Introduction to the Standard Model

December 12/1-5/2022

Markus Elsing's Advice

Given that most of the students are experimentalists, it is suggested that you avoid going into too much mathematical detail and focus on the physics concepts. It is assumed that all students have some pre-knowledge of the topics taught at the School.

Fermi's Advice

Never underestimate the pleasure people get from hearing something they already know

Units

We choose natural units, where the reduced Planck constant and the speed of light are set to one. Thus, out of L,M,T, only one unit is left when making dimensional analysis. We will choose most of the time powers of energy. Converting to regular units is straightforward, they simply follow from wave-particle duality. We can also classify operators between relevant, marginal and irrelevant

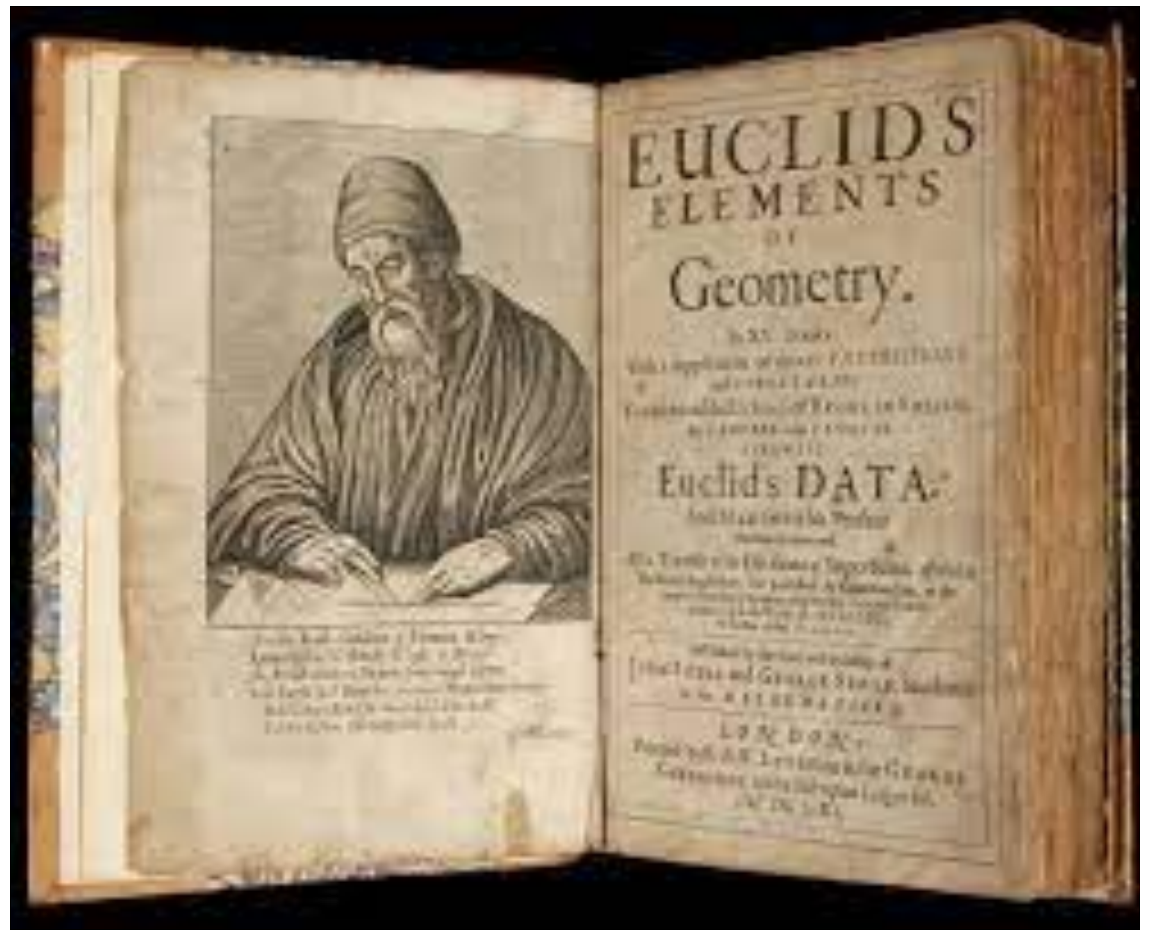
$$m \to \frac{E}{c^2}, \qquad l \to \frac{\hbar c}{E}, \qquad t \to \frac{\hbar}{E} \qquad \hbar = 1, \quad c = 1$$

$$S = \int d^4x\, \mathcal{L} \qquad \mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4 - \frac{g}{M^2}\phi^6$$

S: E^0; d^4x: E^{-4}; $\mathcal{L}$: E^4; $\mathcal{L}$: E^4; ∂_μ: E; ϕ: E; m: E; ϕ: E; $\lambda\phi^4$: E^4; ϕ^6: E^6

Relevant (mass term) — Marginal ($\lambda\phi^4$) — Irrelevant ($\frac{g}{M^2}\phi^6$)

The importance of symmetry



Vergleichende Betrachtungen

über

neuere geometrische Forschungen

von

Dr. Felix Klein,

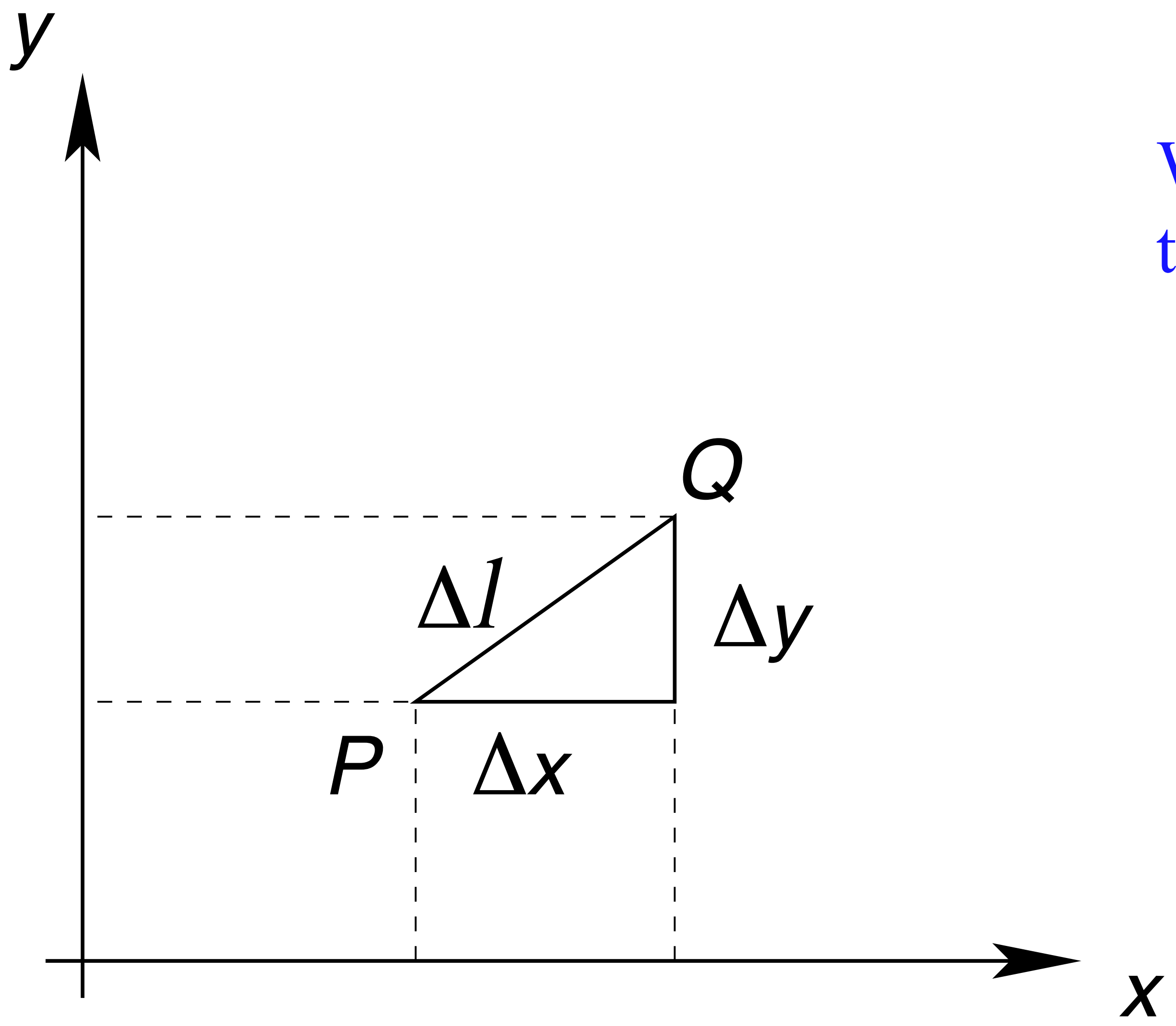
o. ö. Professor der Mathematik an der Universität Erlangen.

Programm

zum Eintritt in die philosophische Facultät und den Senat der k. Friedrich-Alexanders-Universität zu Erlangen.

Felix Klein's Erlangen program translates geometry into groups of symmetry acting in space.

All of Euclidean geometry follows from a symmetry principle. Euclidean geometry on the plane (or in d-dimensional space) is characterized by the set of motions leaving invariant the distance between two points. Group theory is a major mathematical heritage from the XIXth Century, providing the basis for one of the fundamental concepts in XXth Century physics: The importance of symmetry principles



$$(\Delta l)^2 = (\Delta x)^2 + (\Delta y)^2$$

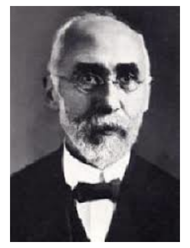
We have invariance wrt translations and rotation, this is the Euclidean group in d-dimensions

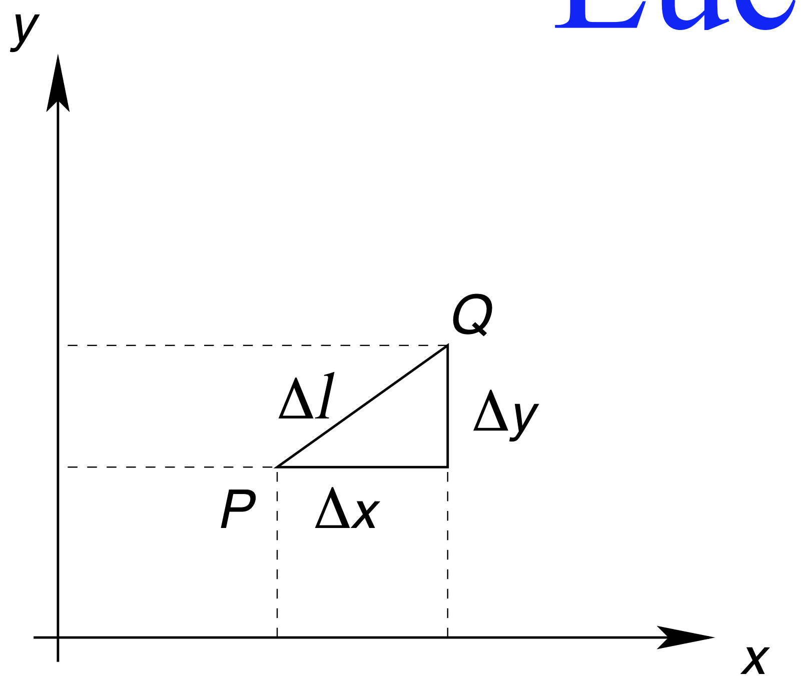
$$X^i \to a^i + R^i_j X^j$$

$$R^T R = 1$$

$$E(d) \equiv \mathrm{ISO}(d)$$

Euclidean vs Relativistic Geometry

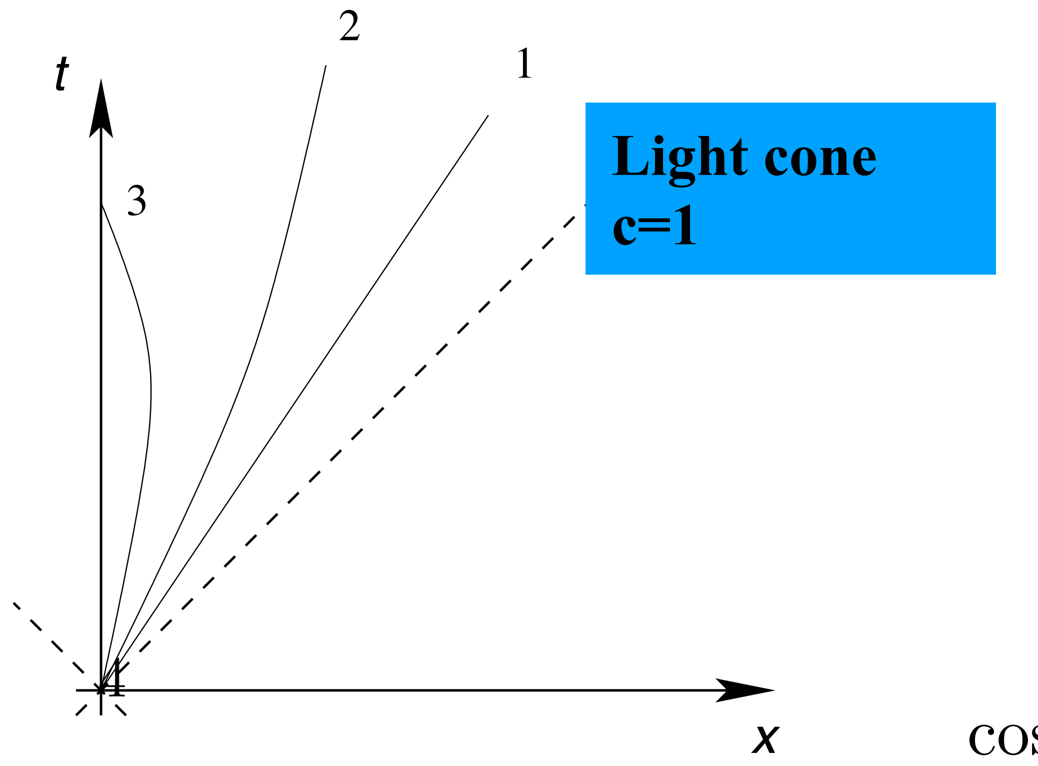





$$x' = \cos(\alpha)x + \sin(\alpha)y$$
$$y' = -\sin(\alpha)x + \cos(\alpha)y$$

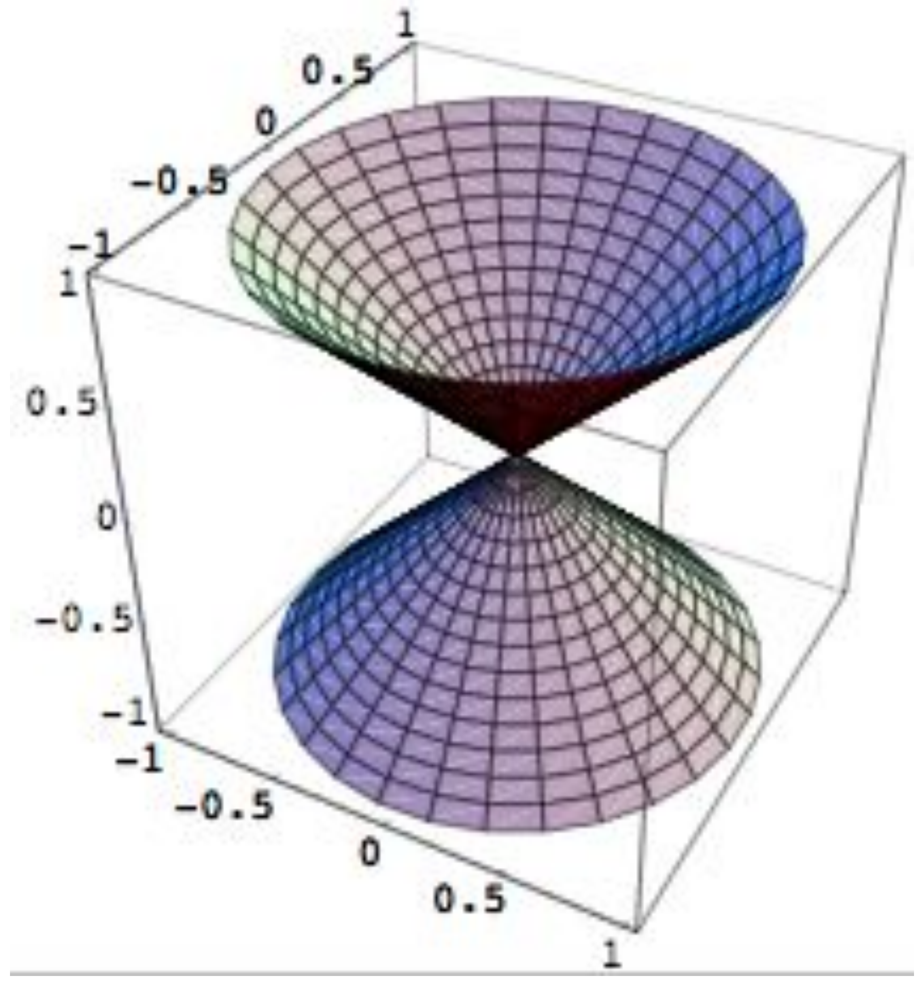
$$(\Delta l)^2 = (\Delta x)^2 + (\Delta y)^2$$

Length becomes proper time



$$t' = \cosh(\alpha)t - \sinh(\alpha)x$$
$$x' = -\sinh(\alpha)t + \cosh(\alpha)y$$
$$\tanh(\alpha) = v$$
$$\cosh(\alpha) = \frac{1}{\sqrt{1-v^2}}, \ \sinh(\alpha) = \frac{v}{\sqrt{1-v^2}}$$

$$(\Delta\tau)^2 = (\Delta t)^2 - (\Delta l)^2$$



Proper time, the light cone determines the causal structure of space-time. A four-dimensional space-time known as Minkowski space-time. The symmetry group that defines the geometry is the Poincaré group, consisting of translations and Lorentz transformations

