Relations between GPDs and TMDs in LFQDM

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Outline

- Introduction: Various aspects of nucleon properties
- LF Quark-Diquark model
- Generalized Parton Distributions (GPDs)
- Transverse Momentum Dependent Parton Distributions (TMDs)
- Possible relations between GPDs and TMDs
- Orbital Angular Momentum of the quarks
- Conclusion
Hofstadter and collaborators interpret elastic ep scattering cross section in terms of a form factor: [-R. Hofstadter, Nobel Prize 1961]

\[ \sigma(\theta_e) = \sigma_M \left| \int \rho(\vec{r}) e^{i \vec{q} \cdot \vec{r}} d^3 \vec{r} \right|^2 = \sigma_M |F(q)|^2 \]

The nucleon EM FFs describe the spatial distributions of electric charge and current inside the nucleon and thus are intimately related to its internal structure.

- The electromagnetic form factors can be probed through elastic scattering.

The Fourier transformation of these form factors \( \Rightarrow \) spatial distributions.

- Dynamical informations (orbital angular momentum ) \( \Rightarrow \) Missing!!
Introduction: nucleon properties

- DIS discovered the existence of quasi-free point-like objects (quarks) inside the nucleon. [Friedman, Kendall, Taylor, Nobel Prize 1990]

- Parton distribution functions (PDFs) are extracted from deep inelastic scattering (DIS) processes.

- PDFs encode the distribution of longitudinal momentum and polarization carried by the constituents.

- The PDFs provide no knowledge of spatial locations of parton
GPDs appear in the exclusive processes like deeply virtual Compton scattering (DVCS) or vector meson productions.

In DVCS a highly virtual photon, radiated by incident electron, interacts with a quark and emit a real photon before going back to the nucleon.

GPDs encode the informations about the three dimensional spatial structure of the nucleon as well as the spin and orbital angular momentum of the constituents.
Proton is considered as a bound state of a quark and a diquark. The diquark can have spin-0 singlet (scalar diquark) or spin-1 triplet (axialvector diquark).

The proton state is written in the spin-flavor SU(4) structure as

\[ |P; \pm \rangle = C_S |uS^0\rangle^\pm + C_V |uA^0\rangle^\pm + C_{VV} |dA^1\rangle^\pm. \]

Two particle Fock-state expansion for \( J_z = \pm \frac{1}{2} \)

The state with spin-0 scalar diquark

\[ |uS\rangle^\pm = \int \frac{dx d^2 p_{\perp}}{2(2\pi)^3 \sqrt{x(1-x)}} \sum_{\lambda} \psi^{N(u)}_{\lambda}(x, p_{\perp}) |\lambda S; xP^+, p_{\perp}\rangle \bigg|_{\Lambda_S=0} \]
The state with spin-1 axialvector diquarks

\[ |\nu A\rangle^\pm = \int \frac{dx d^2 p_\perp}{2(2\pi)^3 \sqrt{x(1-x)}} \sum \sum \psi^{\lambda N(\nu)}_{\lambda A}(x, p_\perp) \left| \psi^{\lambda A}_{\lambda A}; x P^+, p_\perp \right|_{\Lambda_A = 1, 0, -1} \]

The light-front wave functions:

\[ \psi^{\pm(\nu)}_{\lambda A}(x, p_T) = N^{\nu} f(x, p_T, \lambda, \Lambda) \phi^{(\nu)}_i(x, p_T) |_{i=1,2} \]

Modified soft-wall AdS/QCD wave function for two particle bound state: [-Brodsky and Teramond arXiv:1203.4025]

\[ \varphi_i^{(\nu)}(x, p_\perp) = \frac{4\pi}{\kappa} \sqrt{\frac{\log(1/x)}{1-x}} x^{a_i^{\nu}} (1-x)^{b_i^{\nu}} \exp \left[ -\delta^{\nu} \frac{p^2_\perp}{2\kappa^2} \frac{\log(1/x)}{(1-x)^2} \right] \]

with the AdS/QCD scale parameter \( \kappa = 0.4 \text{GeV} \).

