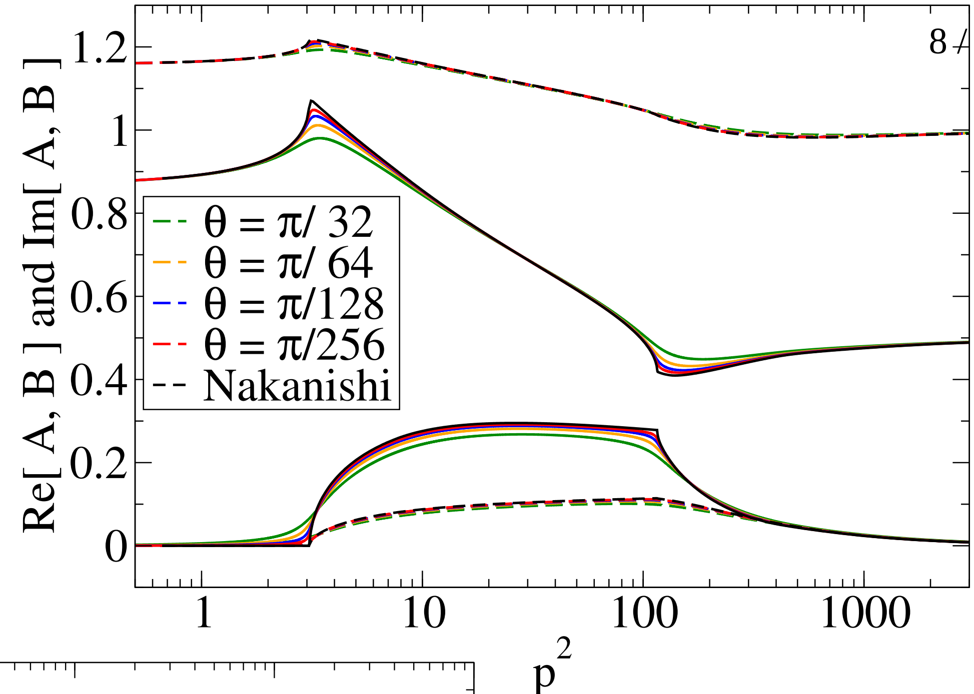
$$\theta = 0, \begin{cases} p_0^2 = 0, \quad \vec{p}^2 > 0 & \text{spacelike region} \\ p_0^2 > 0, \quad \vec{p}^2 = 0 & \text{timelike region} \end{cases}$$





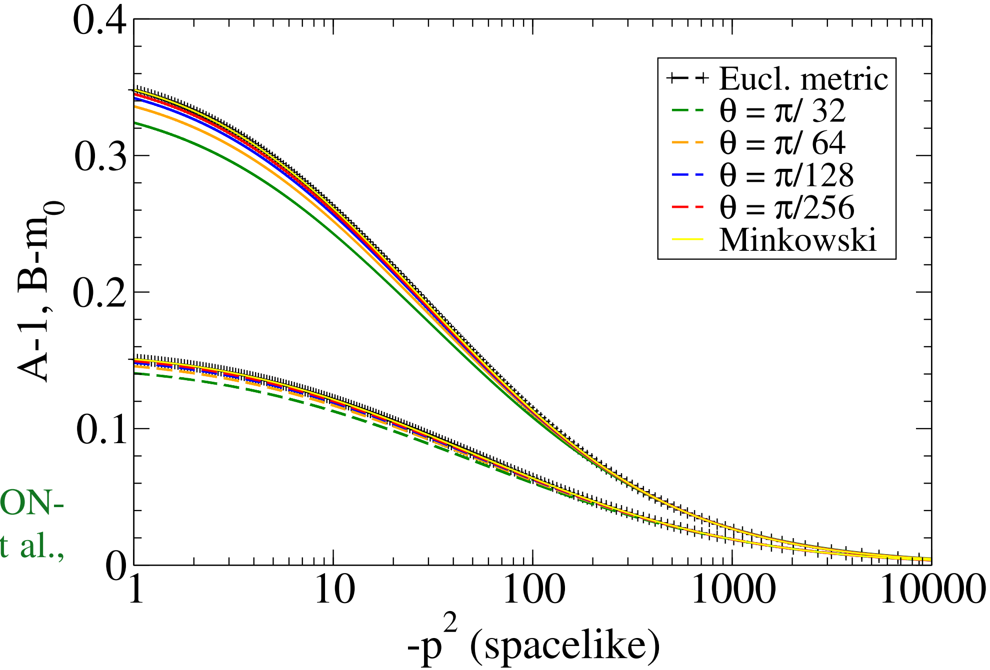
*S. Jia et al., Proceedings of HADRON-2019, arXiv:1912.00063, T. Frederico et al., Proceedings of NTSE-2018, arXiv:1905.00703.