$$\mathcal{P} \equiv ISO(1,3)$$

$$x^\mu \to a^\mu + \Lambda^\mu_\nu x^\nu$$

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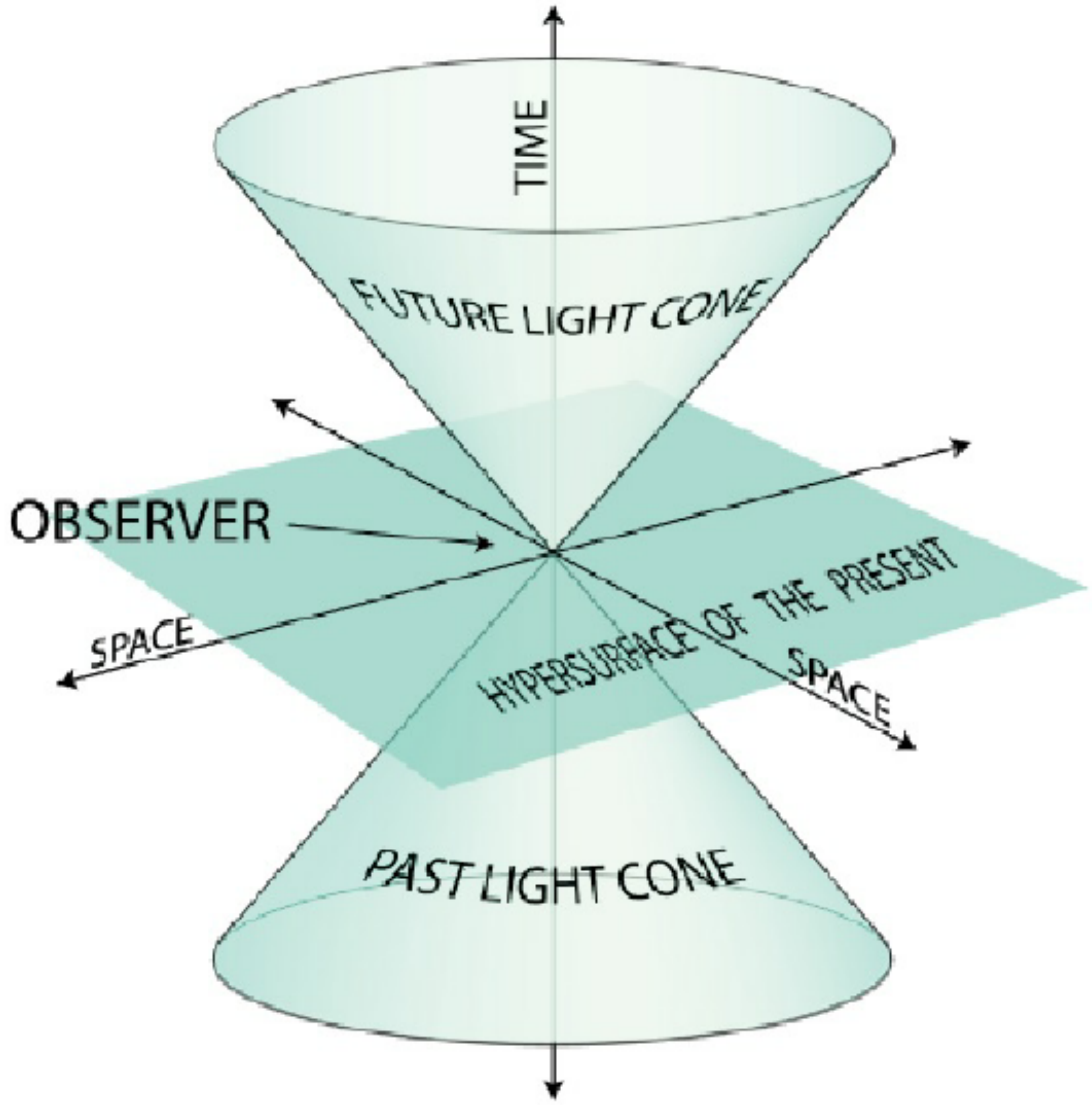
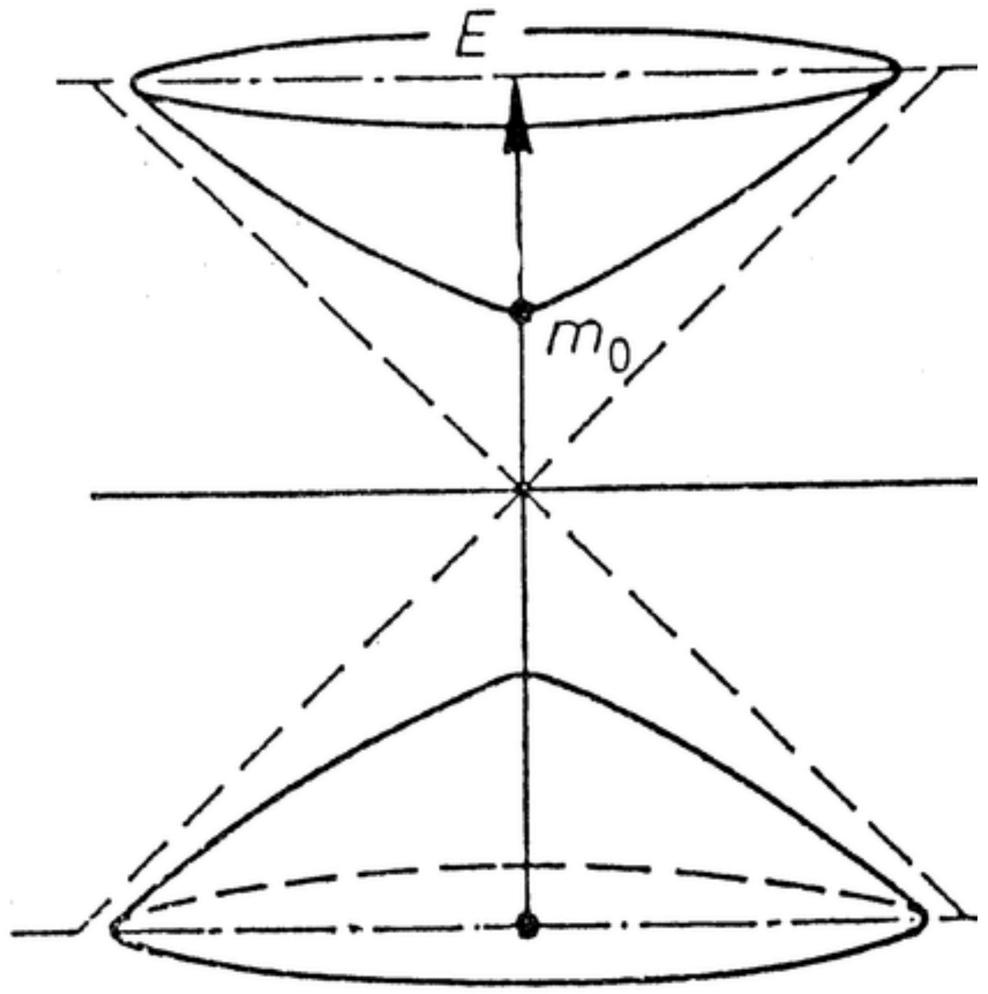
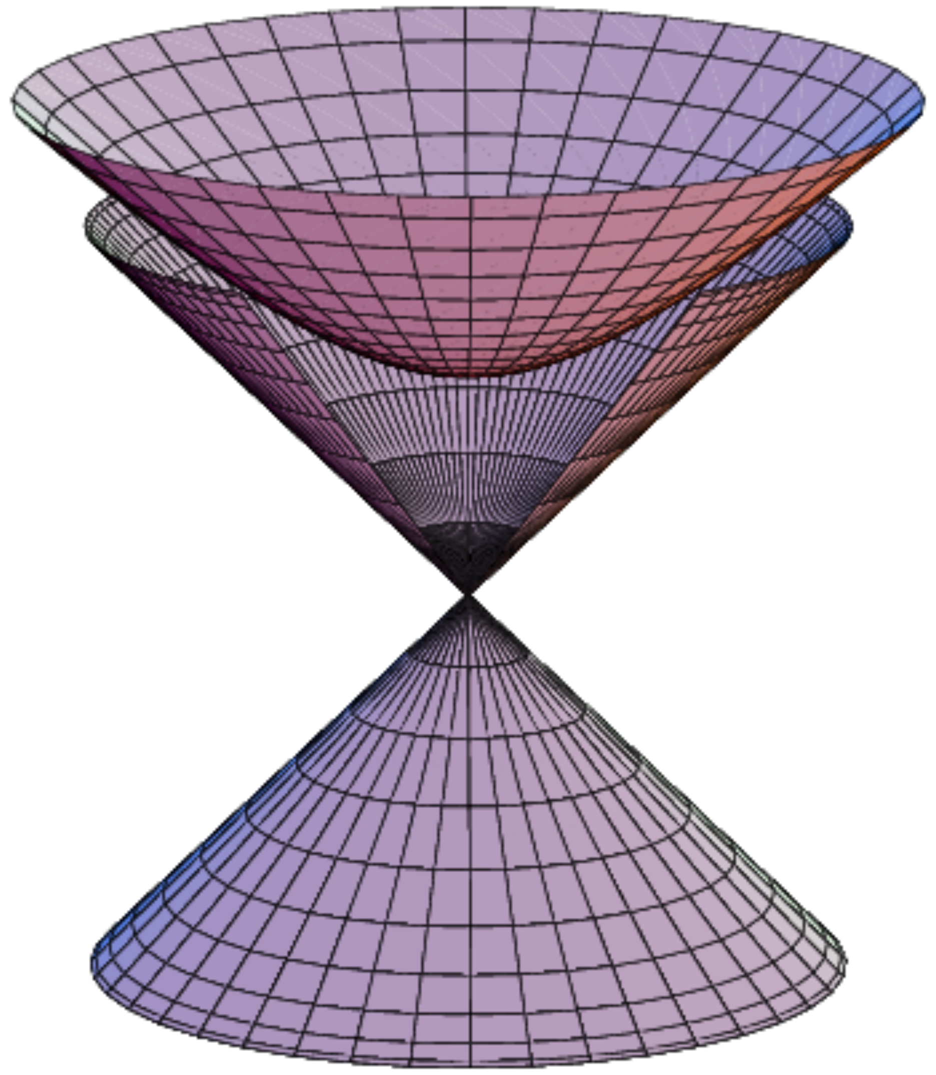
The light cone and the mass hyperboloid, position space versus momentum space



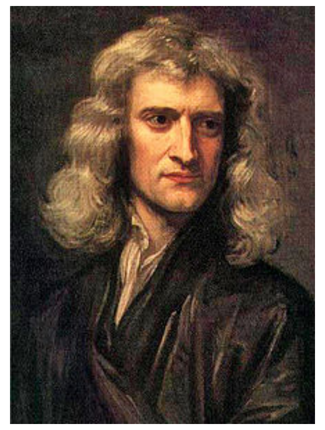


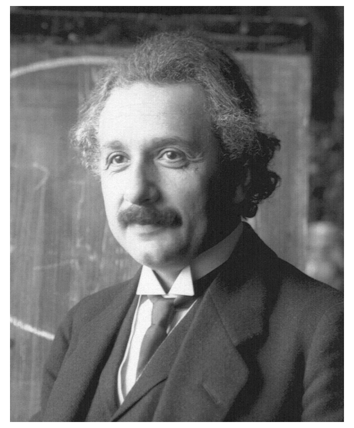


$$E^2 - \mathbf{p}^2 = m_0^2 \quad \Rightarrow \quad E = m\,c^2 \qquad m = \frac{E}{c^2}$$

All of relativistic mechanics follows from implementing these properties

A few useful remarks





Mechanics reminder

$$
\begin{aligned}
p_N &= m\,v \\
p_E &= m\,v\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}
\end{aligned}
$$

$$
\begin{aligned}
E_N &= \frac{m}{2}\,v^2 \\
E_E &= m\,c^2\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}
\end{aligned}
$$

$$m \to 0, \qquad v \to c \qquad p = \frac{E}{c}$$

Massless particles can only be dealt with in relativity

Mass (inertia) represents resistance to acceleration

Nothing to do with friction

Viscosity is resistance to velocity

Einstein and Heisenberg complicate our lives

Useful basic formulae. A reminder. Just this once, we reintroduce h and c

$$p^2 = \left(\frac{E}{c}\right)^2 - \mathbf{p}^2 = m^2c^2$$

$$E = \pm\sqrt{\mathbf{p}^2c^2 + m^2c^4} \approx \pm(mc^2 + \frac{\mathbf{p}^2}{2m} + \dots)$$

$$\Delta x \, \Delta p \geq \frac{\hbar}{2}$$

$$\lambda = \frac{h}{mc} \quad \text{Compton wavelength}$$

$$E = \frac{mc^2}{\sqrt{1 - \mathbf{v}^2/c^2}} \qquad \mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - \mathbf{v}^2/c^2}}$$

$$\Delta p \geq mc \qquad \Delta E \geq mc^2$$

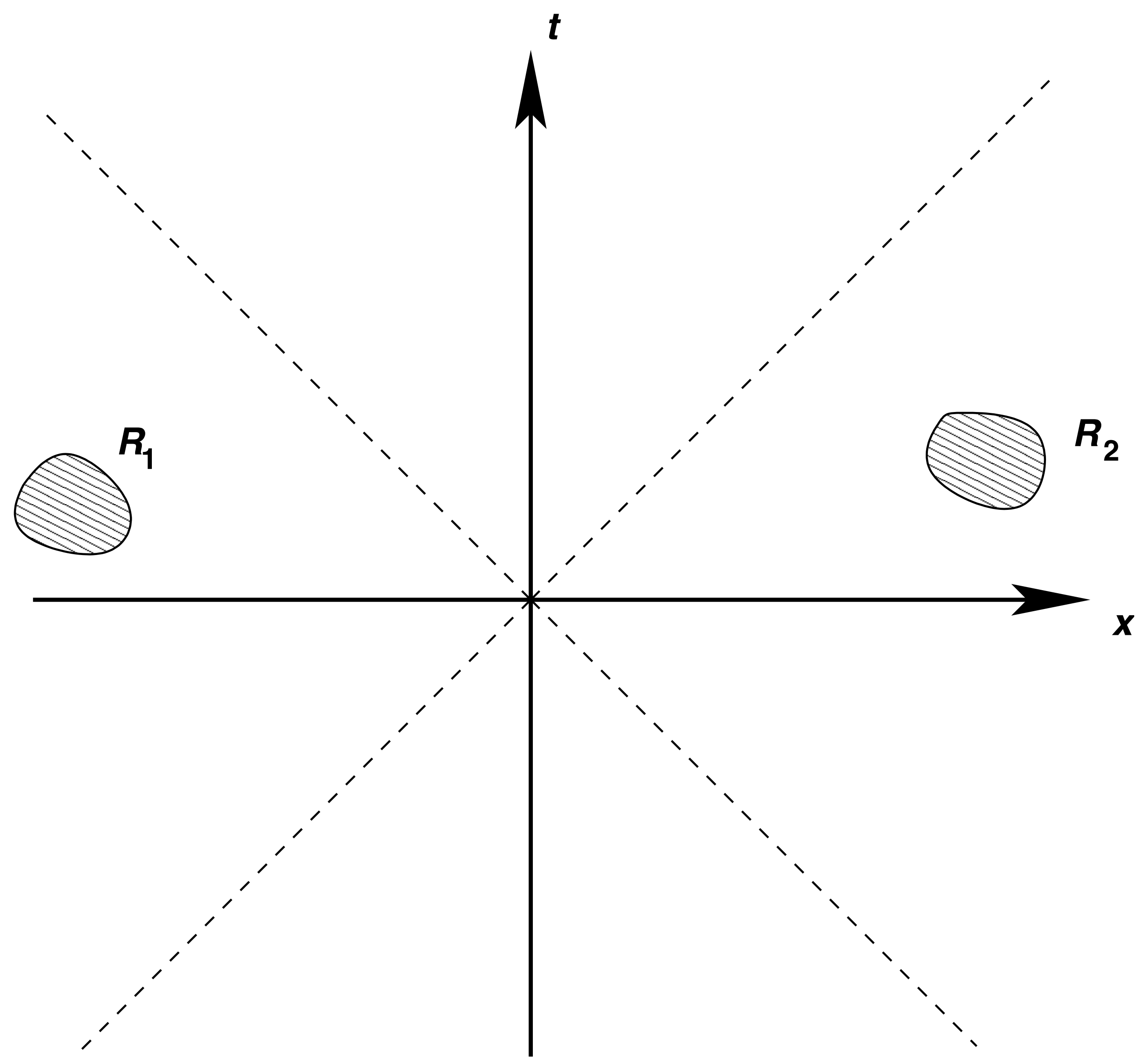
$$(\Delta x)_{\min} \geq \frac{1}{2}\left(\frac{\hbar}{mc}\right)$$

When the uncertainly in momentum is bigger than mc, the uncertainty in energy is larger than mc^2, hence there is enough energy to produce another particle of the same type. In Relativity mass and energy are interchangeable. Hence we cannot localize a particle below its Compton wavelength. If we do, we will not find a single particle, but rather a fairly complicated quantum state with no well-defined number of particles.

Particle production by physical processes should be a central part of the theory.

Particle number is not conserved. We need fields

Relativistic causality



$$[\mathcal{O}(x), \mathcal{O}(y)] = 0, \qquad \text{if } (x-y)^2 < 0.$$

The world is Quantum

Particle Wave Duality

Special Relativity

Microscopic Causality

→ LQFT

Microscopic causality, Locality in Special Relativity imposes important constraints into what are observables. The light-cone decrees the causal structure of space-time.Physical measurements should be compatible with it

Lorentz and Poincaré Groups

In trying to systematically construct viable QFTs it is useful to understand the representations of the Lorentz (and Poincaré) groups.

The Hilbert space of states has to carry a unitary representation of the Lorentz group, so that quantum amplitudes are consistent with Unitarity and Relativistic Invariance. The fields themselves however, transform under finite dimensional representations. They are much easier to study. Just recall the usual rotation group SU(2). The Lorentz group, also known as SO(3,1) preserves the Minkowski metric

In a moment we will look at the representations of the group.

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu \qquad \mu,\nu = 0,1,2,3$$

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \qquad \eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta}$$

$$\det\Lambda = \pm 1 \qquad (\Lambda^0_0)^2 - \sum_{i=1}^{3} (\Lambda^i_0)^2 = 1$$

- $\mathfrak{L}^\uparrow_+$: proper, orthochronous transformations with $\det\Lambda = 1, \Lambda^0{}_0 \geqslant 1$.
- $\mathfrak{L}^\uparrow_-$: improper, orthochronous transformations with $\det\Lambda = -1, \Lambda^0{}_0 \geqslant 1$.
- $\mathfrak{L}^\downarrow_-$: improper, non-orthochronous transformations with $\det\Lambda = -1, \Lambda^0{}_0 \leqslant -1$.
- $\mathfrak{L}^\downarrow_+$: proper, non-orthochronous transformations with $\det\Lambda = 1, \Lambda^0{}_0 \leqslant -1$.

$$\mathfrak{L}^\uparrow_+ \xrightarrow{\mathscr{P}} \mathfrak{L}^\uparrow_-, \qquad \mathfrak{L}^\uparrow_+ \xrightarrow{\mathscr{T}} \mathfrak{L}^\downarrow_-, \qquad \mathfrak{L}^\uparrow_+ \xrightarrow{\mathscr{P}\mathscr{T}} \mathfrak{L}^\downarrow_+$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

E&M in Quantum Mechanics

Classical EM

$$\nabla \cdot \mathbf{E} = 0 \quad (\rho)$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}$$
$$\nabla \times \mathbf{B} = \frac{\partial}{\partial t}\mathbf{E} \quad (+\mathbf{j})$$

$$\partial_\mu F^{\mu\nu} = j^\mu$$
$$\varepsilon^{\mu\nu\sigma\eta}\partial_\nu F_{\sigma\eta} = 0,$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\mathbf{E} = -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t}$$
$$\mathbf{B} = \nabla \times \mathbf{A}.$$

Classical EM in relativistic form

$$\nabla \cdot (\mathbf{E} + i\,\mathbf{B}) = \rho_e + (i\,\rho_m)$$
$$\nabla \times (\mathbf{E} + i\,\mathbf{B}) = i\,\frac{\partial}{\partial t}(\mathbf{E} + i\,\mathbf{B}) + \mathbf{j}_e + (i\,\mathbf{j}_m)$$

$$(\mathbf{E} + i\,\mathbf{B})^2 = \mathbf{E}^2 - \mathbf{B}^2 + 2\,i\,\mathbf{E}\cdot\mathbf{B}$$

The two invariants we can make with F

In this form, relativistic invariance is manifest (see later), but also duality if we ignore charges and currents, and also scale invariance

$$(\mathbf{E} + i\,\mathbf{B}) \to e^{i\theta}(\mathbf{E} + i\,\mathbf{B}) \qquad \phi \to \lambda^{-D}\phi(\lambda^{-1}x)$$

With QM, gauge symmetry creeps in

Coupling to QM requires the gauge potentials and a non-trivial transformation of the wave function, this gives subtle consequences to gauge symmetry

$$i\frac{\partial}{\partial t}\Psi = \left[-\frac{1}{2m}(\nabla - ie\mathbf{A})^2 + e\varphi\right]\Psi$$

$$\Psi(t,\mathbf{x}) \longrightarrow e^{-ie\,\varepsilon(t,\mathbf{x})}\Psi(t,\mathbf{x})$$

$$\varphi(t,\mathbf{x}) \to \varphi(t,\mathbf{x}) + \frac{\partial}{\partial t}\varepsilon(t,\mathbf{x}), \qquad \mathbf{A}(t,\mathbf{x}) \to \mathbf{A}(t,\mathbf{x}) + \nabla\varepsilon(t,\mathbf{x}).$$

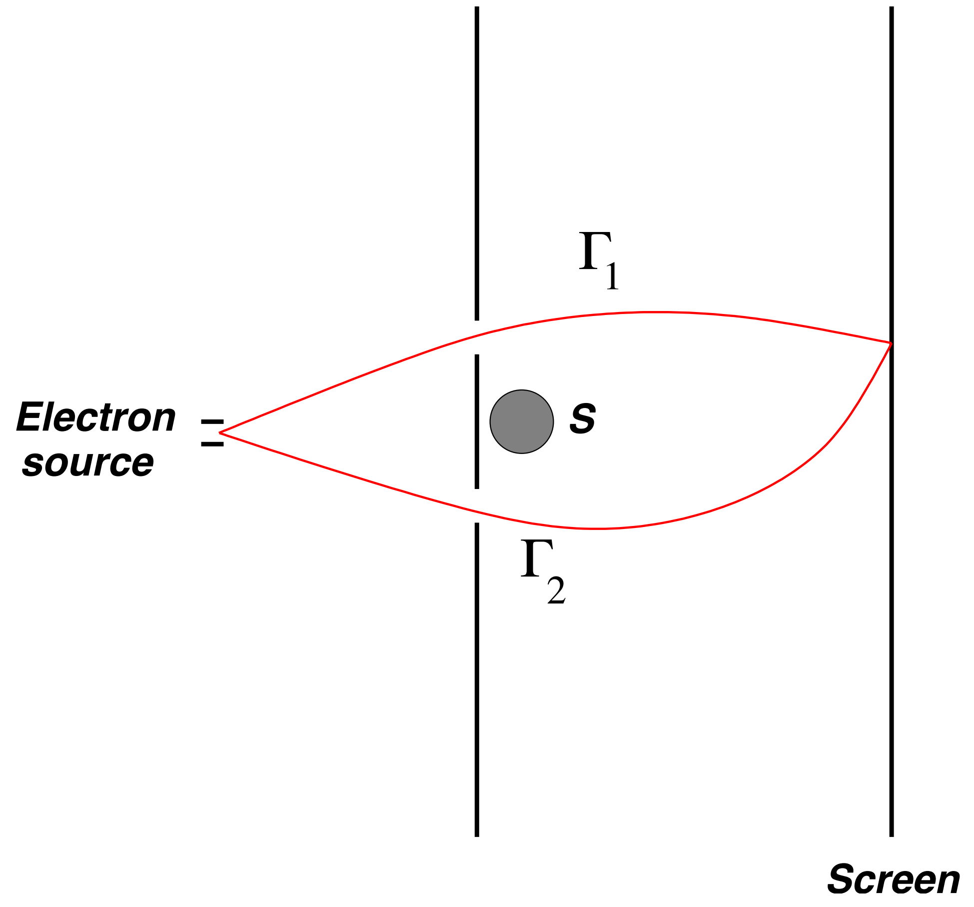
$$A_\mu \longrightarrow A_\mu + \partial_\mu \varepsilon$$

Gauge invariance is the guiding principle in the construction of SM interactions
It is crucial in the consistent quantization of the theory
Global symmetries are accidents of low-dimension Lagrangians…

Advanced Topic… Skip

Non-local observables

$$\Psi = e^{ie\int_{\Gamma_1} \mathbf{A}\cdot d\mathbf{x}}\Psi_1^{(0)} + e^{ie\int_{\Gamma_2} \mathbf{A}\cdot d\mathbf{x}}\Psi_2^{(0)}$$
$$= e^{ie\int_{\Gamma_1} \mathbf{A}\cdot d\mathbf{x}}\left[\Psi_1^{(0)} + e^{ie\oint_{\Gamma} \mathbf{A}\cdot d\mathbf{x}}\Psi_2^{(0)}\right]$$



$$U = \exp\left[ie\oint_{\Gamma} \mathbf{A}\cdot d\mathbf{x}\right]$$

This is the Aharonov-Bohm effect. The phase factor, and its non-abelian generalisation are known as "Wilson loops" or holonomies of the gauge field. Note that classically there would be no effect. The Lorentz force equation only involves E,B hence the electrons would not see the solenoid at all!!

$$m\frac{du^\mu}{d\tau} = eF^{\mu\nu}u_\nu$$

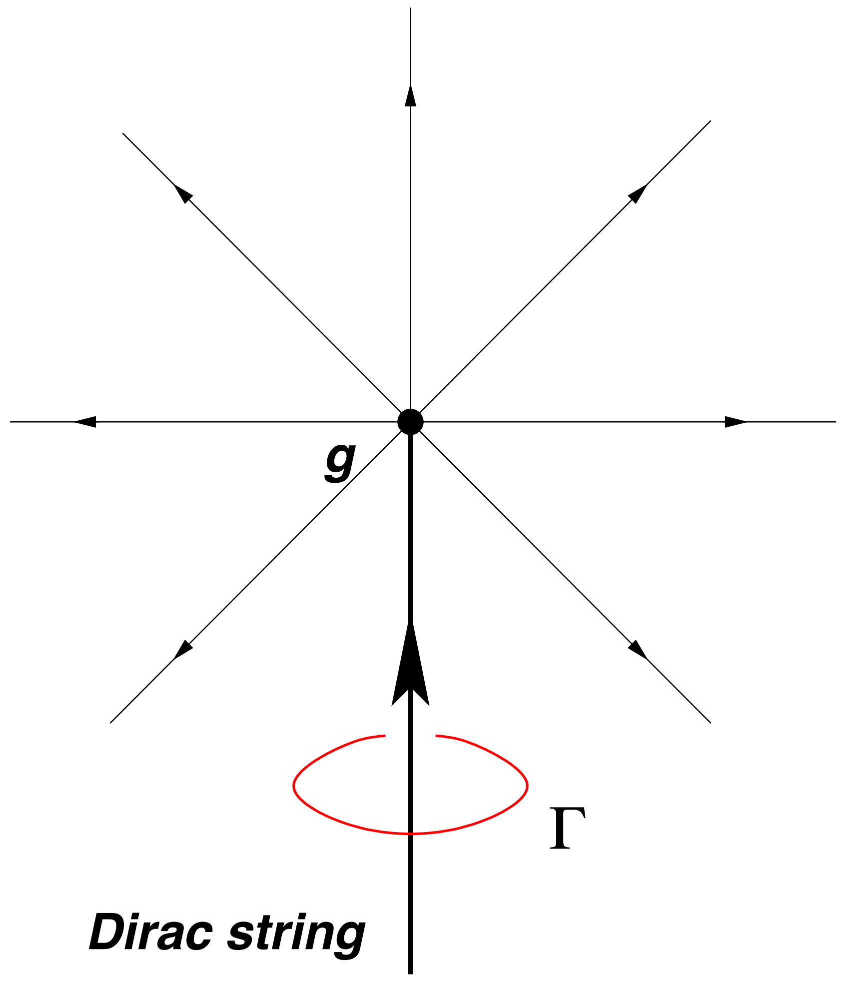
Advanced Topic… Skip

Magnetic monopoles: Dirac and charge quantisation

$$\nabla\cdot\mathbf{E} = 0$$
$$\nabla\cdot\mathbf{B} = 0$$
$$\nabla\times\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}$$
$$\nabla\times\mathbf{B} = \frac{\partial}{\partial t}\mathbf{E}$$

$$\mathbf{E} - i\mathbf{B} \longrightarrow e^{i\theta}(\mathbf{E} - i\mathbf{B})$$