The parameters \( a_i^{\nu}, b_i^{\nu}, \delta^{\nu} \) are fixed by fitting the experimental data of the Dirac \( F_1^{\nu}(Q^2) \) and Pauli \( F_2^{\nu}(Q^2) \) form factors.
Nucleon Generalized Parton Distributions (GPDs)

- GPDs can be defined through off-forward matrix elements of quarks and gluon operators at a light-cone separations

\[ F^q[\Gamma] (x, \Delta; \lambda, \lambda') = \frac{1}{2} \int \frac{dz^-}{2\pi} e^{ik\cdot z} \langle p'; \lambda' | \bar{\psi} \left( -\frac{1}{2} z \right) \Gamma \times \mathcal{W}_{GPD} \left( -\frac{1}{2} z; \frac{1}{2} z \right) \psi \left( \frac{1}{2} z \right) | p; \lambda \rangle \bigg|_{z^+=0^+, z_T=0^+} \]

- For three different Dirac, \( \Gamma = \gamma^+ \) (vector), \( \Gamma = \gamma^+\gamma_5 \) (axial vector) and \( \Gamma = i\sigma^+j\gamma_5 \) (tensor) structures, The GPD correlator

\[
\begin{align*}
F^q[\gamma^+] (x, \Delta; \lambda, \lambda') &= \frac{1}{2P^+} \bar{u} \left( p', \lambda' \right) \left[ H^q \gamma^+ + E^q \frac{i}{2M} \sigma^+\alpha \Delta_\alpha \right] u(p, \lambda) \\
F^q[\gamma^+\gamma_5] (x, \Delta; \lambda, \lambda') &= \frac{1}{2P^+} \bar{u} \left( p', \lambda' \right) \left[ \bar{H}^q \gamma^+\gamma_5 + \bar{E}^q \frac{\gamma_5\Delta^+}{2M} \right] u(p, \lambda) \\
F^q[\sigma^+j\gamma_5] (x, \Delta; \lambda, \lambda') &= \frac{1}{2P^+} \bar{u} \left( p', \lambda' \right) \left[ H^q \sigma^+\gamma_5 + \bar{H}^q \frac{\epsilon^{j+\alpha\beta} \Delta_\alpha P_\beta}{M^2} + E^q \frac{\epsilon^{j+\alpha\beta} \Delta_\alpha \gamma_\beta}{2M} \right. \\
&\quad \left. + \bar{E}^q \frac{\epsilon^{j+\alpha\beta} P_\alpha \gamma_\beta}{M} \right] u(p, \lambda)
\end{align*}
\]

- Project out quark polarizations: 
  - Unpolarized: \( \text{Tr}[F \gamma^+] \rightarrow (H, E)(x, \zeta, t) \), Longitudinally
  - Polarized: \( \text{Tr}[F \gamma^+\gamma_5] \rightarrow (\bar{H}, \bar{E}) \), Transversely polarized [chiral-odd]: \( \text{Tr}[F \sigma^+\gamma_5^\perp] \rightarrow (H_T, E_T, \bar{H}_T, \bar{E}_T) \)
GPDs sum rules and relations

- First moment of the GPDs are related to the electromagnetic FF.
  \[ \int_0^1 dx \ H_0^q(x, t) = F_1^q(t), \quad \int_0^1 dx \ E_0^q(x, t) = F_2^q(t) \]

- Second moment of GPDs give the gravitational FFs.
  \[ \int_0^1 dx \ xH_0^q(x, t) = A_1^q(t), \quad \int_0^1 dx \ xE_0^q(x, t) = B_1^q(t) \]

- At \( t = -Q^2 = 0 \), Gravitational FFs \( \Rightarrow \) Angular momentum.
  \[ J^q = \frac{1}{2}[A^q(0) + B^q(0)] \quad [X. J. PRL 78, 610(1997)] \]
  \[ J^T = \frac{1}{2}[A_T^q(0) + 2A_T^q(0) + B_T^q(0)] \quad [M. Burkardt, PRD 72, 094020(2005)] \]

- GPDs \( \Rightarrow \) not probabilistics!! (in momentum space).
- GPDs in impact parameter space[\( \zeta = 0 \)]:
  \[ \mathcal{F}(x, b_T) = \int \frac{d^2\Delta_T}{(2\pi)^2} e^{-i\Delta_T \cdot b_T} F(x, 0, \Delta_T) \]

- GPDs in Impact parameter space(IPDs) \( \Rightarrow \) Parton distribution in transverse position space.
- Impact parameter space \( \Rightarrow \) ”digonal” matrix elements \( \Rightarrow \) Density interpretations!
TMDs give probability of finding a parton with the longitudinal momentum fraction $x$ and the transverse momentum $p_\perp$ inside a nucleon.