$$M^2(p^2) = \frac{B^2(p^2)}{A^2(p^2)}$$

$$Z(p^2) = \frac{1}{A(p^2)}$$



*S. Jia et al., Proceedings of HADRON-2019, arXiv:1912.00063, T. Frederico et al., Proceedings of NTSE-2018, arXiv:1905.00703.

Feynman gauge results!

$m_\sigma = 1.0$

$\Lambda = 10$

$\alpha = 0.5$

$\overline{m}_0 = 0.759$

$m_B = 0.5$

Recent developments and perspectives: Phenomenological model

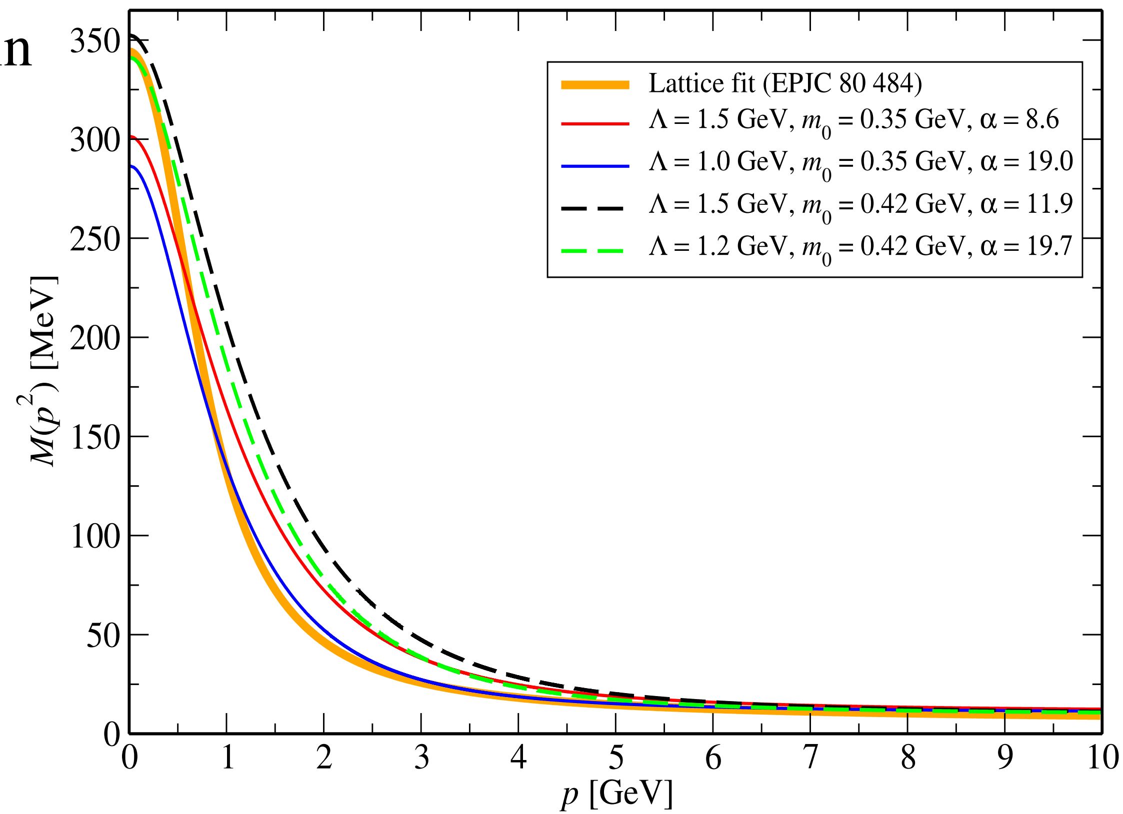
Calibration of the model: Possibility to explore the chiral symmetry breaking region!