For angle = 90 E and B get exchanged



The symmetry extend to matter if we have magnetic sources:

$$\rho - i\rho_m \longrightarrow e^{i\theta}(\rho - i\rho_m), \qquad \mathbf{j} - i\mathbf{j}_m \longrightarrow e^{i\theta}(\mathbf{j} - i\mathbf{j}_m).$$

Consider a magnetic pole:

$$\nabla\cdot\mathbf{B} = g\,\delta(\mathbf{x}). \qquad B_r = \frac{1}{4\pi}\frac{g}{|\mathbf{x}|^2}, \qquad B_\varphi = B_\theta = 0$$

$$A_\varphi = \frac{1}{4\pi}\frac{g}{|\mathbf{x}|}\tan\frac{\theta}{2}, \qquad A_r = A_\theta = 0.$$

The Dirac string can be changed by gauge transformations, in doing QM it has to be unobservable. Then we can do a “A-B” like argument (Dirac did it 20 years earlier). We should not forget the fact that there is a factor of $\hbar c$

$$e^{ieg} = 1 \qquad eg = 2\pi n$$

$$q_1 g_2 - q_2 g_1 = 2\pi n.$$

Electromagnetic Fields and Photons

The classical theory of radiation needs a profound revision after the photon was discovered, which together with the wave-particle duality, yields the standard quantization of the electromagnetic field

Ignoring sources, the E&M field is a "free field"

$$\mathbf{E} = -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t}$$
$$\mathbf{B} = \nabla \times \mathbf{A}.$$

The electric field is the momentum p and the vector potential the "coordinate" q

$$\mathscr{L}_{\text{Maxwell}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad A_\mu \longrightarrow A_\mu + \partial_\mu \varepsilon$$

$$\partial_\mu F^{\mu\nu} = 0 \qquad 0 = \partial_\mu\partial^\mu A^\nu - \partial_\nu\left(\partial_\mu A^\mu\right) = \partial_\mu\partial^\mu A^\nu$$

To be able to invert, we need to fix the gauge: $\partial_\mu A^\mu = 0.$

$$\varepsilon_\mu(\mathbf{k},\lambda)e^{-i|\mathbf{k}|t + i\mathbf{k}\cdot\mathbf{x}}$$
$$k^\mu \varepsilon_\mu(\mathbf{k},\lambda) = 0$$

As usual, we look for plane wave solutions Residual gauge transformation used to fully fix the gauge

$$\epsilon_\mu(\mathbf{k},\lambda) \to \epsilon_\mu(\mathbf{k},\lambda) + k_\mu\,\chi(\mathbf{k}),\;\; k^2 = 0$$
$$k^2 = k_\mu k^\mu = (k^0)^2 - \mathbf{k}^2 = 0$$

Now, as usual we expand the field in oscillator and apply CCR. After fully fixing the gauge there are only two physical polarisations. Gauge invariance seems more a redundancy rather than a symmetry in the description of the theory

$$[\hat{a}(\mathbf{k},\lambda), \hat{a}^\dagger(\mathbf{k}',\lambda')] = (2\pi)^3(2|\mathbf{k}|)\delta(\mathbf{k}-\mathbf{k}')\delta_{\lambda\lambda'}$$

$$\hat{A}_\mu(t,\mathbf{x}) = \sum_{\lambda=\pm1}\int \frac{d^3k}{(2\pi)^3}\frac{1}{2|\mathbf{k}|}\Big[\varepsilon_\mu(\mathbf{k},\lambda)\hat{a}(\mathbf{k},\lambda)e^{-i|\mathbf{k}|t+i\mathbf{k}\cdot\mathbf{x}} + \varepsilon_\mu(\mathbf{k},\lambda)^*\hat{a}^\dagger(\mathbf{k},\lambda)e^{i|\mathbf{k}|t-i\mathbf{k}\cdot\mathbf{x}}\Big].$$

If we keep all four polarisation by partial gauge fixing, then we get negative probabilities (Gupta-Bleuler, BRST)

$$\delta_{\lambda,\lambda'} \to -\eta_{\lambda,\lambda'}$$

Implementing wave-particle duality

For every matter particle we construct asymptotic states, and the corresponding creation and annihilation operators, an equivalent way of thinking of canonical variables Q,P. The construction automatically generates a unitary representation of ISO(1,3)

$$|p\rangle = (2\pi)^{\frac{3}{2}}\sqrt{2E_{\mathbf{p}}}\,|\mathbf{p}\rangle \qquad \langle p|p'\rangle = (2\pi)^3(2E_{\mathbf{p}})\delta(\mathbf{p}-\mathbf{p}') \qquad \hat{P}^\mu = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}|p\rangle p^\mu \langle p| \qquad \mathscr{U}(\Lambda)|p\rangle = |\Lambda^\mu{}_\nu p^\nu\rangle \equiv |\Lambda p\rangle$$

$$\langle 0|0\rangle = 1$$

$$\alpha(\mathbf{p}) \equiv (2\pi)^{\frac{3}{2}}\sqrt{2E_{\mathbf{p}}}\,a(\mathbf{p}) \qquad [\alpha(\mathbf{p}),\alpha^\dagger(\mathbf{p}')] = (2\pi)^3(2E_{\mathbf{p}})\delta(\mathbf{p}-\mathbf{p}'),$$

$$\alpha^\dagger(\mathbf{p}) \equiv (2\pi)^{\frac{3}{2}}\sqrt{2E_{\mathbf{p}}}\,a^\dagger(\mathbf{p}) \qquad [\alpha(\mathbf{p}),\alpha(\mathbf{p}')] = [\alpha^\dagger(\mathbf{p}),\alpha^\dagger(\mathbf{p}')] = 0. \qquad |f\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}} f(\mathbf{p})\alpha^\dagger(\mathbf{p})|0\rangle$$

Let us construct some observable in this theory. It will be an operator depending on space time, and satisfying some simple conditions:

Hermiticity $\qquad \phi(x)^\dagger = \phi(x).$

Microcausality $\qquad [\phi(x),\phi(y)] = 0, \qquad (x-y)^2 < 0.$

Translational invariance $\qquad e^{i\hat{P}\cdot a}\phi(x)e^{-i\hat{P}\cdot a} = \phi(x-a)$

Lorentz invariance $\qquad \mathscr{U}(\Lambda)^\dagger\phi(x)\mathscr{U}(\Lambda) = \phi(\Lambda^{-1}x).$

Linearity $\qquad \phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left[f(\mathbf{p},x)\alpha(\mathbf{p}) + g(\mathbf{p},x)\alpha^\dagger(\mathbf{p})\right].$

This is the standard quantization of the Klein-Gordon field, but it can be generalized to other fields with charge, spinor indices, etc. The procedure is basically the same (see appendix)

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left[e^{-iE_{\mathbf{p}}t+i\mathbf{p}\cdot\mathbf{x}}\alpha(\mathbf{p}) + e^{iE_{\mathbf{p}}t-i\mathbf{p}\cdot\mathbf{x}}\alpha^\dagger(\mathbf{p})\right]$$

+ve energy   -ve energy

What is the aether? The answer could not be obtained in the 19th century…

$$G(x,y) = \langle 0|T A(x) A(y)|0\rangle$$

Coupling matter

We imitate the coupling in the Schrödinger equation, this is what used to be called minimal coupling. We make derivatives covariant with respect to space-time dependent changes of phases in the wave-function

$$i\frac{\partial}{\partial t}\Psi = \left[-\frac{1}{2m}(\nabla - ie\mathbf{A})^2 + e\varphi\right]\Psi \qquad D_\mu\left[e^{ie\varepsilon(x)}\psi\right] = e^{ie\varepsilon(x)}D_\mu\psi.$$

$$\Psi(t,\mathbf{x}) \longrightarrow e^{-ie\varepsilon(t,\mathbf{x})}\Psi(t,\mathbf{x}) \qquad D_\mu = \partial_\mu - ieA_\mu.$$

$$A_\mu \longrightarrow A_\mu + \partial_\mu\varepsilon$$

The rigid phase rotation invariance of the Dirac Lagrangian for electrons is transformed into local phase rotations, a physically more satisfactory concept. This defines the coupling of the electron to the E&M field:

$$\mathscr{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\not{D} - m)\psi, \qquad \mathscr{L}_{\text{QED}}^{(\text{int})} = -eA_\mu\overline{\psi}\gamma^\mu\psi.$$

$$\psi \longrightarrow e^{ie\varepsilon(x)}\psi, \qquad A_\mu \longrightarrow A_\mu + \partial_\mu\varepsilon(x).$$

This is QED, the best tested theory in the history of science, an example is the gyromagnetic ratio of the electron,

$$g\frac{e}{8m}[\gamma^\mu, \gamma^\nu]\, F_{\mu\nu}$$

$$g/2 = 1.00115965218085(76)$$
$$\alpha^{-1} = 137.035999070(98)$$

$$\vec{\mu} = g_\mu \frac{e\hbar}{2m_\mu c}\vec{s}, \qquad \underbrace{g_\mu = 2}_{\text{Dirac}}(1 + a_\mu)$$

Lorentz and Poincaré Groups

$$R(\mathbf{e},\varphi) = e^{-i\varphi\,\mathbf{e}\cdot\mathbf{J}}$$

$$B(\mathbf{u},\lambda) = e^{-i\lambda\,\mathbf{u}\cdot\mathbf{M}}$$

Rotations and boosts generate Lorentz transformation, hence six parameter and six generators of infinitesimal transformations. The algebra is easy to obtain and "diagonalise"

$$[J_i, J_j] = i\varepsilon_{ijk}J_k,$$
$$[J_i, M_k] = i\varepsilon_{ijk}M_k,$$
$$[M_i, M_j] = -i\varepsilon_{ijk}J_k$$

$$J_k^{\pm} = \frac{1}{2}(J_k \pm iM_k) \qquad [J_i^{\pm}, J_j^{\pm}] = i\varepsilon_{ijk}J_k^{\pm}, \quad [J_i^{+}, J_j^{-}] = 0. \qquad (\mathbf{s}_+, \mathbf{s}_-)$$

$$\mathbf{J} = \mathbf{J}^+ + \mathbf{J}^-$$

$$(\mathbf{s}_+, \mathbf{s}_-) = \sum_{\mathbf{j}=|\mathbf{s}_+-\mathbf{s}_-|}^{\mathbf{s}_++\mathbf{s}_-} \mathbf{j}$$

The representations of each SU(2) are labelled by a single integer or half integer "angular" momentum s=0, 1/2, 1, 3/2, … Under parity

$$\begin{array}{ccc} \mathbf{J} & \stackrel{P}{\rightarrow} & \mathbf{J} \\ \mathbf{M} & \rightarrow & -\mathbf{M} \\ \mathbf{J}^{\pm} & \rightarrow & \mathbf{J}^{\mp} \\ (\mathbf{s_1}, \mathbf{s_2}) & \rightarrow & (\mathbf{s_2}, \mathbf{s_1}) \end{array}$$

Representation	Type of field
$(\mathbf{0},\mathbf{0})$	Scalar
$(\frac{\mathbf{1}}{\mathbf{2}},\mathbf{0})$	Right-handed spinor
$(\mathbf{0},\frac{\mathbf{1}}{\mathbf{2}})$	Left-handed spinor
$(\frac{\mathbf{1}}{\mathbf{2}},\frac{\mathbf{1}}{\mathbf{2}})$	Vector
$(\mathbf{1},\mathbf{0})$	Selfdual antisymmetric 2-tensor
$(\mathbf{0},\mathbf{1})$	Anti-selfdual antisymmetric 2-tensor

← $\mathbf{E} \pm i\,\mathbf{B}$

Weyl spinors

The simplest representations have fundamental physical importance, they are called Weyl spinors. Clearly they are representations of the connected component of SO(3,1), but not of parity, since parity interchanges the representations

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$J_i^+ = \frac{1}{2}\sigma_i, \qquad J_i^- = 0 \qquad \text{for} \qquad (\tfrac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}),$$
$$J_i^+ = 0, \qquad J_i^- = \frac{1}{2}\sigma_i \qquad \text{for} \qquad (\mathbf{0}, \tfrac{\mathbf{1}}{\mathbf{2}}).$$

$$u_\pm \longrightarrow e^{-\frac{i}{2}(\theta\mathbf{n}\mp i\beta)\cdot\sigma} u_\pm \qquad u_\pm \longrightarrow e^{i\theta} u_\pm$$

Consider for simplicity this global symmetry: fermion number

$$\sigma_\pm^\mu = (\mathbf{1}, \pm\sigma_i) \qquad u_+^\dagger \sigma_+^\mu u_+ \qquad u_-^\dagger \sigma_-^\mu u_-$$

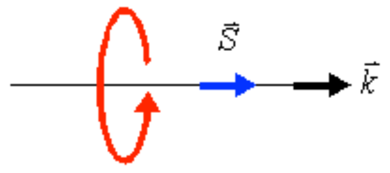
$$\mathscr{L}_{\text{Weyl}}^\pm = i u_\pm^\dagger (\partial_t \pm \sigma\cdot\nabla) u_\pm = i u_\pm^\dagger \sigma_\pm^\mu \partial_\mu u_\pm$$

$$(\partial_0 \pm \sigma\cdot\nabla) u_\pm = 0$$

$$u_\pm(x) = u_\pm(k) e^{-ik\cdot x}$$
$$k^2 = k_0^2 - \mathbf{k}^2 = 0$$

$$(|\mathbf{k}| \mp \mathbf{k}\cdot\sigma) u_\pm = 0$$

$$u_+ : \qquad \frac{\sigma\cdot\mathbf{k}}{|\mathbf{k}|} = 1,$$



positive helicity, right handed antineutrinos

$$u_- : \qquad \frac{\sigma\cdot\mathbf{k}}{|\mathbf{k}|} = -1$$

negative helicity, left handed, neutrinos

Charge conjugation and Majorana masses

We know that under parity, the L,R Weyl spinors are exchanged. Another way to exchange them is via complex conjugation, later to be related to charge conjugation

$$M_L = e^{-\frac{i}{2}\theta\cdot\sigma - \frac{1}{2}\beta\cdot\sigma} \qquad \det M_L = 1 \qquad \det M = \epsilon_{ab} M_{a1} M_{b2}$$
$$M_R = e^{-\frac{i}{2}\theta\cdot\sigma + \frac{1}{2}\beta\cdot\sigma} \qquad \det M_R = 1 \qquad \det M \epsilon_{ab} = \epsilon_{cd} M_{ca} M_{db}$$

$$\epsilon = i\,\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Using $\sigma^* = -\sigma_2\,\sigma\,\sigma_2$

$$\psi_L^c = \sigma_2\,\psi_L^* \quad \text{transforms like} \quad \psi_R$$
$$\psi_R^c = \sigma_2\,\psi_R^* \quad \text{transforms like} \quad \psi_L$$

$$\mathscr{L}^{\pm}_{\text{Weyl}} = iu_{\pm}^{\dagger}\sigma_{\pm}^{\mu}\partial_{\mu}u_{\pm} + \frac{m}{2}\left(\varepsilon_{ab}u_{\pm}^{a}u_{\pm}^{b} + \text{h.c.}\right) \qquad \epsilon_{ab}u^a\,u^b = u^1\,u^2 - u^2\,u^1$$

We can express any theory fully in terms of L or R fermions.

Charge conjugation and parity exchange L and R

A parity invariant theory requires L,R spinors at the same time

We can construct a mass for pure L spinors if we ignore fermion number

Fermions anticommuting

Most general Majorana mass, Takagi factorisation

$$\frac{1}{2}\left(M_{IJ}\,\epsilon_{ab}\,u^{a,I}\,u^{b,J} + \text{h.c.}\right),$$
$$I, J = 1, \ldots N_F, \quad M_{IJ} = M_{JI} \text{ complex}$$

$$M = U \begin{pmatrix} m_1 & \ldots & 0 \\ 0 & \ddots & 0 \\ 0 & \ldots & m_{N_F} \end{pmatrix} U^T$$

m_i are positive square roots of $MM^\dagger$

This is the most general fermion mass matrix!!! It includes CKM, in fact it is more general

Weyl + parity $\longrightarrow$ Dirac

$$(\tfrac{1}{2},\mathbf{0})\oplus(\mathbf{0},\tfrac{1}{2})$$

$$P: u_\pm \longrightarrow u_\mp$$

$$\psi = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}$$

$$\left.\begin{array}{l} i\sigma_+^\mu \partial_\mu u_+ = m u_- \\ i\sigma_-^\mu \partial_\mu u_- = m u_+ \end{array}\right\} \Longrightarrow \quad i\begin{pmatrix} \sigma_+^\mu & 0 \\ 0 & \sigma_-^\mu \end{pmatrix}\partial_\mu \psi = m\begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}\psi$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma_-^\mu \\ \sigma_+^\mu & 0 \end{pmatrix} \qquad \overline{\psi} \equiv \psi^\dagger\gamma^0 = \psi^\dagger \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \qquad \mathcal{L}_{\text{Dirac}} = \overline{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi$$

DIRACOLOGY

$$\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \qquad P_\pm = \tfrac{1}{2}(1\pm\gamma_5)$$

$$P_+\psi = \begin{pmatrix} u_+ \\ 0 \end{pmatrix} \qquad P_-\psi = \begin{pmatrix} 0 \\ u_- \end{pmatrix}$$

$$\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}$$

$$\text{Tr}\,\gamma^\mu\gamma^\nu = 4\,\eta^{\mu\nu}$$

$$\text{Tr}\,\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta = 4\,\eta^{\mu\nu}\eta^{\alpha\beta} - 4\,\eta^{\mu\alpha}\eta^{\beta\nu} + 4\,\eta^{\mu\beta}\eta^{\alpha\nu}$$

$$\text{Tr}\,\gamma_5\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu = 4\,i\,\epsilon^{\alpha\beta\mu\nu}$$

We look for +ve and -ve energy solutions as usual

$$u(k,s)e^{-ik\cdot x}$$

$$(\not{k}-m)u(k,s) = 0$$

$$v(k,s)e^{ik\cdot x}$$

$$(\not{k}+m)v(k,s) = 0$$

$$k^2 = m^2$$

$$\bar{u}(\mathbf{k},s)u(\mathbf{k},s) = 2m, \qquad \bar{v}(\mathbf{k},s)v(\mathbf{k},s) = -2m,$$

$$\bar{u}(\mathbf{k},s)\gamma^\mu u(\mathbf{k},s) = 2k^\mu, \qquad \bar{v}(\mathbf{k},s)\gamma^\mu v(\mathbf{k},s) = 2k^\mu,$$

$$\sum_{s=\pm\frac{1}{2}} u_\alpha(\mathbf{k},s)\bar{u}_\beta(\mathbf{k},s) = (\not{k}+m)_{\alpha\beta} \qquad \sum_{s=\pm\frac{1}{2}} v_\alpha(\mathbf{k},s)\bar{v}_\beta(\mathbf{k},s) = (\not{k}-m)_{\alpha\beta}$$

Group Theory reminder

For the SM all group we will need are:

$$G: \quad U(1),\, SU(2),\, SU(3) \qquad [T^a, T^b] = i f^{abc} T^c \qquad G_{SM} = SU(3)\times SU(2)\times U(1)$$

$$g \in G \qquad g = e^{i\epsilon^a T^a} \qquad \mathrm{tr}(T^a T^b) = T_2(R)\,\delta^{ab}$$

$$\det g = 1 \Rightarrow \mathrm{tr}\, T^a = 0$$ (for SU(2), SU(3) not for U(1) of course)

U(1) is of course the simplest, just phase multiplication, i.e. as in QED

SU(2): angular momentum, isospin, and also weak isospin

$$[T^a, T^b] = i\varepsilon^{abc} T^c. \qquad T^{\pm} = \frac{1}{\sqrt{2}}(T^1 \pm iT^2), \qquad T^3$$

$$[T^3, T^{\pm}] = \pm T^{\pm}$$

$$[T^+, T^-] = T^3$$

$T^a = \frac{1}{2}\sigma^a$ For spin ½ $\mathrm{tr}\frac{\sigma^a}{2}\frac{\sigma^a}{2} = \frac{1}{2}\delta^{ab} \qquad a, b = 1, 2, 3$

$$J^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad J^2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad J^3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

For spin 1

For SU(3) the generators are the eight Gell-Mann 3x3 traceless hermitean matrices chosen to satisfy:

$$\mathrm{tr}\frac{\lambda^a}{2}\frac{\lambda^a}{2} = \frac{1}{2}\delta^{ab};\ a, b = 1, \ldots, 8$$

SU(3) of color, an exact gauge symmetry, also flavor SU(3), which is global (see later)

There are very few representations we will need for color SU(3):