In the light front formalism, the unintegrated quark-quark correlator for SIDIS

$$
\Phi_\nu^{[\Gamma]}(x, p_\perp; S) = \frac{1}{2} \int \frac{dz - d^2z_T}{2(2\pi)^3} e^{ip\cdot z} \left\langle P; S \middle| \bar{\psi}_\nu^{(-\frac{z}{2})} W_{TMD}(\frac{-z}{2}, \frac{z}{2}) \Gamma \psi_\nu^{(\frac{z}{2})} \middle| P; S \right\rangle \bigg|_{z^+ = 0}
$$

Leading twist TMDs are projected out using different Dirac structure

$$
\Phi_\nu^{[\gamma^+]}(x, p_\perp; S) = f_1^\nu(x, p_\perp^2) - \frac{\epsilon^{ij} p_\perp^i S_T^j}{M} f_{1T}^\nu(x, p_\perp^2)
$$

$$
\Phi_\nu^{[\gamma^+\gamma^5]}(x, p_\perp; S) = \lambda_N g_1^\nu(x, p_\perp^2) + \frac{p_\perp \cdot S_T}{M} g_{1T}^\nu(x, p_\perp^2)
$$

$$
\Phi_\nu^{[i\sigma^j\gamma^5]}(x, p_\perp; S) = S_T^j h_1^\nu(x, p_\perp^2) + \lambda_N \frac{p_\perp^j}{M} h_{1L}^\nu(x, p_\perp^2)
$$

$$
+ \frac{2p_\perp^j p_\perp \cdot S_T - S_T^2 p_\perp^2}{2M^2} h_{1T}^\nu(x, p_\perp^2) - \frac{\epsilon^{ij} p_\perp^i}{M} h_{1T}^\nu(x, p_\perp^2)
$$
Depending on the different polarization of nucleon and the interior partons, there are altogether 8 TMDs at the leading twist: 6 T-even and 2 T-odd.

Project out quark polarizations:
- **Unpolarized**: $\text{Tr}[\Phi \gamma^+] \rightarrow (f_1, f_1^T)(x, p_T^2)$.
- **Longitudinally polarized**: $\text{Tr}[\Phi \gamma^+\gamma_5] \rightarrow (g_{1L}, g_{1T})$.
- **Transversely polarized [chiral-odd]**: $\text{Tr}[\Phi \sigma^{+-}] \rightarrow (h_1, h_1^T, h_{1L}^T, h_1^L)$.

### Leading Twist TMDs

<table>
<thead>
<tr>
<th>Quark polarization</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Un-Polarized</strong></td>
</tr>
<tr>
<td><strong>Longitudinally Polarized</strong></td>
</tr>
<tr>
<td><strong>Transversely Polarized</strong></td>
</tr>
<tr>
<td><strong>(U)</strong></td>
</tr>
<tr>
<td><strong>(L)</strong></td>
</tr>
<tr>
<td><strong>(T)</strong></td>
</tr>
<tr>
<td>$f_1 = \bullet$</td>
</tr>
<tr>
<td>$g_1 = \bullet$</td>
</tr>
<tr>
<td>$f_{1T}^\perp = \bullet$</td>
</tr>
<tr>
<td>$g_{1T}^\perp = \bullet$</td>
</tr>
<tr>
<td>$h_1^\perp = \bullet$ - $\bullet$</td>
</tr>
<tr>
<td>$h_{1L}^\perp = \bullet$</td>
</tr>
<tr>
<td>$h_{1T}^\perp = \bullet$</td>
</tr>
</tbody>
</table>