- Using a value of the gluon mass m_σ in the region $1.5m_0 \leq m_\sigma \leq 2m_0$:

Appropriate behavior in the infrared require a large enough Kernel

$\Downarrow$

Λ cannot be large compared to m_σ, and as a consequence, α must increase!



*W. de Paula et al., Phys. Rev. D 103 (2021) no.1, 014002; O. Oliveira et al., Eur. Phys. J. C 80:484 (2020).

Recent developments and perspectives

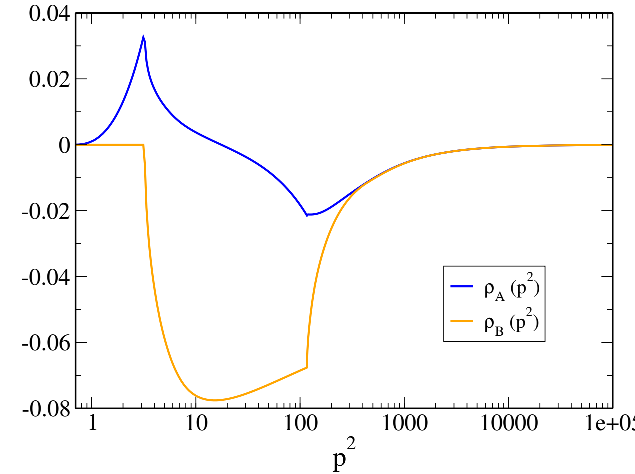
Spectral densities evaluated by solving the DSE the method described previously as inputs for the pion Bethe-Salpeter equation.

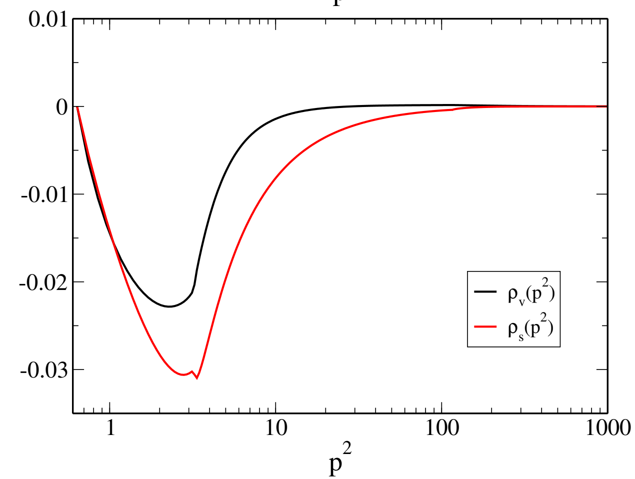
$$\Psi_\pi(k;p) = S_F(k_q)\Gamma_\pi(k;p)S_F(k_{\bar{q}})$$

$$\begin{aligned} S_F(k) &= \frac{i}{A(k^2)\not{k} - B(k^2)} \\ &= i\left[S_v(k^2)\not{k} + S_s(k^2)\right] \end{aligned}$$

$$S_v(k^2) = \frac{R}{k^2 - \overline{m}_0^2 + i\epsilon} + \int_0^\infty ds \frac{\rho_v(s)}{k^2 - s + 1\epsilon}$$

$$S_s(k^2) = \frac{R\overline{m}_0}{k^2 - \overline{m}_0^2 + i\epsilon} + \int_0^\infty ds \frac{\rho_s(s)}{k^2 - s + i\epsilon}$$