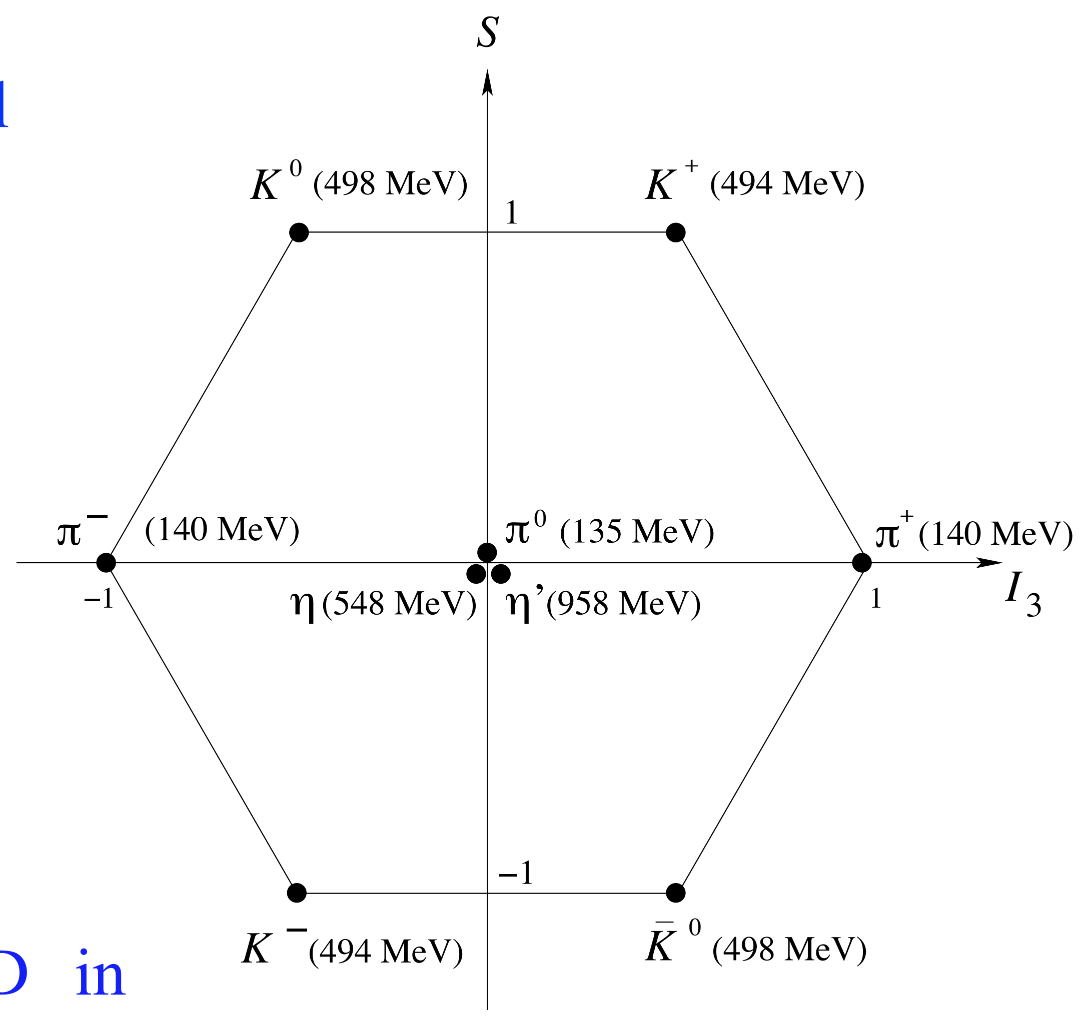
$$\mathbf{3},\ \bar{\mathbf{3}},\ \mathbf{8}$$

quarks antiquarks gluons

For flavor SU(3) more needed: mesons, baryons

$$\mathbf{3},\ \bar{\mathbf{3}},\ \mathbf{8},\ \mathbf{10},\ \overline{\mathbf{10}},\ \mathbf{27}\dots$$

A remarkable fact about the SM and QCD in particular is the fact that once we write the most general Lagrangian compatible with color gauge symmetry, flavor appears as an approximate global symmetry of the problem, although it was theorized earlier.



pseudo-scalar meson octet

$$Q = I_3 + \frac{B+S}{2},$$

$$|\Delta^{++}; s_z = \tfrac{3}{2}\rangle = |uuu\rangle \otimes |\uparrow\uparrow\uparrow\rangle \equiv |u\uparrow, u\uparrow, u\uparrow\rangle.$$

$$|uud\rangle_S = \frac{1}{\sqrt{6}}\Big(|uud\rangle + |udu\rangle - 2|duu\rangle\Big),$$

$$|uud\rangle_A = \frac{1}{\sqrt{2}}\Big(|uud\rangle - |udu\rangle\Big).$$

$$|\Uparrow\rangle_S = \frac{1}{\sqrt{6}}\Big(|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle - 2|\uparrow\uparrow\downarrow\rangle\Big).$$

$$|\Uparrow\rangle_A = \frac{1}{\sqrt{2}}\Big(|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle\Big).$$

$$|p\uparrow\rangle = \frac{1}{\sqrt{2}}\Big(|uud\rangle_S \otimes |\Uparrow\rangle_A + |uud\rangle_A \otimes |\Uparrow\rangle_S\Big).$$

$$|p\downarrow\rangle = \frac{1}{\sqrt{2}}\Big(|uud\rangle_S \otimes |\Downarrow\rangle_A + |uud\rangle_A \otimes |\Downarrow\rangle_S\Big).$$

Range of applications of QFT

QFT expresses physics compatible with special relativity and QM

Hence, it is the natural arena to formulate the basic laws of Nature

Non-relativistic theories: Many body Physics, e.g. Quantum Liquids

Interaction of radiation with matter (QED)

Precise description of the strong interactions (QCD)

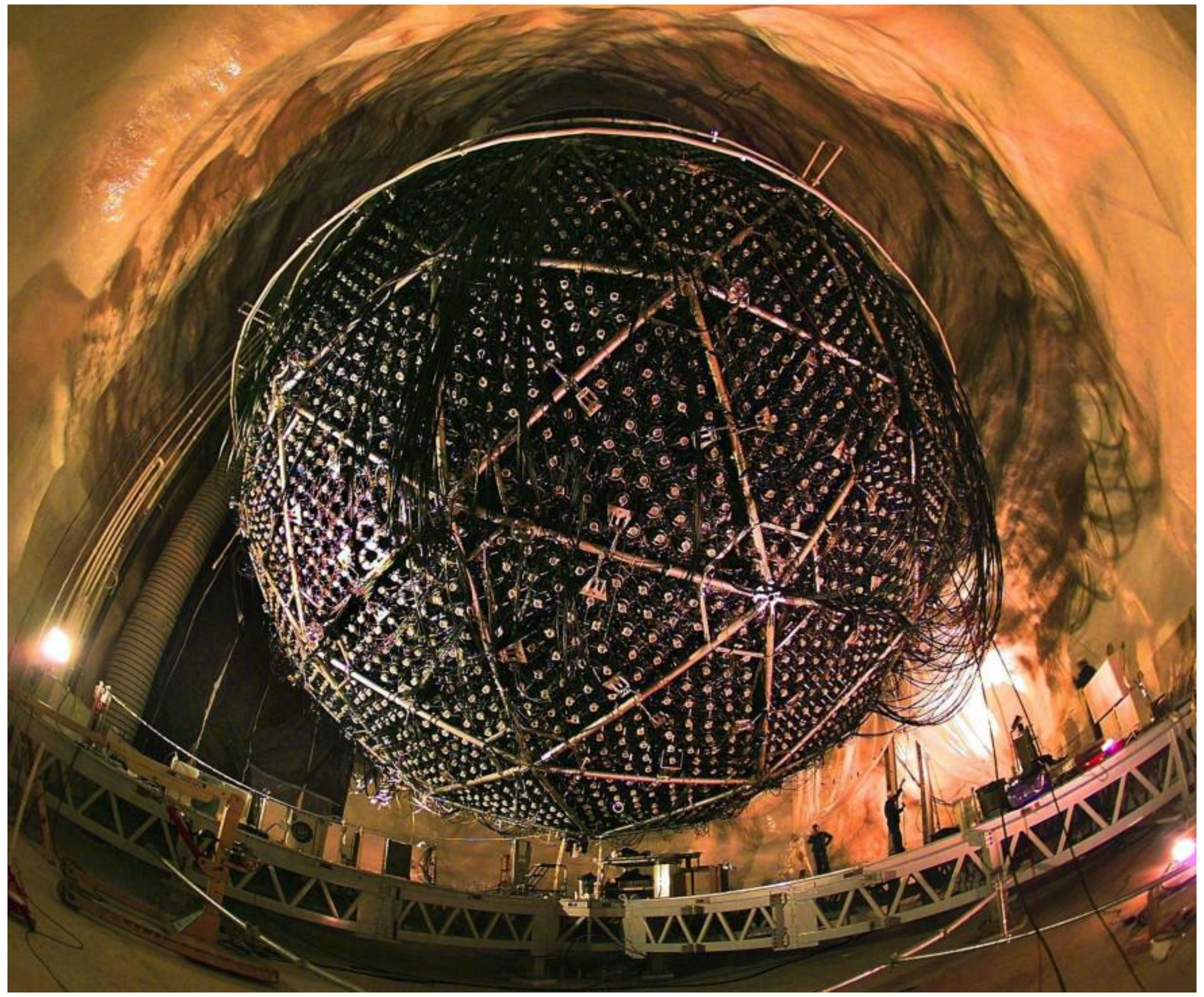
Successful EW unification leading to the GWS formulation of the SM

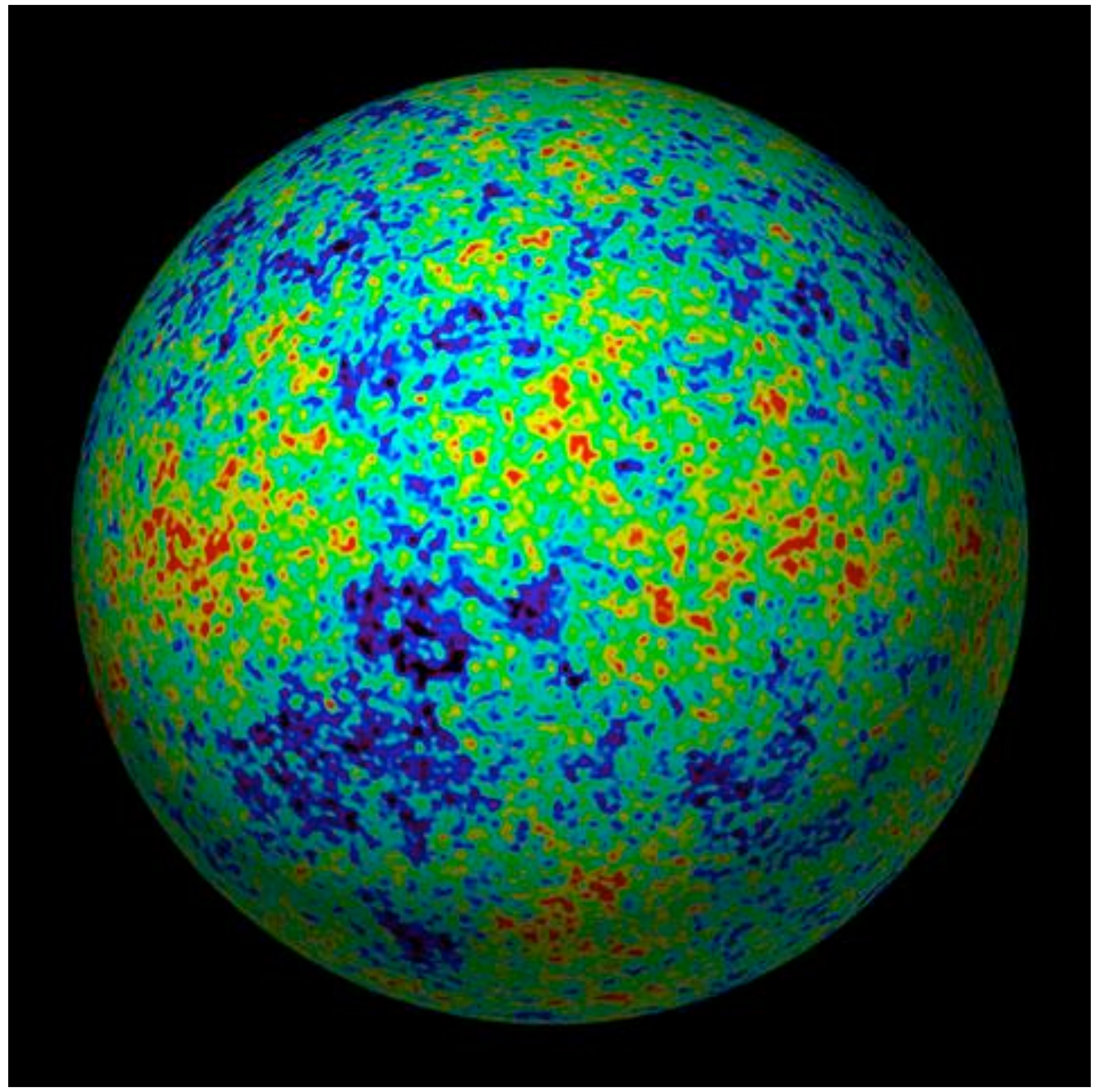
Routinely tested at all known accelerators, present and past

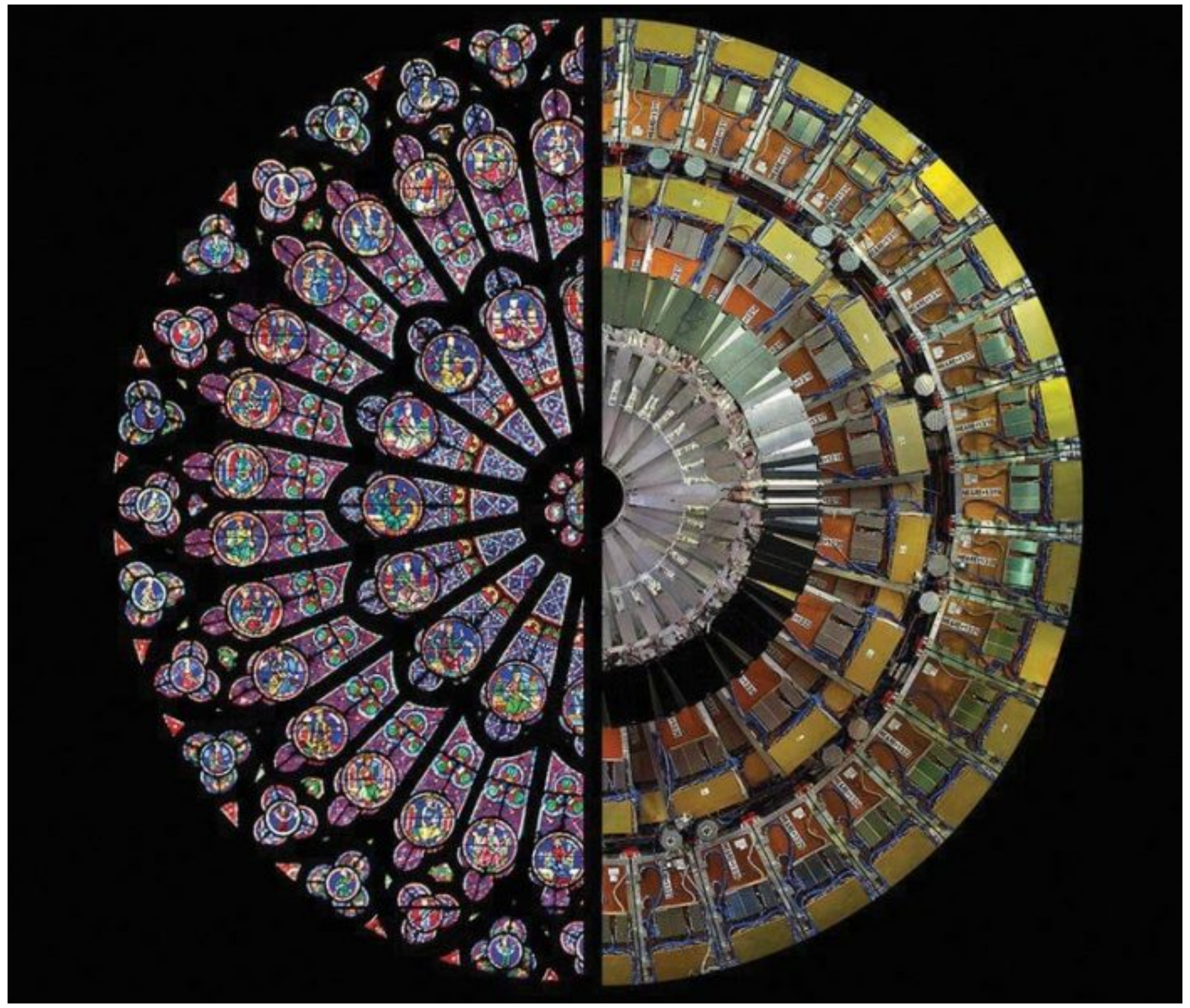
Spectacular application to the Large Scale Structure of the Universe

Basic description of symmetries, unbroken and broken...

Range of applications







Picture courtesy of Phil Palmedo

Some general remarks

The 19th century gave us group theory and electromagnetism, but symmetry arguments rarely entered the discussions

After 100 years, the study of symmetries plays a central role in the discussion of physical theories. We will present the SM by pursuing different aspects of symmetry related to it

Kinematic symmetries: Lorentz, Poincaré, scale and conformal (we have covered them already)

Local (gauge) or global. Our current view is that global symmetries are accidental properties of low energy Lagrangians, in the UV all symmetries should be local

It seems that all fundamental interactions are mediated by suitable generalizations of the EM field. They are gauge (gravity) theories. There is something unnatural and rigid about fundamental global symmetries. We do not really know why this is so (string theory?), because we do not have a full UV completion of the SM including gravity

We have learned to live with EFTs, the old goal of fully renormalizable theories is too naive. EFTs are as good, if used properly, and far more flexible

Explicit, spontaneous or softly broken

Anomalous symmetries: those that do not survive quantization, but play a fundamental role in determining consistency conditions, and in the end provide the basis for Renormalization and the Renormalization Group.

Discrete symmetries: C, P, T, CP, CPT… We start our exploration of symmetries…

Noether's Theorem

Quantum mechanical realization of Symmetries (Wigner's theorem). In QM theory physical symmetries are maps among states that preserve probability amplitudes (their modulus). They can be unitary or anti-unitary

$$|\alpha\rangle \longrightarrow |\alpha'\rangle, \qquad |\beta\rangle \longrightarrow |\beta'\rangle$$

$$|\langle\alpha|\beta\rangle| = |\langle\alpha'|\beta'\rangle|.$$

$$\langle \mathscr{U}\alpha|\mathscr{U}\beta\rangle = \langle\alpha|\beta\rangle$$ unitary

$$\langle \mathscr{U}\alpha|\mathscr{U}\beta\rangle = \langle\alpha|\beta\rangle^*$$ anti-unitary T-reversal, CPT

For continuous symmetries we have Noether's celebrated theorem: If under infinitesimal transformations, AND WITHOUT USING THE EQUATIONS OF MOTION you can show that:

$$\delta_\varepsilon \mathscr{L} = \partial_\mu K^\mu \qquad S[\phi] = \int d^4x\, \mathcal{L}(\phi, \partial_\mu\phi)$$

then there is a conserved current in the theory

In formulas:

$$\begin{aligned}\delta_\varepsilon \mathscr{L} &= \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \partial_\mu \delta_\varepsilon \phi + \frac{\partial \mathscr{L}}{\partial \phi} \delta_\varepsilon \phi \\ &= \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \delta_\varepsilon \phi \right) + \left[\frac{\partial \mathscr{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \right) \right] \delta_\varepsilon \phi \\ &= \partial_\mu K^\mu .\end{aligned}$$

$$\partial_\mu J^\mu = 0$$

$$\boxed{J^\mu = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \delta_\varepsilon \phi - K^\mu}$$

With a conserved charge that generates the symmetry:

$$Q \equiv \int d^3x J^0(t, \mathbf{x}) \qquad \frac{dQ}{dt} = \int d^3x\, \partial_0 J^0(t, \mathbf{x}) = -\int d^3x\, \partial_i J^i(t, \mathbf{x}) = 0,$$

$$\delta \phi = i[\phi, Q].$$

Space-time translations -- Energy-Momentum
Lorentz transformation-- Angular momentum and CM motion
Phase rotation -- abelian and non-abelian charges

Useful examples

Massive Dirac fermions:

$$\mathscr{L} = i\overline{\psi}_j \partial\!\!\!/ \psi_j - m\overline{\psi}_j \psi_j$$

$\psi_i \longrightarrow U_{ij}\psi_j$ $U \in U(N)$ N the number of fermions

$U = \exp(i\alpha^a T^a)$, $(T^a)^\dagger = T^a$

$$j^{\mu a} = \overline{\psi}_i T^a_{ij} \gamma^\mu \psi_j \qquad \partial_\mu j^\mu = 0 \qquad Q^a = \int d^3x \, \psi_i^\dagger T^a_{ij} \psi_j$$

$$[Q^a, H] = 0. \qquad \mathscr{U}(\alpha) = e^{i\alpha^a Q^a}.$$

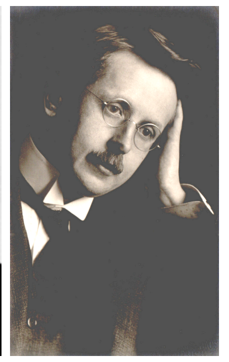
When U is the identity, we have fermion number, or charge

In the m=0 limit we have more symmetry: CHIRAL SYMMETRY, rotate L,R fermions independently. If the mass parameters are small, we can always consider approximate chiral symmetry, as it is the case in QCD

$$\mathcal{L} = i\bar{\psi}_{jL} \partial\!\!\!/ \psi_{Lj} + i\bar{\psi}_{jR} \partial\!\!\!/ \psi_{Rj}$$

$$\psi_{L,R} \to U_{L,R} \psi_{L,R} \qquad U(N)_L \times U(N)_R$$

Wigner-Weyl mode



Imagine we have a symmetry that is a symmetry of the ground state

$$[Q^a, H] = 0. \qquad \mathscr{U}(\alpha)|0\rangle = |0\rangle \qquad Q^a|0\rangle \;=\; 0$$

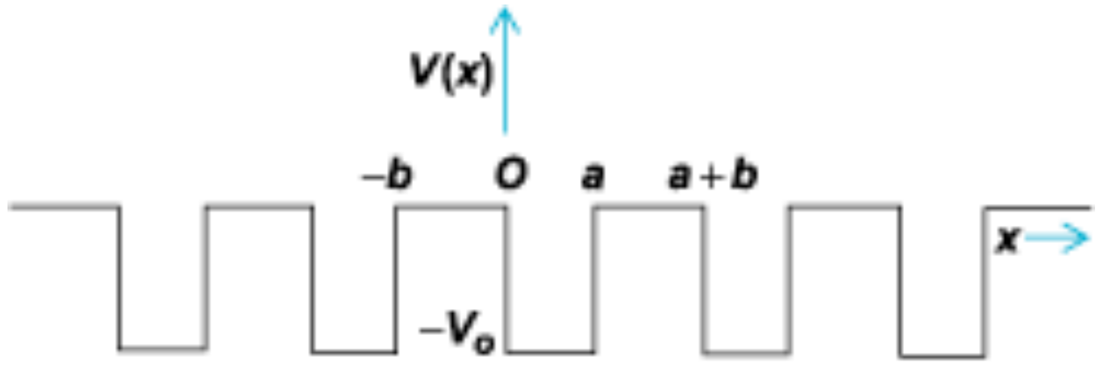
Then the states of the theory fall into multiplets of the symmetry group