- **Boer-Mulder**
- **Helicity**
- **Sivers**
- **Transversity**
Gauge link for TMDs

- Quark field operators are at two different points \( \Rightarrow \) breaks the gauge invariance of the correlator!!
- Gauge invariance restored \( \Rightarrow \) Gauge link/ Wilson line \( \mathcal{W}[z_1; z_2] \)
- Gauge link \( \mathcal{W}[z_1; z_2] \) for T-even TMDs \( \Rightarrow \) Unity.
- Gauge link for T-odd TMDs \( \not\Rightarrow \) Unity[more complicated function].

\[
\mathcal{W}[z_1; z_2] = \mathcal{P}e^{-ig \int_{z_1}^{z_2} ds \cdot A(s)}
\]

- Describes initial (DY) and Final(SIDIS) state interactions

- Time-reversal: switches Wilson-line \( ISI \rightarrow FSI \)
  \[ f_{1T}^{+}|_{DIS} = -f_{1T}^{+}|_{DY}, \quad h_{1}^{+}|_{DIS} = -h_{1}^{+}|_{DY} \Rightarrow \text{process dependence!!} \]
Final state interactions for SIDIS and T-odd TMDs

- Gluon exchange between outgoing quark and the target spectator system $\Rightarrow$ FSI

- T-odd TMDs $\Rightarrow$ FSI $\rightarrow$ complex phase!

- Leads SSA at leading twist.

- The modified LFWFs for T-odd TMDs

$$\psi_{\lambda q} = \tilde{\psi}_{\lambda q} + i \tilde{\phi}_{\lambda q}$$

$$\psi_{\lambda q}^{\perp} = \tilde{\psi}_{\lambda q}^{\perp} + i \tilde{\phi}_{\lambda q}^{\perp}$$

$$g_1 = \int_0^1 d\alpha \frac{1}{\alpha(1-\alpha)|p_\perp|^2 + \alpha m_g^2 + (1-\alpha)B}$$

$$g_2 = \int_0^1 d\alpha \frac{-\alpha}{\alpha(1-\alpha)|p_\perp|^2 + \alpha m_g^2 + (1-\alpha)B}$$

$$B = x(1-x) \left( -M^2 + \frac{m_q^2}{x} + \frac{m_D^2}{1-x} \right)$$

$$e_1 e_2 \rightarrow -4\pi C_F \alpha_s$$  [-T. Maji, D. Chakrabarti, and A.Mukherjee Phys. Rev. D 97, 014016]
The non-trivial and non-perturbative correlations between the intrinsic transverse momentum of the partons and the spin of the nucleon ⇒ Sivers/Boer Mulders (T-odd) TMDs

T-odd *(Sivers and Boer Mulder) TMDs,

\[
f_{1T}^{\perp, \nu} (x, p_T^2) = \left( C_S^2 N_S^{\nu 2} - C_A^2 \frac{1}{3} N_0^{\nu 2} \right) f^{\nu} (x, p_T^2)
\]

\[
h_{1}^{\perp, \nu} (x, p_T^2) = \left( C_S^2 N_S^{\nu 2} + C_A^2 \left( \frac{1}{3} N_0^{\nu 2} + \frac{2}{3} N_1^{\nu 2} \right) \right) f^{\nu} (x, p_T^2)
\]

\[
f^{\nu} (x, p_T^2) = (-C_F \alpha_s) \times \frac{1}{x} \left[ \frac{p_T^2 + B(x)}{p_T^2} \right] \log \left[ \frac{p_T^2 + B(x)}{B(x)} \right]
\]

\[
\times \frac{1}{16 \pi^3} A_1^{\nu'} (x) A_2^{\nu'} (x) \exp \left[ -a(x)p_T^2 \right]
\]

Boer Mulders function \(\propto\) Sivers function

\[
h_{1}^{\perp, \nu} (x, p_T^2) \simeq \lambda^{\nu} f_{1T}^{\perp, \nu} (x, p_T^2)
\]