Recent developments and perspectives

- First approximation: Chiral limit! In this case, the pion quark-antiquark vertex is given by**

$$E_\pi(k,p) = \frac{i}{f_\pi^0} B(k^2) \ , \qquad B\left(k^2\right) = \int_0^\infty d\gamma \frac{\rho_B\left(\gamma\right)}{k^2 - \gamma + i\epsilon}$$

- Next steps: Calculation of observables, more rigorous study of chiral symmetry breaking, ingredients from LQCD...

$$i p^\mu f_\pi = N_c \int \frac{d^4}{(2\pi)^4} \mathrm{Tr}[\gamma^\mu \gamma^5 \Psi_\pi(k,p)]$$

**C. S. Mello, et al., Phys. Lett. B, 766 86–93 (2017), L. Chang et al., PRL 110, 132001 (2013).

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$$ip^\mu f_\pi = N_c \int \frac{d^4}{(2\pi)^4} \mathrm{Tr}[\gamma^\mu \gamma^5 \Psi_\pi(k,p)]$$

Numerical implementation in progress!

**C. S. Mello, et al., Phys. Lett. B, 766 86–93 (2017), L. Chang et al., PRL 110, 132001 (2013).

Recent developments and perspectives/possibilities

- The NIR is a very important tool to solve DSE (and BSE) directly in Minkowski space.

- Inclusion of more sophisticated ingredients, as quark-gluon vertex, Lattice QCD (self energy, vertex, ...) $\Rightarrow$ more realistic theories! See W. de Paula and E. Ydrefors talks!

- Wide range of applications: Form factors, parton distribution functions, analytic structure of pion, kaon, nucleon, Nakanishi weight functions ...

Thanks for your attention!

Kernels

$$\mathcal{K}_A^{\xi=1}(a,m_\sigma,\gamma)=\frac{2g^2}{(4\pi)^2}\frac{\Theta\left[\gamma-(a+m_\sigma)^2\right]}{\gamma}\sqrt{m_\sigma^4-2m_\sigma^2(\gamma+a^2)+(\gamma-a^2)^2}$$

$$-\frac{g^2}{(4\pi)^2}(1+\xi)\int_0^1 d\alpha\,\alpha\Theta\left[\alpha_1(1-\alpha_1)\gamma-\alpha_1 a^2-m_\sigma^2(1-\alpha_1)\right]$$

$$\mathcal{K}_A^{\xi}(a,m_\sigma,\gamma)=\frac{g^2}{(4\pi)^2 m_\sigma^2}\int_0^1 d\alpha_1\left[3\gamma\alpha_1^2+\alpha_1\left(a^2-\xi m_\sigma^2-\gamma\right)-\xi m_\sigma^2\right]$$

$$\times\Theta\left[\alpha_1(1-\alpha_1)\gamma-\alpha_1 a^2-\xi m_\sigma^2(1-\alpha_1)\right]\Theta\left[m_\sigma^2(1-\alpha_1)+\alpha_1 a^2-\alpha_1(1-\alpha_1)\gamma\right]$$

$$\mathcal{K}_B^{\xi=1}(a,m_\sigma,\gamma)=\frac{(3+\xi)g^2}{(4\pi)^2}\frac{\Theta\left[\gamma-(a+m_\sigma)^2\right]}{\gamma}\sqrt{a^4-2a^2(\gamma+m_\sigma^2)+(\gamma-m_\sigma^2)^2}$$