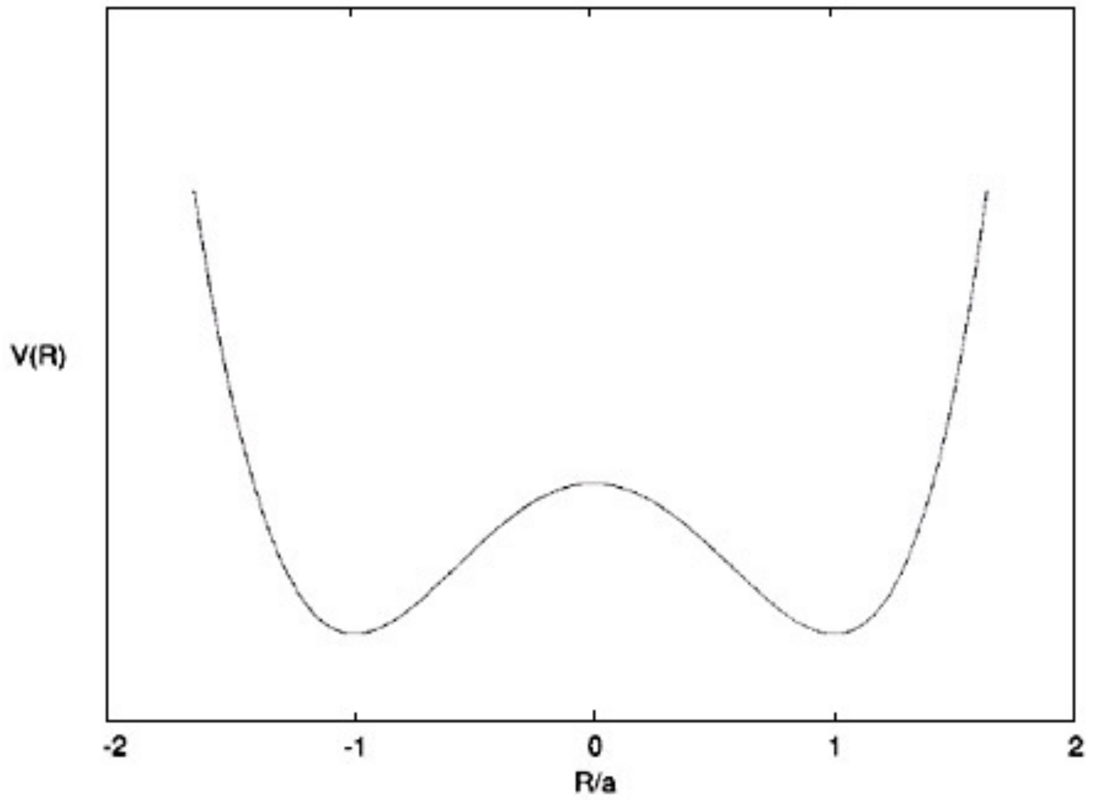
$$\mathscr{U}(\alpha)\phi_i\mathscr{U}(\alpha)^{-1} = U_{ij}(\alpha)\phi_j$$

$$|i\rangle = \phi_i|0\rangle$$

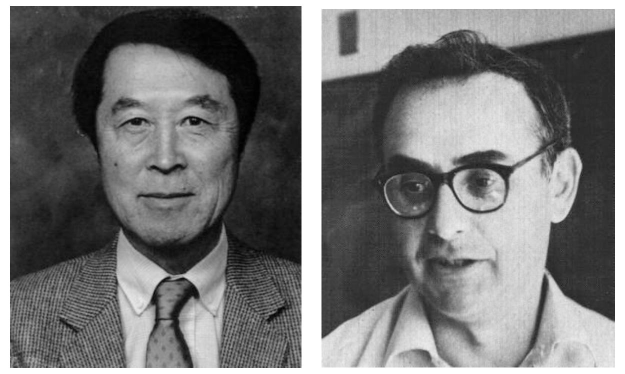
$$\mathscr{U}(\alpha)|i\rangle = \mathscr{U}(\alpha)\phi_i\mathscr{U}(\alpha)^{-1}\mathscr{U}(\alpha)|0\rangle = U_{ij}(\alpha)\phi_j|0\rangle = U_{ij}(\alpha)|j\rangle$$

The spectrum of the theory is classified in terms of multiplets of the symmetry group. This is the case of the Hydrogen atom. The Hamiltonian is rotational invariant, the ground state is an s-wave state, hence all excited states fall into degenerate representations of the rotation group: 1s, 2s, 2p, 2s, 3p, 3d,… In QM (finite number of d.o.f.) this is (almost) always the case (tunnelling, band theory in solids)





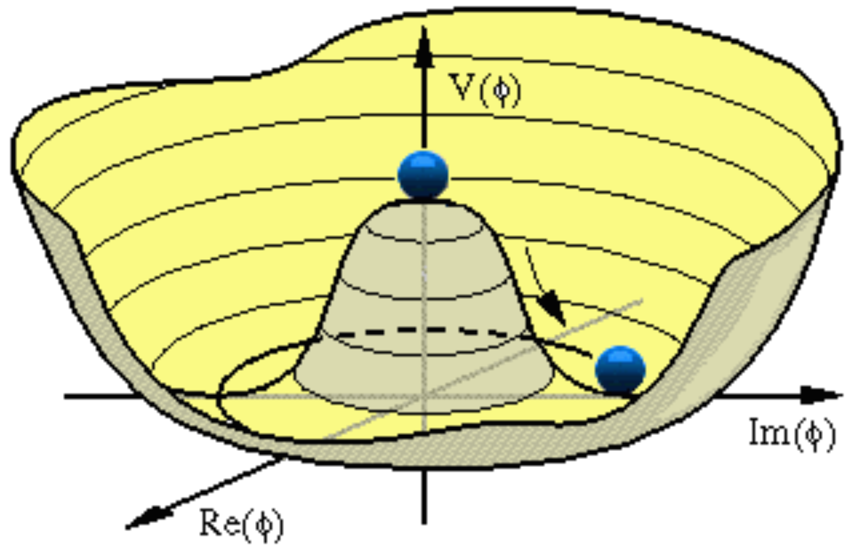
Nambu-Goldstone mode



Sometimes also called hidden symmetry. The symmetry is spontaneously broken by the vacuum

$$[Q^a, H] = 0. \quad Q^a|0\rangle \neq 0.$$

Consider a collection of N scalar fields with a global symmetry group G



Famous Mexican hat potential

$$\mathscr{L} = \frac{1}{2}\partial_\mu \varphi^i \partial^\mu \varphi^i - V(\varphi^i) \qquad \delta\varphi^i = \varepsilon^a (T^a)^i_j \varphi^j$$

$$H[\pi^i, \varphi^i] = \int d^3x \left[\frac{1}{2}\pi^i\pi^i + \frac{1}{2}\nabla\varphi^i \cdot \nabla\varphi^i + V(\varphi^i)\right]$$

$$\mathscr{V}[\varphi^i] = \int d^3x \left[\frac{1}{2}\nabla\varphi^i \cdot \nabla\varphi^i + V(\varphi^i)\right]$$

The minima satisfy

$$\langle\varphi^i\rangle \qquad V(\langle\varphi^i\rangle) = 0, \quad \nabla\varphi = \mathbf{0} \qquad \left.\frac{\partial V}{\partial\varphi^i}\right|_{\varphi^i = \langle\varphi^i\rangle} = 0$$

$$T^a = \{H^\alpha, K^A\} \qquad (H^\alpha)^i_j\langle\varphi^j\rangle = 0 \qquad (K^A)^i_j\langle\varphi^j\rangle \neq 0.$$

unbroken broken

The masses are given by the second derivatives of the potential (assuming canonical normalisation)

$$M_{ij}^2 \equiv \left.\frac{\partial^2 V}{\partial\varphi^i \partial\varphi^j}\right|_{\varphi=\langle\varphi\rangle}$$

Invariance

$$\delta V(\varphi) = \varepsilon^a \frac{\partial V}{\partial \varphi^i}(T^a)^i_j \varphi^j = 0 \qquad \frac{\partial^2 V}{\partial\varphi^i\partial\varphi^k}(T^a)^i_j\varphi^j + \frac{\partial V}{\partial\varphi^i}(T^a)^i_k = 0$$

$$M_{ik}^2 (T^a)^i_j \langle\varphi^j\rangle = 0. \qquad M_{ik}^2 (K^A)^i_j \langle\varphi^j\rangle = 0$$

For every broken generator there is a massless scalar field

The argument works at the full quantum level

The fields acquiring a VEV need not be elementary

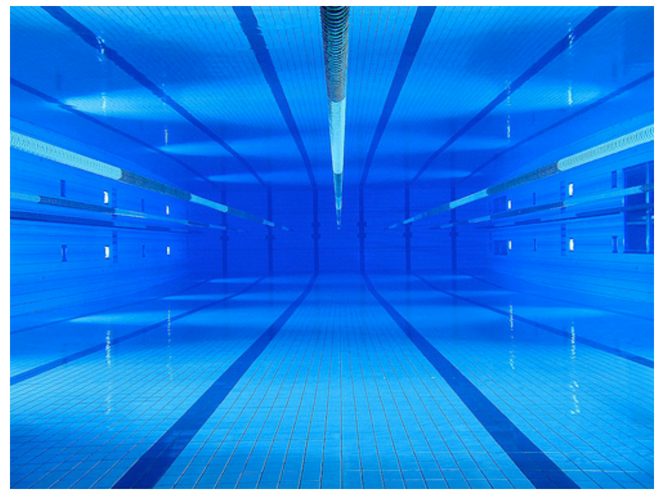
Simplest example:

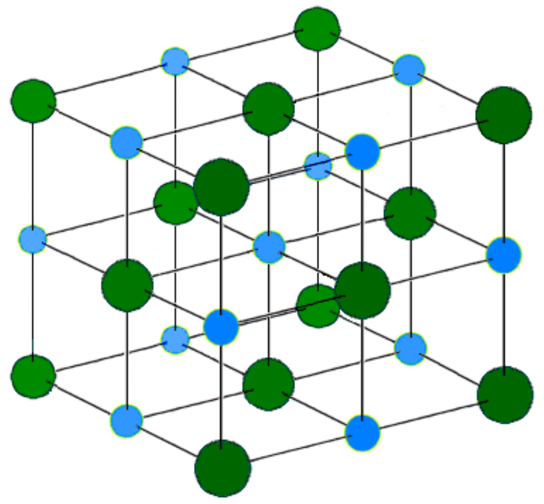
$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi$$

$$\phi \to \phi + c$$

Its own NG-boson

Phonons are NG bosons





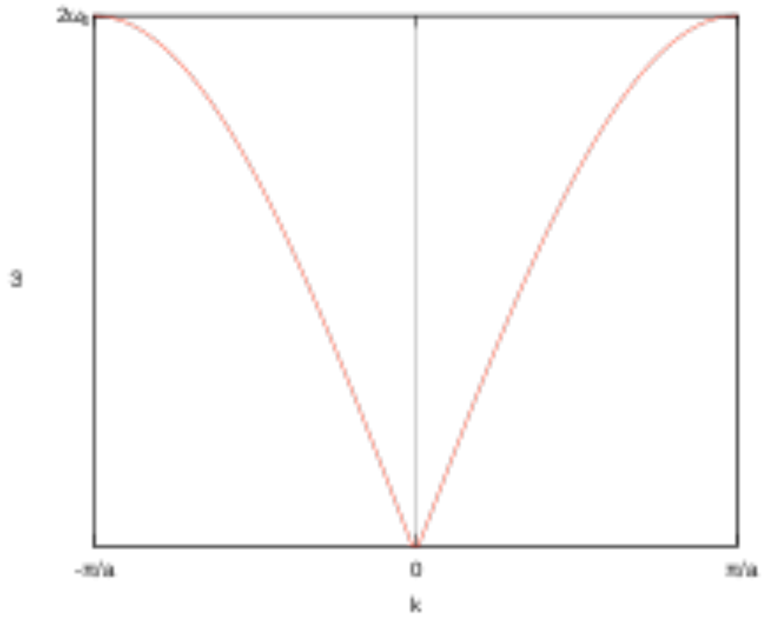
A liquid is translationally invariant

The crystal after solidification has discrete translational symmetry

The low energy excitation of the lattice contain acoustic phonons

Their dispersion relation is as for NG bosons

They propagate at the speed of sound



$$\omega(k) = 2\omega|\sin(ka/2)|$$

Order parameters

The notion of symmetry breaking is intimately connected with the theory of phase transitions in CMP

It is quite frequent that in going from one phase to another the symmetry of the ground state (vacuum) changes

In real physical systems this is what we see with magnetic domains in magnetic materials below the Curie point

In going from one phase to the other, some parameters change in a noticeable way. These are the order parameters.

In liquid-solid transition it is the density

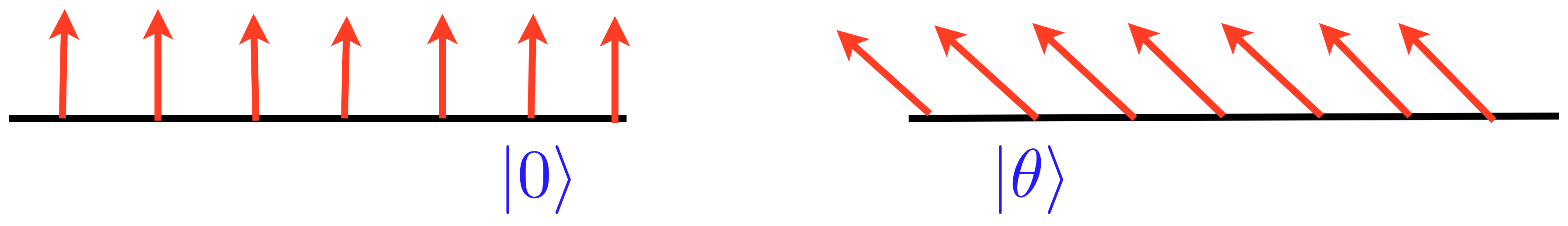
In magnetic materials it is the magnetization

In the Ginsburg-Landau theory of superconductivity, the Cooper pairs acquire a VEV. This breaks U(1) inside the superconductor and thus explains among other things the Meissner effect. The Cooper pairs are pairs of electrons bound by the lattice vibrations. In ordinary superconductors their size is several hundred Angstroms.

The order parameters need not be elementary fields...

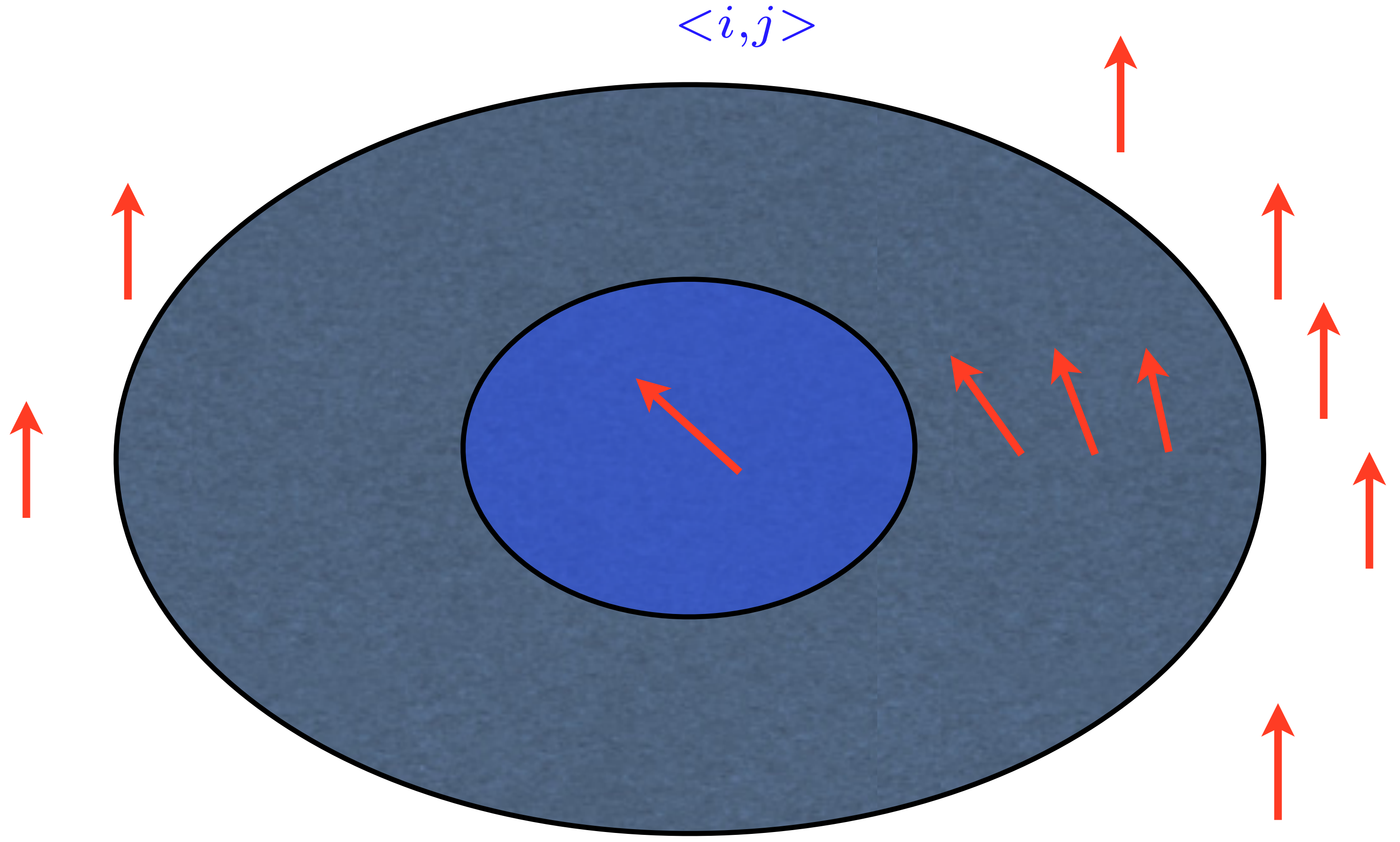
Misconceptions, vacuum degeneracy

By abuse of language we often hear, or say that in theories with SSB there is vacuum degeneracy. This is fact is not the case, at least in LQFT. In understanding this we will also understand why there are massless states in theories with SSB. N is the volume in the example. The Heisenberg model of magnetism. H is rotational invariant above the critical temperature, and magnetized below it



$$H = -J \sum_{<i,j>} \mathbf{s}_i \cdot \mathbf{s}_j$$

$$\langle 0|\theta\rangle = (\cos(\theta/2))^N \rightarrow 0 \qquad N \rightarrow \infty$$



By making the transitions very slowly we can manage to make this configuration to have as small an energy as we wish. Hence we have a continuum spectrum above zero. This is the sign of a massless particle, the NG-boson

Advanced Topic… Skip

No Goldstone bosons in finite volume

This simple example contains the ingredients of the general case. Consider a theory in a box of side L and PBCs, the plane waves solutions are easy to write down

We will not explain the details. They are spelled out for those who want to understand the mathematics behind Goldstone's theorem in finite volume.

$$\Phi = (\phi_1, \phi_2) \qquad \zeta(x) = \frac{1}{\sqrt{2}}[\varphi_1(x) + i\varphi_2(x)] \equiv \frac{1}{\sqrt{2}}[a + h(x)]\, e^{i\theta(x)}.$$

$$\mathscr{L} = \frac{1}{2}\partial_\mu \Phi \cdot \partial^\mu \Phi - \frac{\lambda}{4}\left(\Phi^2 - a^2\right)^2$$

$$= \partial_\mu \zeta^* \partial^\mu \zeta - \lambda\left(|\zeta|^2 - \frac{a^2}{2}\right)^2 = \frac{a^2}{2}\partial_\mu \theta \partial^\mu \theta + \cdots, \qquad \partial_\mu \partial^\mu \theta = 0 \qquad \partial_\mu \partial^\mu \phi = 0$$

$$\varphi_{\mathbf{k}}(t, \mathbf{x}) = \frac{1}{\sqrt{V}} e^{-i|\mathbf{k}|t + i\mathbf{k}\cdot\mathbf{x}}, \quad \mathbf{k} = \frac{2\pi}{L}\mathbf{n} \qquad \varphi(t, \mathbf{x}) = \varphi_0 + \pi_0 t + \sum_{\mathbf{k}\neq\mathbf{0}} \frac{1}{\sqrt{2V|\mathbf{k}|}}\left[\alpha(\mathbf{k}) e^{-i|\mathbf{k}|t + i\mathbf{k}\cdot\mathbf{x}} + \alpha^\dagger(\mathbf{k}) e^{i|\mathbf{k}|t - i\mathbf{k}\cdot\mathbf{x}}\right].$$

$$[\varphi(t, \mathbf{x}_1), \dot{\varphi}(t, \mathbf{x}_2)] = i\delta(\mathbf{x}_1 - \mathbf{x}_2) = \frac{i}{V} + \frac{i}{V}\sum_{\mathbf{k}\neq\mathbf{0}} e^{i\mathbf{k}\cdot(\mathbf{x}_1 - \mathbf{x}_2)}$$

$$[\varphi_0, \pi_0] = \frac{i}{V}. \qquad a = \frac{1}{\sqrt{2}}\left(\varphi_0 + iV^{\frac{1}{3}}\pi_0\right), \quad a^\dagger = \frac{1}{\sqrt{2}}\left(\varphi_0 - iV^{\frac{1}{3}}\pi_0\right), \qquad :H:= \frac{V}{2}\pi_0^2 + \sum_{\mathbf{k}\neq\mathbf{0}} |\mathbf{k}|\alpha^\dagger(\mathbf{k})\alpha(\mathbf{k}).$$

$$[a, a^\dagger] = V^{-\frac{2}{3}}. \qquad Q = \int d^3x\, \partial_0\varphi = V\pi_0 = \frac{V^{\frac{2}{3}}}{i\sqrt{2}}\left(a - a^\dagger\right). \qquad e^{-i\xi Q}\varphi(x)e^{i\xi Q} = \varphi(x) + \xi,$$

$$|\xi\rangle \sim e^{i\xi Q}|0\rangle = e^{-\frac{1}{\sqrt{2}}\xi V^{\frac{2}{3}}(a^\dagger - a)}|0\rangle.$$

$$\boxed{\langle 0|\xi\rangle = e^{-\frac{1}{4}\xi^2 V^{\frac{2}{3}}}\langle 0|0\rangle.}$$

http://carlossicoli.free.fr/A/Alvarez-Gaume_L._Vazquez-Mozo_M.A.-An_Invitation_to_Quantum_Field_Theory_-Springer(2011).pdf

The BEH mechanism

Notice we say the mechanism, not necessary the particle! In gauge theories one cannot just add a mass for the gauge bosons. This badly destroys the gauge symmetry and the theory is inconsistent.

BEH showed that in gauge theories with SSB the NG bosons are “eaten” by the gauge bosons to become massive but preserving the basic properties of the gauge symmetry. Ex. Abelian Higgs model

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\varphi)^\dagger(D^\mu\varphi) - \frac{\lambda}{4}\left(\varphi^\dagger\varphi - \mu^2\right)^2 \qquad \varphi \longrightarrow e^{i\alpha(x)}\varphi, \qquad A_\mu \longrightarrow A_\mu + \partial_\mu\alpha(x)$$

$$\langle\varphi\rangle = \mu e^{i\vartheta_0} \longrightarrow \mu e^{i\vartheta_0 + i\alpha(x)} \qquad \varphi(x) = \left[\mu + \frac{1}{\sqrt{2}}\sigma(x)\right]e^{i\vartheta(x)}$$

Take the unitary gauge

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + e^2\mu^2A_\mu A^\mu + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{1}{2}\lambda\mu^2\sigma^2 - \lambda\mu\sigma^3 - \frac{\lambda}{4}\sigma^4 + e^2\mu A_\mu A^\mu\sigma + e^2A_\mu A^\mu\sigma^2.$$

$$m_\gamma^2 = 2e^2\mu^2$$

The simplest example is the GL and BCS theory of superconductivity, in this case the “Higgs” particle is composite, it is an object of charge made of two bound electrons that get a “VEV” (Cooper pairs) that get a VEV in the superconducting state. The photon is massive in this state. This explains among other things the Meissner effect.