<table>
<thead>
<tr>
<th>(\lambda^{\nu})</th>
<th>Our Model</th>
<th>Phenomenological fits</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\lambda^{u})</td>
<td>2.29</td>
<td>2.1 ± 0.1</td>
</tr>
<tr>
<td>(\lambda^{d})</td>
<td>-1.08</td>
<td>-1.11 ± 0.02</td>
</tr>
</tbody>
</table>

For SIDIS process

The Sivers TMD ⇒

\[ f_{1T}^{\perp, u} < 0, \text{ and } f_{1T}^{\perp, d} > 0 \]

The Boer Mulders TMD ⇒

\[ h_{1}^{\perp, u} < 0, \text{ and } h_{1}^{\perp, d} < 0 \]

Confirmed by HERMES, COMPASS data

[-Barone,Melis, Prokudin,PRD81,114026]
Non-trivial relations for “T-odd” parton distributions:

- **Sivers TMD** ⇒ Unpolarized partons inside transversely polarized nucleon. **Related to the distortion in impact parameter space through ”Lensing function”**

\[
\mathcal{L}_{k_{T}^{q,i}(x)}^{UT} = - \int d^{2}k_{T} k_{T}^{i} \frac{e_{T}^{j} k_{T}^{j} S_{T}^{k}}{M} f_{1T}^{q} (x, k_{T}^{2}) \\
\simeq \int d^{2}b_{T} I_{q,i} (x, b_{T}) \frac{e_{T}^{j} b_{T}^{j} S_{T}^{k}}{M} \left( \mathcal{E}_{q} (x, b_{T}^{2}) \right)'
\]

- **Boer Mulder TMD** ⇒ Transversely polarized partons inside unpolarized nucleon. **The spin-orbit correlation of quarks can be understood from** \( h_{1}^{\perp} (x, p_{\perp}) \)

\[
\mathcal{L}_{k_{T}^{q,i}(x)}^{UT} = - \int d^{2}k_{T} k_{T}^{i} \frac{e_{T}^{j} k_{T}^{j} S_{T}^{k}}{M} h_{1\perp}^{q} (x, k_{T}^{2}) \\
\simeq \int d^{2}b_{T} I_{q,i} (x, b_{T}) \frac{e_{T}^{j} b_{T}^{j} S_{T}^{k}}{M} \times \left( \mathcal{E}_{q}^{q} (x, b_{T}^{2}) + 2 \mathcal{H}_{q}^{q} (x, b_{T}^{2}) \right)'
\]

[ -M. Burkardt [Nucl.Phys. A735, 185], [PRD66, 114005]]
Lensing function for LFQDM

- The Lensing function contains final state interaction information.
- The lensing function SDM/QTM model:

\[ \mathcal{I}_{SDM}^{q,i}(x, b_T) = \frac{e_q e_S}{4\pi} \frac{(1-x)b_T^2}{b_T^2} \]


(1) Axial vector diquark model

\[ \mathcal{I}_{SDM}^{q,i}(x, b_T) = \frac{5F_F\alpha_s}{2\pi} x^{3/2}(1-x)^{-1/5} \log^5(1/x) \frac{b_T^i}{b_T^2}, \quad 0.2 > x > 0 \]

\[ = \frac{2C_F\alpha_s}{\pi} \sqrt{(1-x)\log(1/x)} \frac{b_T^i}{b_T^2}, \quad 1 > x > 0.2 \]

(2) Scalar diquark model

\[ \mathcal{I}_{SDM}^{q,i}(x, b_T) = \frac{5C_F\alpha_s}{\pi} (1-x)b_T^i \]


- The lensing function for our model:
Moment relations between GPDs and TMDs

- Relations between arbitrary moments:
  - \textbf{GPD} : \( X^{(n)}(x) = \frac{1}{2M^2} \int d^2 \Delta_T \left( \frac{\Delta_T^2}{2M^2} \right)^{n-1} X \left( x, 0, -\frac{\Delta_T^2}{(1-x)^2} \right) \)
  - \textbf{TMD} : \( Y^{(n)}(x) = \int d^2 k_T \left( \frac{k_T^2}{2M^2} \right)^n Y \left( x, k_T^2 \right) \)