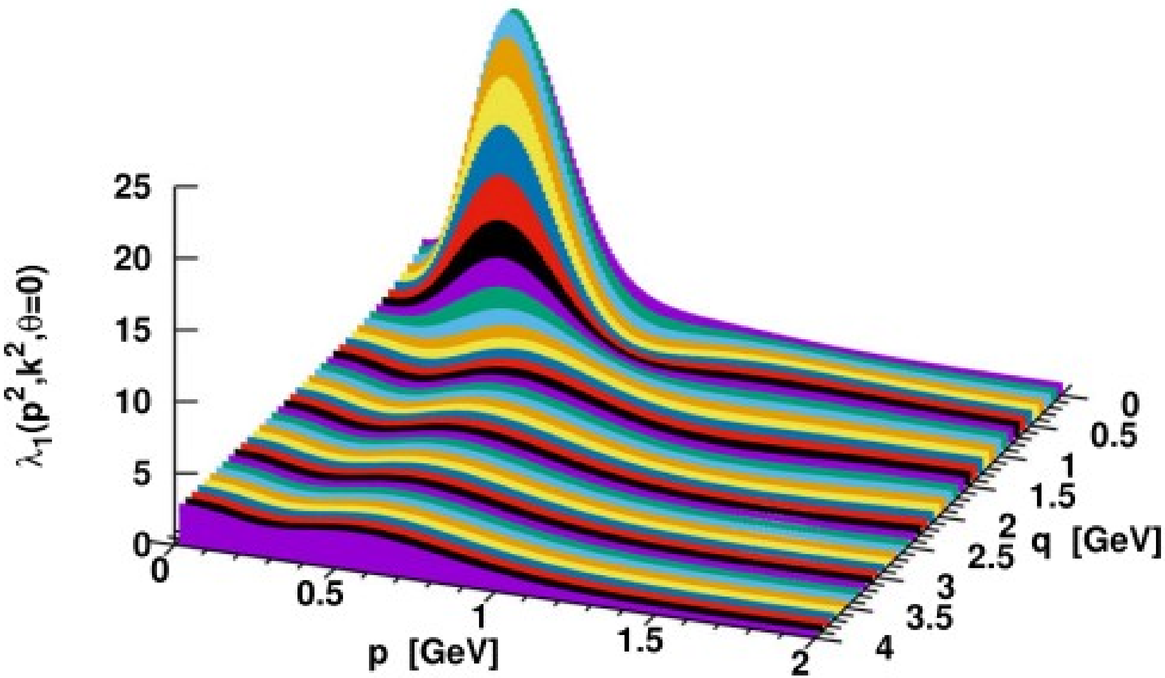
$$\mathcal{K}_B^{\xi}(a,m_\sigma,\gamma)=\frac{g^2\xi}{(4\pi)^2}\int_0^1 d\alpha_1\left[m_\sigma^2(1-\alpha_1)-\gamma(1-\alpha_1)\alpha_1+\alpha_1 a^2\right]$$

$$\times\Theta\left[\gamma(1-\alpha_1)\alpha_1-\alpha_1\left(a^2-\xi m_\sigma^2\right)-\xi m_\sigma^2\right]$$

$$R^{-1} = 1 + \int_{\gamma^{\text{thres}}}^{\infty} d\gamma \frac{\rho_A(\gamma)}{\overline{m}_0^2 - \gamma} - 2\overline{m}_0^2 P\int_{\gamma^{\text{thres}}}^{\infty} d\gamma' \frac{\rho_A(\gamma')}{(\overline{m}_0^2 - \gamma')^2} + 2\overline{m}_0 P\int_{\gamma^{\text{thres}}}^{\infty} d\gamma' \frac{\rho_B(\gamma')}{(\overline{m}_0^2 - \gamma')^2} \quad \gamma^{\text{thres}} = (\overline{m}_0 + \xi m_\sigma)^2$$

Common parameters: $\Lambda = 10$, $m_\sigma = 1$, $\alpha = 0.5$.

ξ	R	m_0	m_B
1(Feynman)	0.884	0.759	0.5
0(Landau)	1.05	0.797	0.5



$\alpha \sim 0.22$

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Regular Article - Theoretical Physics

The soft-gluon limit and the infrared enhancement of the quark-gluon vertex

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