Gauge couplings: colour

There are three gauge groups in the theory, the colour group SU(3) and the electroweak group SU(2)xU(1) of weak isospin and hypercharge. Y and T3 mix to generate electric charge and the photon

$$SU(3)_c \times SU(2) \times U(1)_Y \to SU(3) \times U(1)_Q$$

QCD by itself is a perfect theory in many ways

$$\mathscr{L}_{\mathrm{QCD}} = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + \sum_{f=1}^{6} \overline{Q}^f \left(i\not{D} - m_f \right) Q^f . \qquad Q^f_i \longrightarrow U(g)_{ij} Q^f_j , \qquad \text{with} \quad g \in \mathrm{SU(3)}$$

Isospin as an approximate symmetry:

$$\mathscr{L} = (\overline{u}, \overline{d}) \begin{pmatrix} i\not{D} - \frac{m_u + m_d}{2} & 0 \\ 0 & i\not{D} - \frac{m_u + m_d}{2} \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix} - \frac{m_u - m_d}{2} (\overline{u}, \overline{d}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}$$

Once the electroweak sector is included the story of the masses is far more complicated (see later)
The masses appearing here, and in the ordinary formulation of the SM are called the current algebra masses.
They are rather small compared to the proton or neutron masses. The limit with zero up and down masses makes sense.

Note that Standard Gell-Mann Neeman SU(3) is an accidental symmetry of the QCD Lagrangian

Furthermore, the quark masses provide soft-breaking to the full chiral symmetry (see below)

Chiral Symmetry Breaking in QCD

In HEP they provide the only observed NG bosons

The order parameter is not an elementary field, it is provided by a quark bilinear condensate

To find other NG bosons in the SM we have to go to the Higgs sector, and there they are "eaten" to provide masses for the W and Z vector bosons

In QCD there are no fundamental scalars. Consider just two flavors u,d. We have chiral symmetry

$$\begin{pmatrix} u_{L,R} \\ d_{L,R} \end{pmatrix} \longrightarrow M_{L,R} \begin{pmatrix} u_{L,R} \\ d_{L,R} \end{pmatrix}$$

$$G = \underbrace{SU(2)_L \times SU(2)_R}_{SU(2)_V} \times U(1)_B \times U(1)_A$$

The diagonal vector subgroup is isospin one U(1) is baryon number. The axial U(1) is in fact broken by quantum anomalies and it is very important in the dynamics and vacuum structure of the theory

$$q^f_\alpha \qquad f = u, d, \ \ \alpha = 1, 2, 3$$

$$\langle \bar{q}^f \cdot q^{f'} \rangle = \Lambda^3_{\chi SB} \ \delta^{ff'}$$

$$\bar{q}^f \cdot q^{f'} \simeq \Lambda^3_{\chi SB} \ e^{i\pi^a \sigma^a / f_\pi}$$

These are the pions.
This is an IR property of QCD, not accessible to Pert. Th.
Low-E pion theorems, chiral Lagrangians….
General symmetry principles allow us to work out the low energy description of the Goldstonee bosons

Pion Lagrangians, first look at the anomaly

$$\langle 0|\bar{q}q|0\rangle = \langle 0|\bar{q}_L q_R + \bar{q}_R q_L|0\rangle \neq 0$$

$$j^\mu = \bar{q}\gamma^\mu q, \qquad j^{\mu a} = \bar{q}\gamma^\mu \tau^a q$$

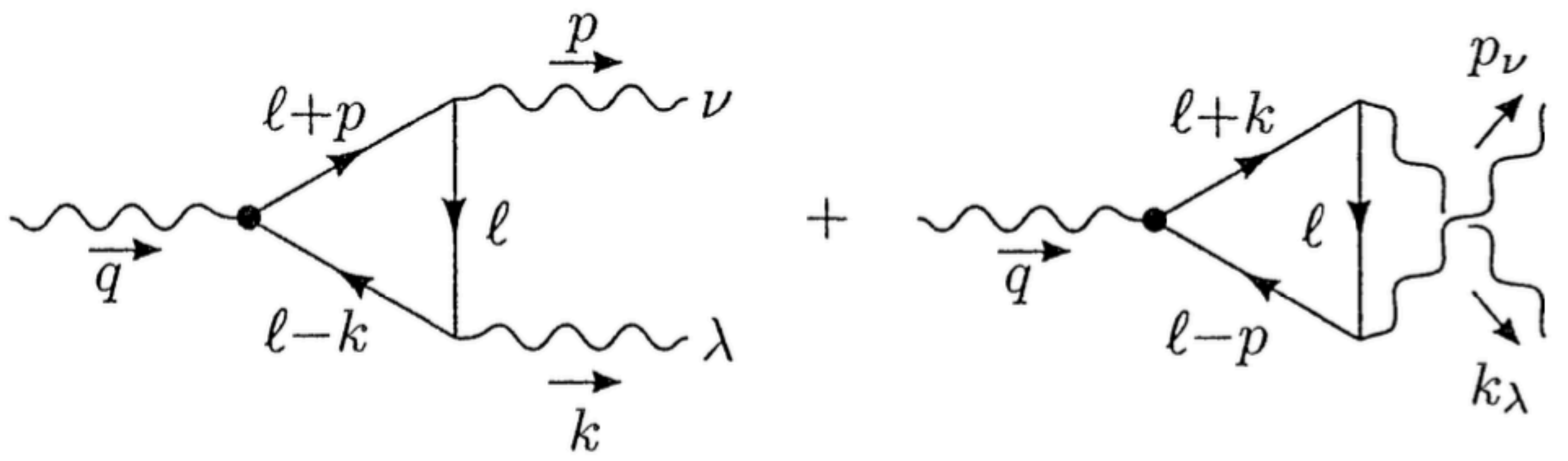
$$j_5^\mu = \bar{q}\gamma^\mu \gamma_5 q, \qquad j_5^{\mu a} = \bar{q}\gamma^\mu \gamma_5 \tau^a q$$

Vector and Axial singlet currents

$$q \to e^{i\alpha} q \qquad q_L \to e^{i\alpha} q_L, q_R \to e^{i\alpha} q_R$$

$$q \to e^{-i\alpha\gamma_5} q \qquad q_L \to e^{i\alpha} q_L, q_R \to e^{-i\alpha} q_R$$

$$\langle 0|j^{\mu a}(x)|\pi^b(p)\rangle = -ip^\mu f_\pi \delta^{ab} e^{-ip\cdot x}, \qquad f_\pi = 93\,\mathrm{MeV}$$



At each vertex we can have:
A current being tested for conservation
Gauge bosons coupling to the internal fermions
Bose symmetry requires adding the graphs
The kinematic structure is the same

If we include in the current conservation laws the small flavor masses,

$$m_f = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} = \mathcal{M}$$

If we include in the current conservation laws the small flavor masses,

$$\partial_\mu j^{\mu 5a} = i\bar{q}\{m, \tau^a\}q$$

From the conservation of the current matrix element defined before if follows that:

$$m_\pi = (m_u + m_d)\frac{v^2}{f_\pi}$$

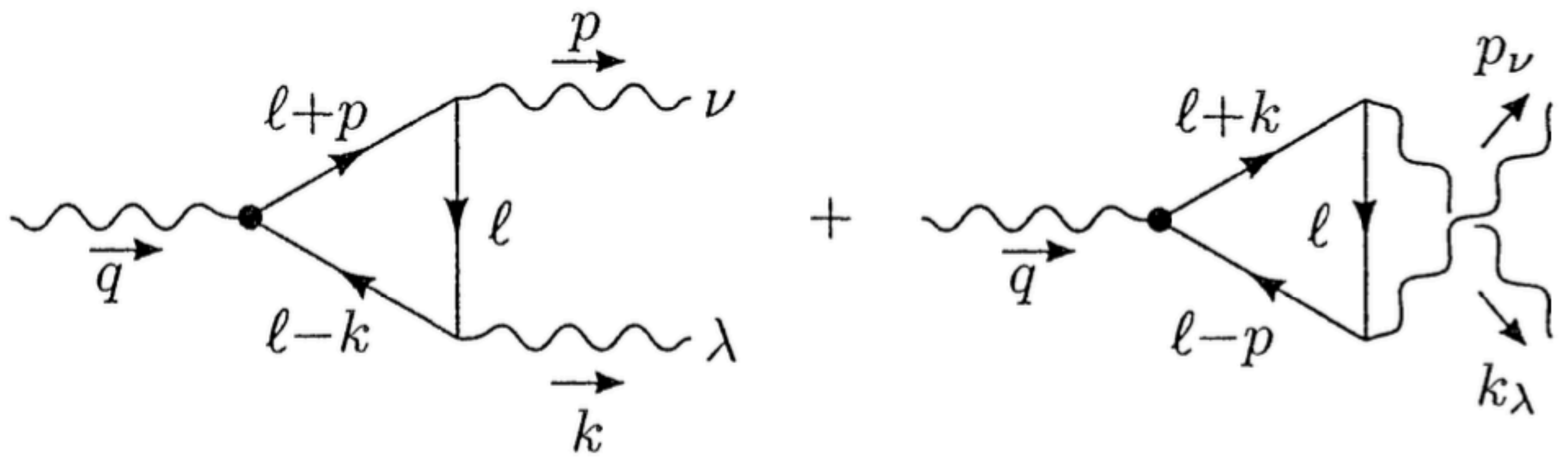
Recall the quark condensate $\bar{q}^f \cdot q^{f'} \simeq \Lambda^3_{\chi SB}\, \underbrace{e^{i\pi^a\sigma^a/f_\pi}}_{U(x)}$

The best way to describe the low-energy chiral Lagrangian for pions, is in terms of a unitary 2x2 matrix, which under L,R transformations. The quark mass matrix breaks softly the symmetry in the expected way

$U(x) \rightarrow LUR^+$

$$\mathcal{L} = -\frac{f_\pi^2}{4}\mathrm{tr}(\partial_\mu U \partial^\mu U^{-1}) + v^3 \mathrm{tr}(\mathcal{M}\mathcal{U} + \mathrm{h.c.})$$

If we select the currents appearing on the vertices of the triangle diagram, we get different interesting results



$\{j^{\mu 5a}, g, g\}$ $$\partial_\mu j^{\mu 5a} = -\frac{g^2}{16\pi^2}\epsilon^{\alpha\beta\mu\nu}F^c_{\alpha\beta}F^d_{\mu\nu}\cdot \mathrm{tr}[\tau^a t^c t^d]$$

The trace vanishes, and their is no anomaly in this current. Its only breaking comes from the quark masses

$\{j^{\mu 5}, g, g\}$ $$\partial_\mu j^{\mu 5a} = -\frac{g^2 n_f}{32\pi^2}\epsilon^{\alpha\beta\mu\nu}F^c_{\alpha\beta}F^d_{\mu\nu}$$

This is a true anomaly, the chiral rotation of phases for the quarks generates a non-trivial change in the action as a consequence of the gluon fields. This is very useful in the formulation of the strong CP problem

Finally, the original ABJ anomaly follows when we consider one isospin current and two photons instead of gluons. This led to the correct computation of the electromagnetic decay of the neutral pion

$\{j^{\mu 5a}, \gamma, \gamma\}$ $$\partial_\mu j^{\mu 5a} = -\frac{e^2}{16\pi^2}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}\cdot \mathrm{tr}[\tau^a Q^2]$$

Gauge theories and their quantization

Imagine we have a theory with a global symmetry

$$\psi \to g\,\psi \qquad \bar\psi \to \bar\psi g^\dagger \qquad \mathcal{L} = \bar\psi i \partial\!\!\!/ \,\psi$$

Imitating electromagnetism:

$$\partial_\mu \to D_\mu \psi = \left(\partial_\mu + ieA_\mu^a T^a\right)\psi \equiv (\partial_\mu + ieA_\mu)\psi \qquad D_\mu\psi \to gD_\mu\psi$$

We can read off the gauge field transformations

$$A_\mu \to \frac{1}{ie} g\,\partial_\mu g^{-1} + gA_\mu\, g^{-1}$$

$$g \approx 1 + \epsilon \qquad A_\mu \to A_\mu + \frac{1}{ie} D_\mu \epsilon \qquad D_\mu\epsilon + ie[A_\mu, \epsilon]$$

$$[D_\mu, D_\nu] = ieT^a F^a_{\mu\nu}, \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - ef^{abc} A^b_\mu A^c_\nu$$

$$F_{\mu\nu} \equiv T^a F^a_{\mu\nu} \to gF_{\mu\nu}\, g^{-1}$$

Nonabelian gauge fields have self-couplings unlike photons. This is responsible for confinement, among other things

General gauge theory
Lagrangian:

$$\mathscr{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu\, a} + i\overline{\psi}\not{D}\psi + (D_\mu\phi)^\dagger D^\mu\phi$$
$$-\overline{\psi}\left[M_1(\phi) + i\gamma_5 M_2(\phi)\right]\psi - V(\phi).$$

We need to provide the gauge group and the matter representations for bosons and fermions and off we go

Quantising a gauge theory is no joke. There are plenty of subtleties. We give you just a taste

We can define chromoelectric and magnetic fields as in QED

$$F^a_{0i} = \partial_0 A^a_i - \partial_i A^a_0 - ief^{abc}A^b_0 A^c_i \equiv E^a_i$$
$$F^a_{ij} = \epsilon_{ijk}B^a_k, \qquad F^a_{0i} = \partial_0 A^a_0 - D_i A^a_0$$

The canonical variables are

$$\mathbf{A}^a, \mathbf{E}^a$$

$$\mathcal{L} = \mathbf{E}^a\,\partial_0\mathbf{A}^a - \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) - A^a_0\,(\mathbf{D}\cdot\mathbf{E})^a$$

A^a_0 implements a constraint

We can read off the Hamiltonian density

$$H = \int d^3x \left(\frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) + A_0^a \, (\mathbf{D}\cdot\mathbf{E})^a \right)$$

$$[A_i^a(\mathbf{x},0), E_j^b(\mathbf{y},0)] \; = \; i\,\delta_{ij}\,\delta^{ab}\delta(\mathbf{x}-\mathbf{y})$$

We can fix the gauge A_0=0 so that we only have time-independent gauge transformations in the Hamiltonian theory, but we are missing one of the equations of motion, Gauss' law that has to be implemented as a constraint.

$$(\mathbf{D}\cdot\mathbf{E})^a = 0$$

Cannot be implemented at the operator level. It generates gauge transformations

$$[Q(\epsilon), A_i^a] \; = \; i(D\epsilon)^a \qquad U(\epsilon) = \exp(i\int d^3x\, \epsilon^a(\mathbf{x})\,(\mathbf{D}\cdot\mathbf{E})^a), \qquad U\,H\,U^{-1} = H$$

Gauss' law becomes a condition on the physical states:

$$U(\epsilon)|\mathrm{phys}\rangle \; = |\mathrm{phys}\rangle$$
$$\mathbf{D}\cdot\mathbf{E}\,|\mathrm{phys}\rangle = 0$$

Each gauge configuration sits in an orbit and we need choose only one element, this is done by "fixing" the gauge for the t-independent gauge transf.

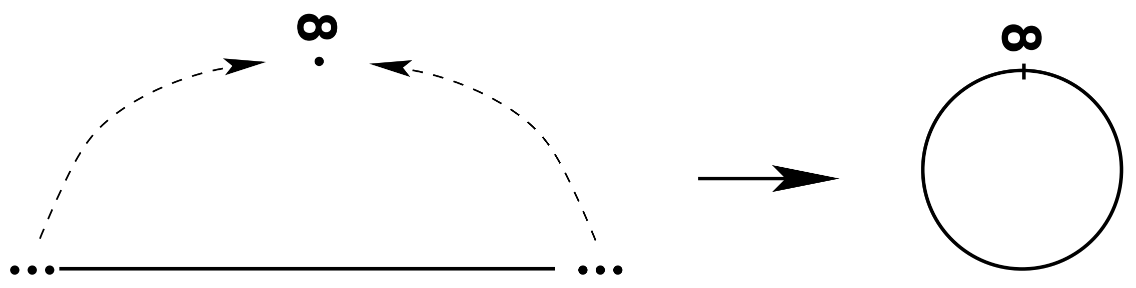
WE HAVE 2-DIM G PHYSICAL DEGREES OF FREEDOM

Some Topology

Gauge symmetry is more a redundant description of the d.o.f.

Gauss' law implements gauge invariance under gauge t. connected to the identity. Consider finite-E configurations

$$g(\mathbf{x}) = e^{i\alpha(\mathbf{x})} \to 1 \qquad |\mathbf{x}| \to \infty$$
$$\alpha(\mathbf{x}) \to 0 \qquad |\mathbf{x}| \to \infty$$



There are others, and Gauss' law cannot impose invariance

$$g(\mathbf{x}) : S^3 \to G, \quad g(\infty) = 1 \qquad \pi_3(G) = Z \text{ the integers}$$

$$g : S^1 \longrightarrow U(1), \qquad g(x) = e^{i\alpha(x)}$$

$$\alpha(2\pi) = \alpha(0) + 2\pi n$$

$$\oint_{S^1} g(x)^{-1} dg(x) = 2\pi n$$

Winding number circle to circle

$$n = \frac{1}{24\pi^2}\int_{S^3} d^3x\, \varepsilon_{ijk} \mathrm{Tr}\left[\left(g^{-1}\partial_i g\right)\left(g^{-1}\partial_i g\right)\left(g^{-1}\partial_i g\right)\right]$$

You cannot comb a sphere

The group SU(2) happens to be topologically a three sphere

$$g = u^0 + i\sigma\cdot\mathbf{u}, \qquad g^\dagger \cdot g = 1 \qquad \det g = 1$$

$$(u^0)^2 + \mathbf{u}^2 = 1$$

Consider the group of gauge transformations for SU(2) in the Hamiltonian formulation of the theory. Consider gauge transformations from space to SU(2)

$$g(x) : R^3 \to SU(2) \equiv S^3$$

If the gauge transformations go to the identity at infinity:

$$R^3 \cup \{\infty\} \simeq S^3$$

Hence the gauge group is the space of maps:

$$\mathcal{G} \equiv \{g : S^3 \to S^3\}$$

Sphere wrap on sphere, like circles, the winding number is n, there is a single generator modulo gauge transformations

A surprise: CP violation

Gauge invariance only requires that under non-trivial transformations, a phase is generated. This is a vacuum angle! In fact it violates CP.

It can be measured by looking for an edm of the neutron. So far no measurement. We will show below how to study the effects of the vacuum angle at low-energies, and the potential solutions proposed so far.

The strong CP problem, axions, invisible axions, axion cosmology, dark matter...

$g_1 \in \mathcal{G}/\mathcal{G}_0$ the generator $\quad \mathscr{U}(g_1)|\text{phys}\rangle = e^{i\theta}|\text{phys}\rangle.$

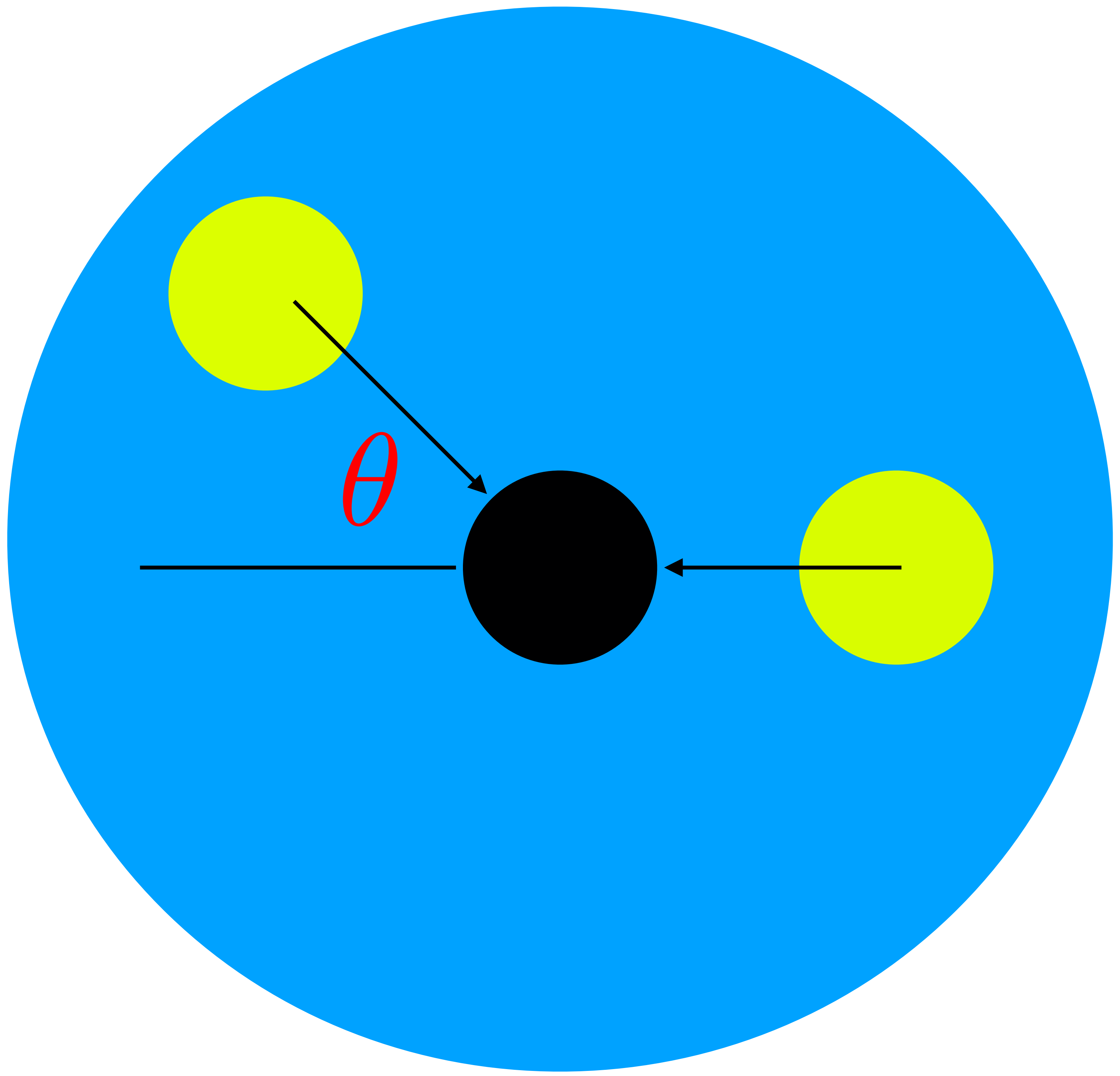
$$S = -\frac{1}{4}\int d^4x F^a_{\mu\nu}F^{\mu\nu a} - \frac{\theta g^2_{\rm YM}}{32\pi^2}\int d^4x F^a_{\mu\nu}\tilde{F}^{\mu\nu a}$$

$$\tilde{F}^a_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\sigma\lambda}F^{\sigma\lambda a} \qquad F^a_{\mu\nu}\tilde{F}^{\mu\nu a} = 4\,\mathbf{E}^a\cdot\mathbf{B}^a$$

$$\frac{g^2_{\rm YM}}{32\pi^2}\int d^4x F^a_{\mu\nu}\tilde{F}^{\mu\nu a} = \frac{1}{24\pi^2}\int d^3x\,\varepsilon_{ijk}\mathrm{Tr}\Big[(g\partial_i g^{-1})(g_+\partial_j g^{-1})(g_+\partial_k g^{-1})\Big].$$