- Moment relations between \( f_{1T}^{\perp q} \) and \( E^q \)
  - \( f_{1T}^{\perp q(0)}(x) = \frac{\pi e_q e_s}{48(1-x)} E^q(x, 0, 0) \)
  - \( f_{1T}^{\perp q(1/2)}(x) = \frac{2e_q e_s \ln(2)}{(2\pi)^3(1-x)} E^{q(1/2)}(x) \)
  - \( f_{1T}^{\perp q(1)}(x) = \frac{e_q e_s}{4(2\pi)^2(1-x)} E^{q(1)}(x) \)

[-S. Meissner, A. Metz, K. Goeke PhysRevD.76.034002, 2007]
Relation between pretzelosity TMD $h_{1T}^{\perp}(n)$ and GPD $\tilde{H}_T^{(n)}$

**LFQDM**

\[
h_{1T}^{\perp q(0)}(x) = \frac{2}{(1-x)^2} \tilde{H}_T^q(x,0,0)
\]

\[
h_{1T}^{\perp q(1/2)}(x) = \frac{1}{2\pi(1-x)^2} \tilde{H}_T^q(1/2)(x)
\]

\[
h_{1T}^{\perp q(1)}(x) = \frac{1}{2\pi(1-x)^2} \tilde{H}_T^q(1)(x)
\]

**SDM/QTM**

\[
h_{1T}^{\perp q(0)}(x) = \frac{3}{(1-x)^2} \tilde{H}_T^q(x,0,0)
\]

\[
h_{1T}^{\perp q(1/2)}(x) = \frac{8}{(2\pi)^2(1-x)^2} \tilde{H}_T^q(1/2)(x)
\]

\[
h_{1T}^{\perp q(1)}(x) = \frac{1}{2\pi(1-x)^2} \tilde{H}_T^q(1)(x)
\]

Quark OAM

- For many years, the spin of the proton $S=1/2 \Rightarrow$ spin of the three valence quarks!
- In 1980, the EMC experiment at CERN $\Rightarrow$ only $13 \pm 16\%$ comes from the spin of the quarks, 'proton spin crisis' or 'proton spin puzzle'.
- Lattice QCD calculations indicates $\Rightarrow$ 50% proton spin come from quark OAM.
- The orbital angular momentum (OAM) of a quark inside the nucleon plays an important role in the spin sum rule of the nucleon.

Jaffe and Manohar showed the decomposition of the nucleon spin (not gauge invariant!)

$$S_{\text{proton}} = \frac{1}{2} = \frac{1}{2} \Sigma + \Delta G + \langle L_q \rangle + \langle L_g \rangle$$


- Ji proposed a gauge invariant decomposition of the nucleon spin as $S^q + L^q + J_g = \frac{1}{2}$

For the diquark model, the above sum rule can be written as

\[ S^q + \ell^q + S^D + \ell^D = \frac{1}{2} \]

The kinetic orbital angular momentum of the quark can extract in terms of GPDs as

\[ L^q_z = \frac{1}{2} \int dx \left[ x \left( H^q(x, 0, 0) + E^q(x, 0, 0) \right) - \tilde{H}^q(x, 0, 0) \right] \]

The quark OAM also can be calculated by using the pretzelosity TMD as

\[ \mathcal{L}^q_z = -\int dx d^2p_\perp \frac{\tilde{p}_\perp^2}{2M^2} h_{1T}^{1q}(x, \tilde{p}_T) \]

The Canonical quark OAM can be given through the GTMDs as

\[ \ell^\nu_z = -\int dx d^2p_\perp \frac{p_\perp^2}{M^2} F_{1,4}^\nu(x, 0, p_\perp^2, 0, 0) \]
Conclusion

- Using LFQDM for the proton we have presented FFs, both the chiral even and odd GPDs and the T-even/T-odd TMDs at leading twist.
- We investigated moment relation between GPDs and TMDs in momentum space using LFQDM.
- Non-trivial relations between GPDs and (T-odd) TMDs were suggested on the basis of a separation between distortion of parton distribution in the transverse plane + Final state interactions.
- We calculate the quark orbital angular momentum and the results are compared with the results in other similar models.

Thank you!!