A simple classical argument

(Thanks to Anson Hook)



Neutron = udd

Using the quark charges, and putting the size of the neutron as a Compton wavelength, and using:

$$\hbar c \approx 200\,\mathrm{MeV\ fermi}$$

$$|d_n| \approx 10^{-13}\sqrt{1-\cos\theta}\; e\,\mathrm{cm}$$

Experimentally:

$$|d_n| \leq 10^{-26}\, e\,\mathrm{cm}$$

$$\theta \leq 10^{-13}$$

Recall the anomaly calculation for the axial current.

$$\{j^{\mu 5}, g, g\} \qquad \partial_\mu j^{\mu 5a} = -\frac{g^2 n_f}{32\pi^2}\epsilon^{\alpha\beta\mu\nu}F^c_{\alpha\beta}F^d_{\mu\nu}$$

What this means is that for each flavor of quarks, if we redefine the chiral phase of the quark, we generate the “topological term

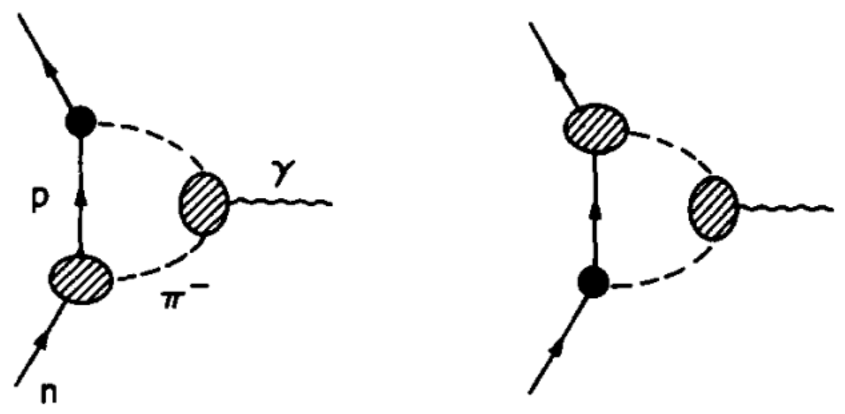
$$\mathbf{E^a}\cdot\mathbf{B^a}$$

By doing this, the theta parameter is redefined:

$$\theta \to \theta + \arg(\det\mathcal{M}) \qquad \mathcal{L} = -\frac{f_\pi^2}{4}\mathrm{tr}(\partial_\mu U\partial^\mu U^{-1}) + v^3\mathrm{tr}(\mathcal{M}\mathcal{U} + \mathrm{h.c.}) \qquad \mathcal{M} \to \begin{pmatrix} m_u\, e^{i\theta} & 0 \\ 0 & m_d \end{pmatrix}$$

And we need to work out the dependence of low energy physics on the vacuum angle. We need to include the nucleon isospin doublet

$$N = \begin{pmatrix} p \\ n \end{pmatrix} \qquad \bar{N}DN - \bar{N}f(U,\theta)N$$



$$d_n \approx 5.2\,10^{-16}\theta\, e\,\mathrm{cm}$$

Why is theta so small? Hadron dynamics depends on it in a non-trivial way

As a function of the vacuum angle, the ground state energy is periodic. It can be shown that:

$$V(0) \leq V(\theta) \ = \ V(\theta + 2\pi)$$

For the full potential $\theta = \pi$ Is a special place, where a cusp may develop for some values of the quark masses

For small vacuum angle, the minimum of the potential is

$$V(\theta) \ = \ \frac{\theta^2}{8} f_\pi^2 m_\pi^2 \frac{m_u m_d}{(m_u + m_d)^2}$$

The axion solution of the strong CP problem consist of making theta into a dynamical field "a" whose potential V(a) will relax its value to zero. A is a pseudo scalar, and also a pseudo Goldstone boson, hence the theory is symmetric approximately under shifts a—> a + constant. The term quadratic in the axion field is:

$$V(\theta) \ = \ \frac{a^2}{8} f_\pi^2 m_\pi^2 \frac{m_u m_d}{(m_u + m_d)^2} \qquad\qquad m_a \ = \ \frac{f_\pi m_\pi}{2 f_a} \frac{\sqrt{m_u m_d}}{m_u + m_d}$$

Normalizing the kinetic term as:

$$\mathcal{L} \ = \ \frac{1}{2} f_a^2 (\partial_\mu a)^2 + \frac{a}{32\pi^2} \epsilon^{\mu\nu\alpha\beta} \, \mathrm{tr} F_{\mu\nu} F_{\alpha\beta} \qquad\qquad m_a \approx 5.4 \times 10^{-10} \mathrm{eV} \cdot \frac{1.1 \times 10^{16} \, \mathrm{GeV}}{f_a}$$

Excellent candidate for dark matter…

The EW theory. Fermion quantum numbers

The fundamental fermions come in three families with the same quantum numbers with respect to the gauge group

Leptons					
i (generation)	1	2	3	T^3	Y
$\mathbf{L}^i$	$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L$	$\begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L$	$\begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L$	$\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$	$-\frac{1}{2}$
ℓ_R^i	e_R^-	μ_R^-	τ_R^-	0	-1

Quarks					
i (generation)	1	2	3	T^3	Y
$\mathbf{Q}^i$	$\begin{pmatrix} u \\ d \end{pmatrix}_L$	$\begin{pmatrix} c \\ s \end{pmatrix}_L$	$\begin{pmatrix} t \\ b \end{pmatrix}_L$	$\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$	$\frac{1}{6}$
U_R^i	u_R	c_R	t_R	0	$\frac{2}{3}$
D_R^i	d_R	s_R	b_R	0	$-\frac{1}{3}$

In principle one could add sterile neutrinos, right handed neutrinos who are singlets under the gauge group. They would generate Dirac masses for the known neutrinos

We can spell out the quantum numbers for each generation. The only difference is related to their masses

$$SU(3) \times SU(2) \times U(1) \to SU(3) \times U(1)_{em}$$

Left handed fermions

$$(3, 2)_{1/6} \oplus (1, 2)_{-1}$$

Right handed fermions

$$(3, 1)_{2/3} \oplus (3, 1)_{-1/3} \oplus (1, 1)_{-1}$$

There is a fundamental chiral nature of these quantum number, generically anomalies spoil gauge invariance and hence the consistency of the theory. It is remarkable that the hypercharge assignments are essentially unique for the consistency of the theory. The exquisite structure of the families is in need of explanation. Let's see how this happens

Anomalous Symmetries

Sometimes symmetries of the classical Lagrangian do not survive quantization

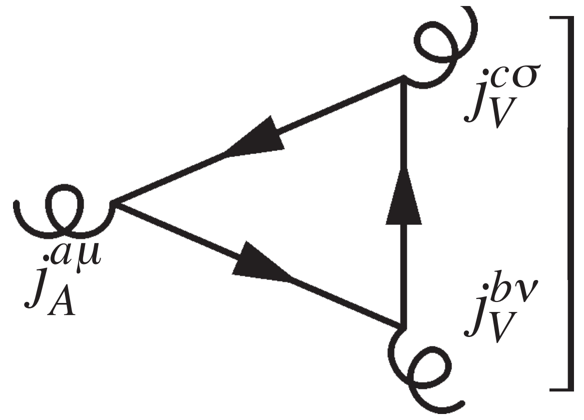
Global chiral symmetries, responsible for the electromagnetic decay of the neutral pion. This is a welcome anomaly, it does not jeopardize consistency of the theory

Gauged chiral symmetries. This happens when left and right multiplets have different representations of the gauge group. At the one-loop level we find a non-trivial conditions among the quantum numbers necessary to maintain gauge invariance. It suffices to satisfy this condition at the one-loop level. This is the example we want to explore now

Scale invariance. The behavior of the theory under scale transformation. Rather, how physics depends on scales is far more interesting that just dimensional analysis

When the gauge current is not conserved, gauge invariance is violated, and unphysical states begin to propagate, or the gauge symmetry is broken by some additional scalars in the theory

$$\langle 0|T\left[j_A^{a\mu}(x)\, j_V^{b\nu}(x')\, j_V^{c\sigma}(0)\right]|0\rangle = \Big[\quad\Big]_{\text{symmetric}} \propto \pm \;\text{tr}\left[\tau^a_{i,\pm}\{\tau^b_{i,\pm},\tau^c_{i,\pm}\}\right]$$



$$\sum_{i=1}^{N_+}\text{tr}\left[\tau^a_{i,+}\{\tau^b_{i,+},\tau^c_{i,+}\}\right] - \sum_{j=1}^{N_-}\text{tr}\left[\tau^a_{j,-}\{\tau^b_{j,-},\tau^c_{j,-}\}\right] = 0.$$

Anomaly cancellation condition, it has highly non-trivial implications for the family structure. In the graph each vertex should be L or R currents, contributing with opposite signs

Deriving Quantum Numbers

Anomalies cancel generation by generation. In fact the hypercharge assignments is completely determined if we also impose the traceless-ness of any U(1), which follows from the mixed anomaly including the energy-momentum tensor.

$SU(N)_c \times SU(2) \times U(1)$ S.M. anomaly

quarks: $\begin{pmatrix} u^\alpha \\ d^\alpha \end{pmatrix}_{L,\frac{1}{6}}$ $u^\alpha_{R,\frac{2}{3}}$ $d^\alpha_{R,\frac{2}{3}}$

leptons: $\begin{pmatrix} \nu_e \\ e \end{pmatrix}_{L,-\frac{1}{2}}$ $e_{R,-1}$

$$(N,2)^L_{q_L} \oplus (1,2)^L_{l_L}$$
$$(N,1)^R_{u_R} \oplus (N,1)^R_{d_R} \oplus (1,1)^R_{e_R}$$

← Single family

Anomaly conditions, we will normalize e_R=-1 as in the SM

$$U(1)SU(2)^2 \qquad 2N\, q_L + 2\, l_L = 0$$
$$U(1)SU(N)^2 \qquad 2\, q_L - (u_R + d_R) = 0$$
$$U(1)^3 \qquad 2\, N\, q_L^3 + 2\, l_L^3 - N\, u_R^3 - N\, d_R^3 - e_R^3 = 0$$
$$U(1) \qquad 2\, q_L + 2\, l_L - N(u_R + d_R) - e_R = 0$$

A simple computation now yields a (nearly) unique solution:

$$q_L = \frac{1}{2\,N} \qquad l_L = -\frac{1}{2} \qquad e_R = -1$$

$$u_R + d_R = 1/N \qquad u_R = \frac{N+1}{2N}$$
$$u_R\, d_R = -\frac{1}{4}(1 - \frac{1}{N^2}) \qquad d_R = -\frac{N-1}{2N}$$

For N=3 we obtain the hypercharges of the SM!!

Gauge couplings: EW sector

The EW group has four generators, and the electric charge is a non-trivial mixing between week hyper charge and weak neutral isospin. The weak mixing angle was first introduced by S. Glashow in 1961, in the original formulation of the SU(2) x U(1) formulation of the standard model. The model did not contain the BEH multiplet, but general physical considerations led to the mixing of the neutral currents.

$$\mathbf{W}_\mu = W_\mu^+ T^- + W_\mu^- T^+ + W_\mu^3 T^3, \qquad \mathbf{B}_\mu = B_\mu Y.$$

$$Q = T^3 + Y.$$

$$A_\mu = B_\mu \cos\theta_w + W_\mu^3 \sin\theta_w,$$
$$Z_\mu = -B_\mu \sin\theta_w + W_\mu^3 \cos\theta_w$$

$$[Q, T^\pm] = \pm T^\pm, \qquad [Q, T^3] = [Q, Y] = 0.$$

$$e = g \sin\theta_w = g' \cos\theta_w.$$

The covariant derivatives become quite intricate:

$$\begin{aligned} D_\mu &= \partial_\mu - ig\mathbf{W}_\mu - ig'\mathbf{B}_\mu \\ &= \partial_\mu - igW_\mu^+ T_R^- - igW_\mu^- T_R^+ - igW_\mu^3 T_R^3 - ig' B_\mu Y_R, \end{aligned}$$

$$\begin{aligned} D_\mu &= \partial_\mu - igW_\mu^+ T_R^- - igW_\mu^- T_R^+ - iA_\mu(g\sin\theta_w T_R^3 + g'\cos\theta_w Y_R) \\ &- iZ_\mu(gT_R^3 \cos\theta_w - g' Y_R \sin\theta_w). \end{aligned}$$

$$\mathscr{L}_{\text{gauge}} = -\frac{1}{2}W^{+}_{\mu\nu}W^{-\,\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{ig}{2}\cos\theta_w W^{+}_{\mu}W^{-}_{\nu}Z^{\mu\nu} + \frac{ie}{2}W^{+}_{\mu}W^{-}_{\nu}F^{\mu\nu} - \frac{g^2}{2}\left[(W^{+}_{\mu}W^{+\,\mu})(W^{-}_{\mu}W^{-\,\mu}) - (W^{+}_{\mu}W^{-\,\mu})^2\right]$$

Gauge couplings

$$W^{\pm}_{\mu\nu} = \partial_\mu W^{\pm}_{\nu} - \partial_\nu W^{\pm}_{\mu} \mp ie\left(W^{\pm}_{\mu}A_\nu - W^{\pm}_{\nu}A_\mu\right) \mp ig\cos\theta_w\left(W^{\pm}_{\mu}Z_\nu - W^{\pm}_{\nu}Z_\mu\right)$$

Convenient notation

$$Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu$$

$$\mathscr{L}_{\text{matter}} = \sum_{i=1}^{3}\left(i\overline{\mathbf{L}}^j \not{D}\mathbf{L}^j + i\bar{\ell}^j_R \not{D}\ell^j_R + i\overline{\mathbf{Q}}^j \not{D}\mathbf{Q}^j + i\overline{U}^j_R \not{D}U^j_R + i\overline{D}^j_R \not{D}D^j_R\right)$$

Matter couplings without the Higgs

Higgs couplings

The Higgs couplings responsible for the masses of the leptons and the current algebra masses of the quarks are below. Notice that the Higgs field allows allows us to accommodate the quark and lepton masses in the standard model, but unlike the masses of the W and Z, it does not explain them. This is the flavor problem…

$$\mathbf{H} = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} \qquad \tilde{\mathbf{H}} \equiv i\sigma^2 \mathbf{H}^* = \begin{pmatrix} H^{0*} \\ H^{+*} \end{pmatrix}$$

$$Y(\mathbf{H}) = \tfrac{1}{2}$$

$$\mathscr{L}_{\text{Higgs}} = (D_\mu \mathbf{H})^\dagger D^\mu \mathbf{H} - V(\mathbf{H}, \mathbf{H}^\dagger), \qquad V(\mathbf{H}, \mathbf{H}^\dagger) = \frac{\lambda}{4}\left(\mathbf{H}^\dagger \mathbf{H} - \mu^2\right)^2$$

$$\mu \sim \sqrt{\langle \mathbf{H}^+ \cdot \mathbf{H} \rangle} \approx 246\,\text{GeV}$$

$$\mathscr{L}^{(\ell)}_{\text{Yukawa}} = -\sum_{i,j=1}^{3}\left(C^{(\ell)}_{ij}\,\overline{\mathbf{L}}^i\,\mathbf{H}\,\ell^j_R + C^{(\ell)*}_{ji}\,\overline{\ell}^i_R \mathbf{H}^\dagger\,\mathbf{L}^j\right)$$

$$\mathscr{L}^{(q)}_{\text{Yukawa}} = -\sum_{i,j=1}^{3}\left(C^{(q)}_{ij}\,\overline{\mathbf{Q}}^i\,\mathbf{H}\,D^j_R + C^{(q)*}_{ji}\,\overline{D}^i_R\,\mathbf{H}^\dagger\,\mathbf{Q}^j\right)$$

$$-\sum_{i,j=1}^{3}\left(\tilde{C}^{(q)}_{ij}\,\overline{\mathbf{Q}}^i\,\tilde{\mathbf{H}}\,U^j_R + \tilde{C}^{(q)*}_{ji}\,\overline{U}^i_R\,\tilde{\mathbf{H}}^\dagger\,\mathbf{Q}^j\right).$$

The most general Lagrangian compatible with the gauge symmetry and up to dimension 4, so that the theory is renormalizable. Once H gets its VEV the masses are generated from the Yukawa couplings. Use unitary gauge. The gauge fields get masses from the kinetic term

$$\mathbf{H}(x) = e^{i\mathbf{a}(x)\cdot\frac{\sigma}{2}}\begin{pmatrix} 0 \\ \mu + \frac{1}{\sqrt{2}}h(x) \end{pmatrix} = \begin{pmatrix} 0 \\ \mu + \frac{1}{\sqrt{2}}h(x) \end{pmatrix}$$

After the scalar gets its VEV, we find that the Higgs coupling to the fermions goes as:

$$\sim \frac{m_f}{v}$$

The lighter the fermion, the weaker the coupling…

EXERCISE: Write a dim 5 operator respecting gauge invariance, but giving a Majorana mass to the neutrinos

Fundamental property

In a theory with a single scalar doublet, using SU(2) x U(1) transformations, it is ALWAYS possible to represent the doublet in the form:

$$SU(2)_W \times U(1)_Y \rightarrow U(1)_{em}$$

$$\mathbf{H}(x) = e^{i\mathbf{a}(x)\cdot\frac{\sigma}{2}} \begin{pmatrix} 0 \\ \mu + \frac{1}{\sqrt{2}} h(x) \end{pmatrix}$$

In other words, EM is always preserved. If we have several scalar doublets, or other representations, the scalar potential has to be chosen judiciously because it may lead to no E&M at low energies. With a single doublet this is guaranteed. Other representations for the Higgs can be envisaged, but none work to reproduce the expected properties of the SM.

In the unitary gauge, we can fix a(x)=0, showing that the only physical degree of freedom associated to the H doublet is the neutral BEH scalar boson h(x)

CKM matrix

$$\mathscr{L}^{(\ell)}_{\rm mass} = -(\bar e_L, \bar\mu_L, \bar\tau_L)\, M^{(\ell)} \begin{pmatrix} e_R \\ \mu_R \\ \tau_R \end{pmatrix} + \text{h.c.} \qquad \mathscr{L}^{(q)}_{\rm mass} = -(\bar d_L, \bar s_L, \bar b_L)\, M^{(q)} \begin{pmatrix} d_R \\ s_R \\ b_R \end{pmatrix} - (\bar u_L, \bar c_L, \bar t_L)\, \tilde M^{(q)} \begin{pmatrix} u_R \\ c_R \\ t_R \end{pmatrix} + \text{h.c.}$$

$$M^{(\ell,q)}_{ij} = \mu C^{(\ell,q)}_{ij}, \qquad \tilde M^{(q)}_{ij} = \mu \tilde C^{(q)}_{ij}$$

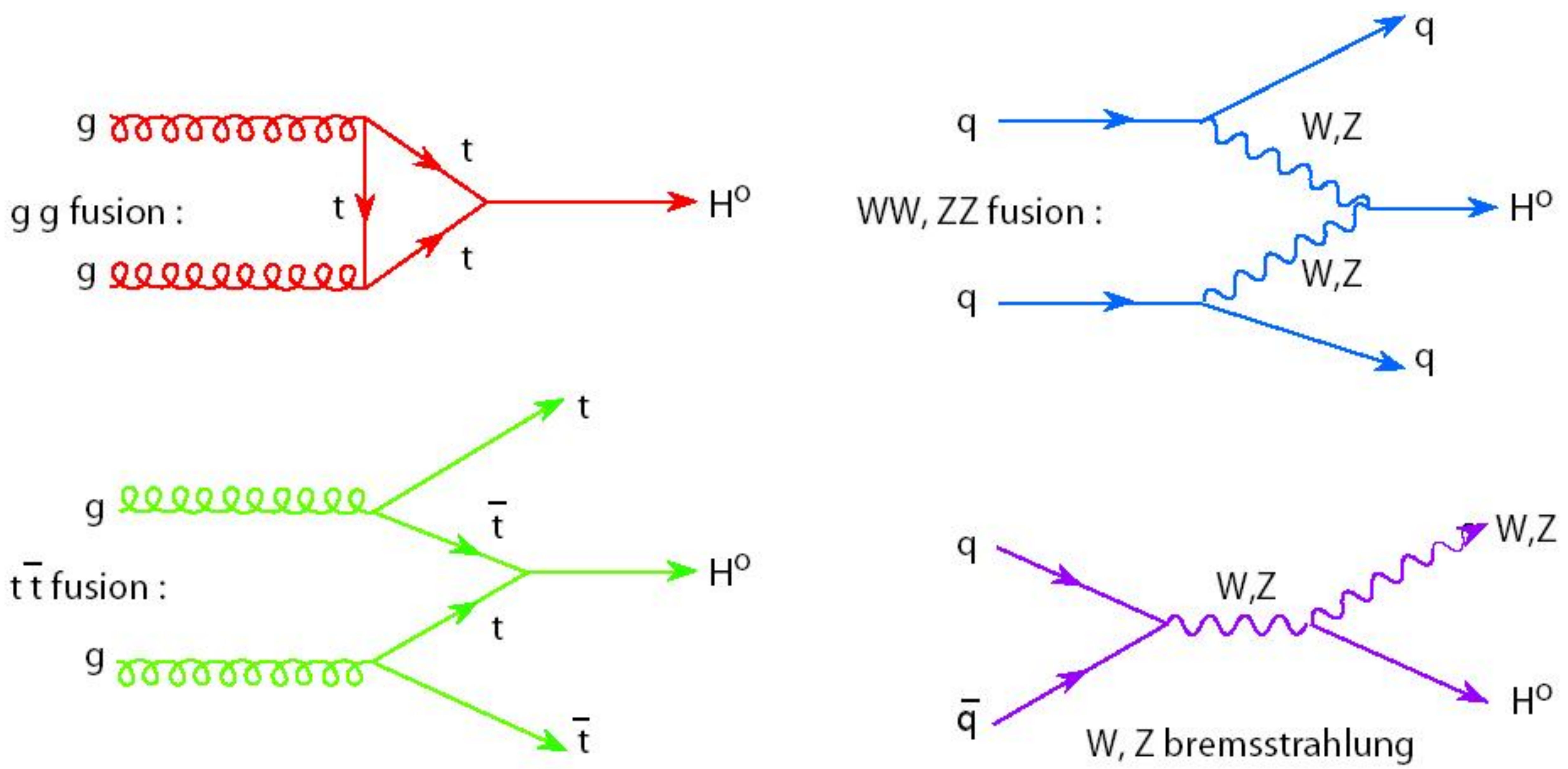
$$V_L^{(\ell)\dagger} M^{(\ell)} V_R^{(\ell)} = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} \qquad V_L^{(q)\dagger} M^{(q)} V_R^{(q)} = \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix} \qquad \tilde V_L^{(q)\dagger} \tilde M^{(q)} \tilde V_R^{(q)} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}$$

In the quark sector going from gauge to mass eigenstates leaves a matrix of phases in the charged currents, the CKM matrix. Not for neutral currents (GIM)

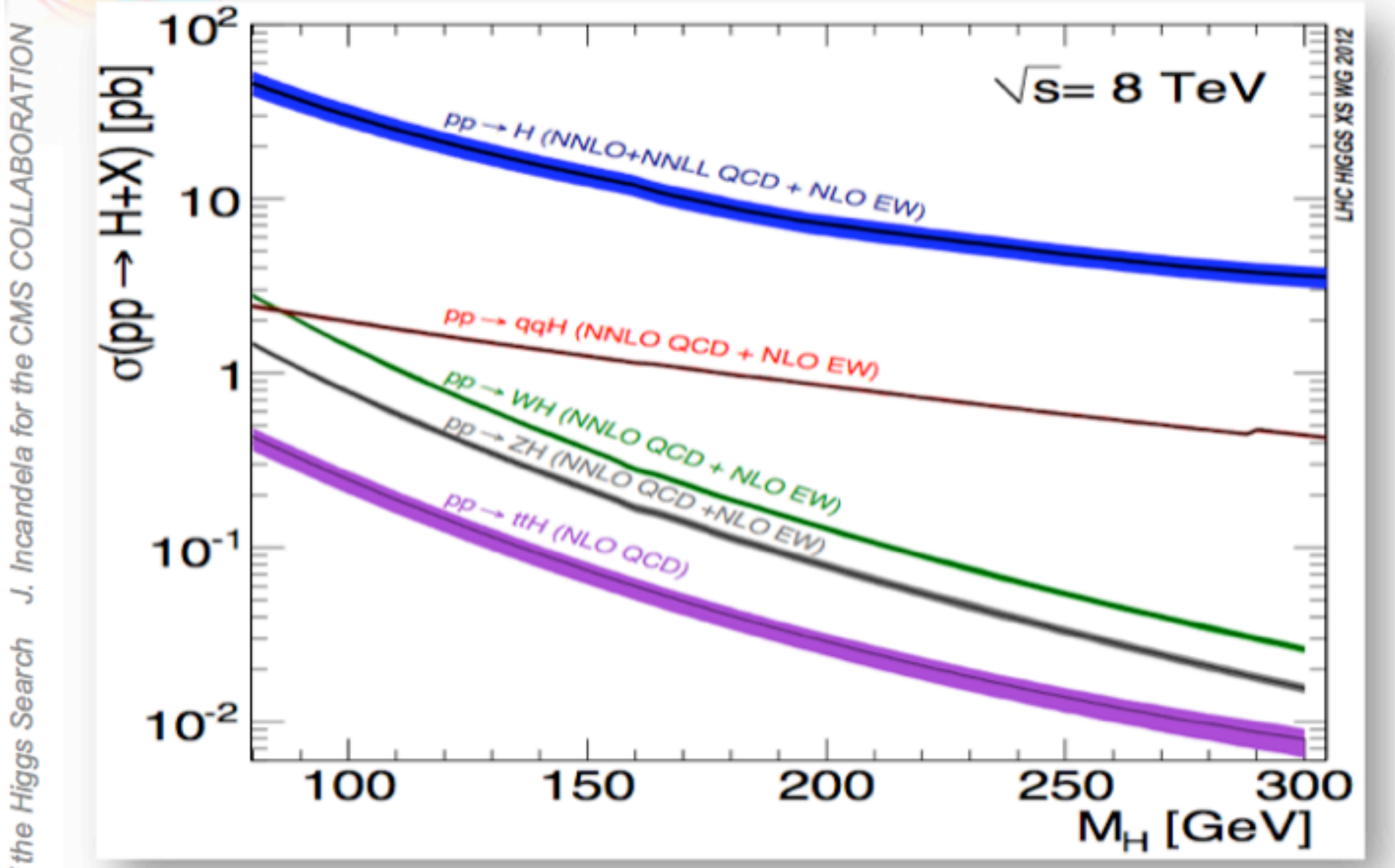
We can add sterile neutrinos, and write completely similar matrices in the neutrino sector. See below, for some remarks.

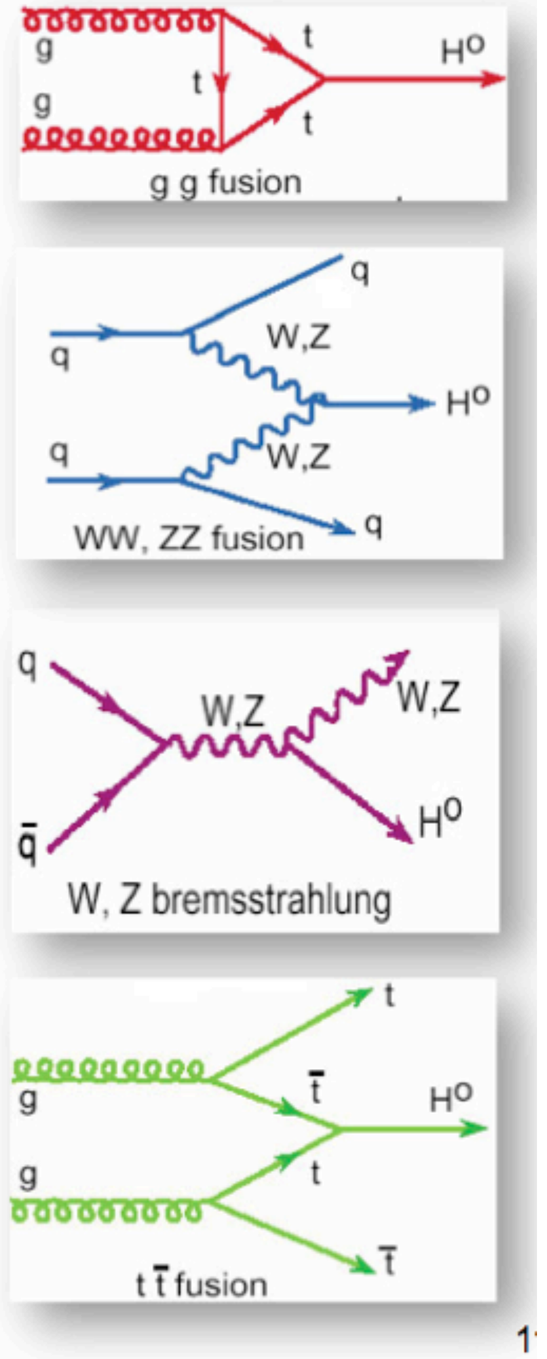
$$j^\mu_+ = (\bar u_L, \bar c_L, \bar t_L)\gamma^\mu \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix} = (\bar u'_L, \bar c'_L, \bar t'_L)\gamma^\mu \tilde V_L^{(q)\dagger} V_L^{(q)} \begin{pmatrix} d'_L \\ s'_L \\ b'_L \end{pmatrix} \qquad V \equiv \tilde V_L^{(q)\dagger} V_L^{(q)}$$

Higgs production



Higgs boson production





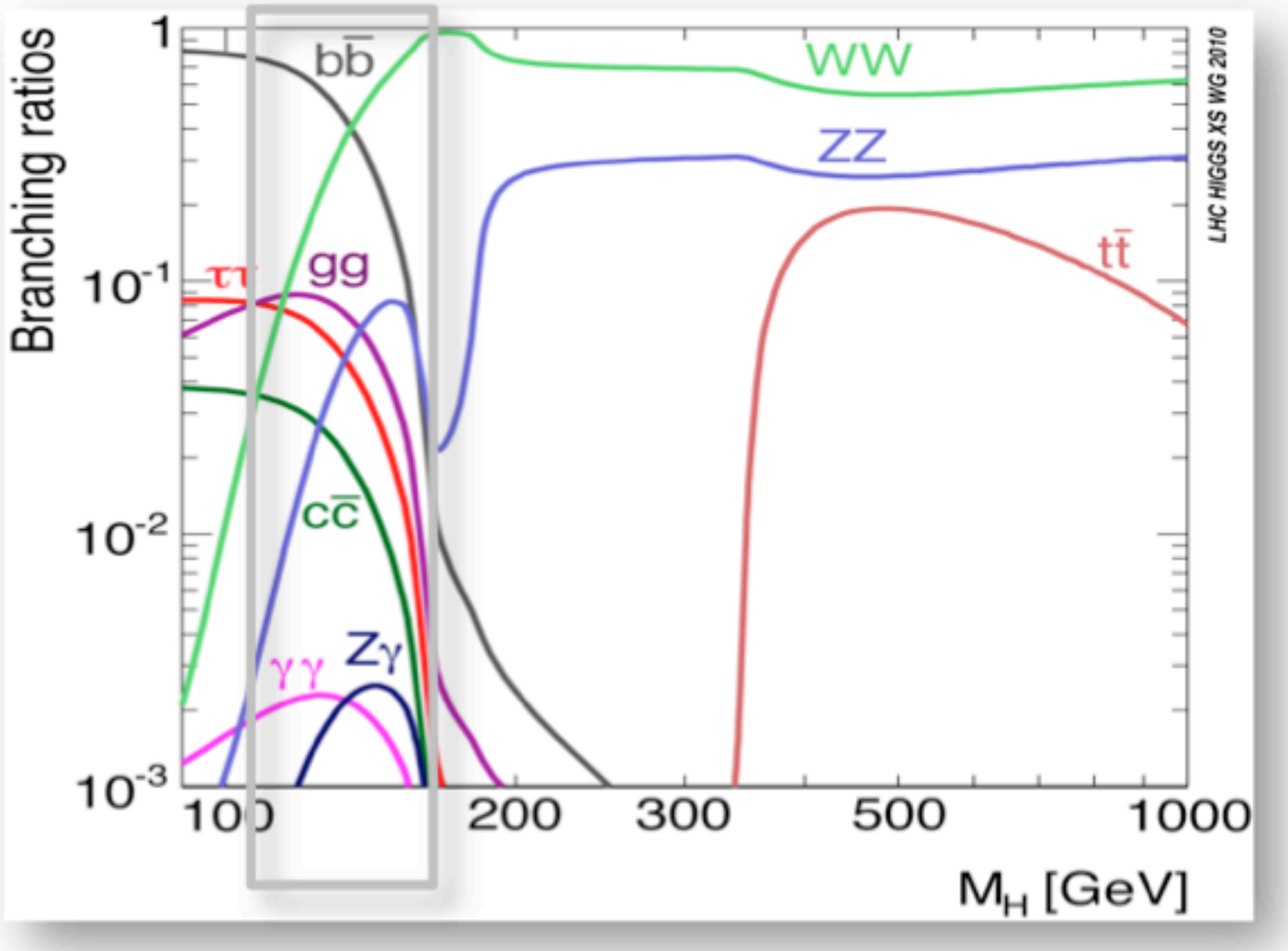
- $\sqrt{s}$=8 TeV: 25-30% higher σ than $\sqrt{s}$=7 TeV at low m_H
- All production modes to be exploited
  - gg VBF VH ttH
  - Latter 3 have smaller cross sections but better S/B in many cases

Courtesy J. Incancella

Higgs boson decays

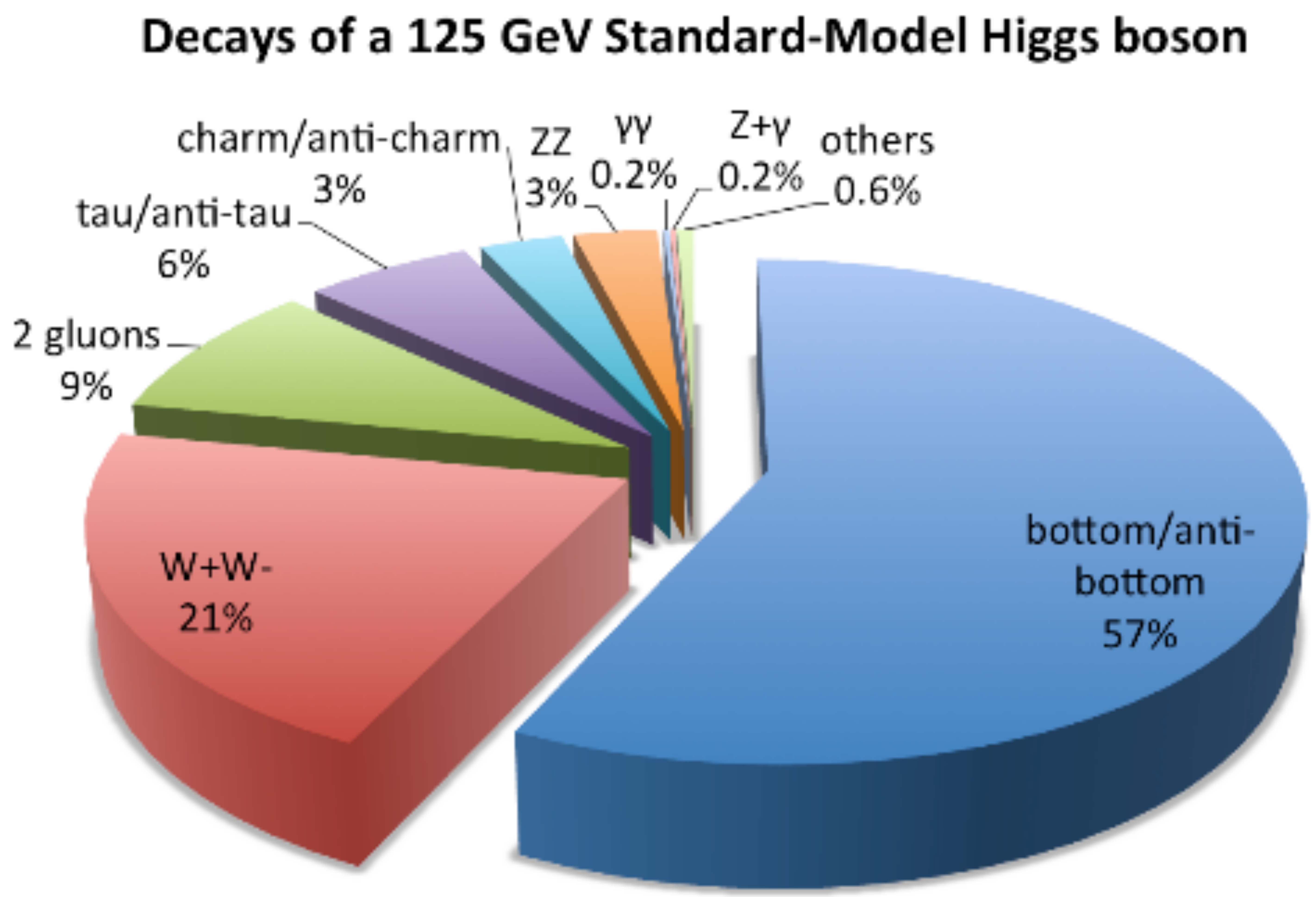
5 decay modes exploited

- High mass: WW, ZZ
- Low mass: $b\bar{b}$, ττ, WW, ZZ, γγ
- Low mass region is very rich but also very challenging:
  main decay modes ($b\bar{b}$, ττ) are hard to identify in the huge background
- Very good mass resolution (1%): H→γγ and H→ZZ→4l



July 4th 2012 The Status of the Higgs Search J. Incandela for the CMS COLLABORATION

Courtesy J. Incandela



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CKM and PMNS mixing matrices

Other lecturers will cover in more details the mixing matrices for quarks and for neutrinos, but I would like to make some simple remarks, that will provide food for thought, and will show how profoundly different are the properties of leptons and quarks. This is a major mystery. Like most properties of the mass matrices in the SM.

It is interesting to look at the textures of the two matrices, which are as different as one could imagine. If they have a common origin, there must be some very fundamental aspects of flavor to be understood.

For simplicity we did not add the couplings to sterile right handed neutrinos, but if we did, we would obtain a matrix similar to CKM, usually knowns as PMNS after their creators. This is what we will do now. You can also show that using dimension five operators it is possible to write a Majorana mass for the SM neutrinos.

Counting CP phases in the mixing matrix is relatively simple. Imagine we have N-generations, then the mixing matrix is in U(N) with NxN real parameters. By redefining the phases of the weak eigenstates and the mass eigenstates, we can remove 2 N-1 phases (the common one is irrelevant). A SO(N) rotation has N (N-1)/2 angles, hence the number of possible CP-violating angles is given below. For N=3 just 1!

$$N^2-(2N-1)-\frac{N(N-1)}{2}=\frac{(N-1)(N-2)}{2}$$

There are several parametrization for the mixing matrices, but let's start with the dumbest one

$$\begin{bmatrix} d' \\ s' \\ b' \end{bmatrix} = \begin{bmatrix} V_{\rm ud} & V_{\rm us} & V_{\rm ub} \\ V_{\rm cd} & V_{\rm cs} & V_{\rm cb} \\ V_{\rm td} & V_{\rm ts} & V_{\rm tb} \end{bmatrix} \begin{bmatrix} d \\ s \\ b \end{bmatrix}.$$

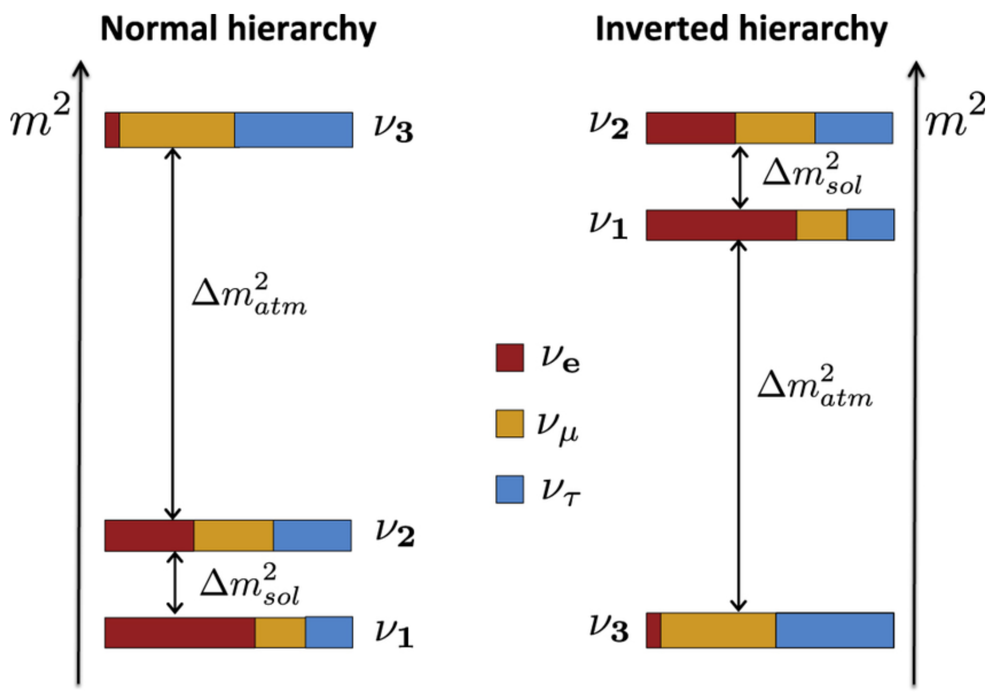
$$\begin{bmatrix} |V_{ud}| & |V_{us}| & |V_{ub}| \\ |V_{cd}| & |V_{cs}| & |V_{cb}| \\ |V_{td}| & |V_{ts}| & |V_{tb}| \end{bmatrix} = \begin{bmatrix} 0.97370\pm0.00014 & 0.2245\pm0.0008 & 0.00382\pm0.00024 \\ 0.221\pm0.004 & 0.987\pm0.011 & 0.0410\pm0.0014 \\ 0.0080\pm0.0003 & 0.0388\pm0.0011 & 1.013\pm0.030 \end{bmatrix}.$$

$$\begin{bmatrix} c_1 & -s_1c_3 & -s_1s_3 \\ s_1c_2 & c_1c_2c_3-s_2s_3e^{i\delta} & c_1c_2s_3+s_2c_3e^{i\delta} \\ s_1s_2 & c_1s_2c_3+c_2s_3e^{i\delta} & c_1s_2s_3-c_2c_3e^{i\delta} \end{bmatrix}.$$

KM original parametrization

Using the similar, dumbest parametrization, the experimental fits give the numbers below, and there have been discussions about normal or inverted hierarchies.

$$\begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} = \begin{bmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix} \quad \begin{bmatrix} |U_{e1}| & |U_{e2}| & |U_{e3}| \\ |U_{\mu 1}| & |U_{\mu 2}| & |U_{\mu 3}| \\ |U_{\tau 1}| & |U_{\tau 2}| & |U_{\tau 3}| \end{bmatrix} = \begin{bmatrix} 0.801 \ldots 0.845 & 0.513 \ldots 0.579 & 0.143 \ldots 0.156 \\ 0.232 \ldots 0.507 & 0.459 \ldots 0.694 & 0.629 \ldots 0.779 \\ 0.260 \ldots 0.526 & 0.470 \ldots 0.702 & 0.609 \ldots 0.763 \end{bmatrix}$$



The textures are so diverse, that one is tempted to take as the zeroth approximation the matrices below. The first is clearly hierarchical, the second is “democratic”, with one nonzero eigenvalue and two zero. So this matrix or its negative are a good zeroth order approximation to the neutrino hierarchy.

Of course this is all wishful thinking, but perhaps there are some clues for BSM model building.

$$[|V|] \sim \begin{pmatrix} 1 & \lambda & \lambda^2 \\ 1 & 1 & \lambda \\ \lambda^2 & \lambda & 1 \end{pmatrix} \qquad [|U|] \sim \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

The Higgs mechanism, how much does it contribute to your weight?

The real answer is: “not much”. We are mostly made of protons and neutrons (and light electrons) in a few dozen nuclei. They are made of up and down quarks, and a quick perusal of the PDG shows the current algebra masses to be rather small, just a few MeV.

The Higgs mechanism is not responsible for most of the mass of the observable matter in the universe…. You are a macroscopic quantum object!!

The mass parameters obtained for the light quarks are too small to explain the masses of protons and neutrons that make up nuclei. From elementary nuclear physics we know:

$$M(Z,A) = Z\,m_p + (A-Z)\,m_n + \Delta M(Z,A)$$

The largest contribution come from the fact that quarks and gluons are highly relativistic objects confined in a space of the order of a fermi. A purely quantum phenomenon due to QCD: the confinement of colour, gluon condensation… A new scale is generated dynamically. Generated with the breaking of scale invariance. Most of the mass of nucleons come from this. Even if the mass parameters of the u,d quarks were set to zero, we would still have nucleons. What makes the study of the strong interactions hard is the fact that:

$$\Lambda >> m_u,\, m_d$$

A large fraction of our mass has its origin in this quantum phenomenon of confinement. We are indeed macroscopic quantum objects! There is also a beautiful analogy with the BEH mechanism, but of a more subtle type as a dual superconductor.

So when somebody says that the Higgs mechanism is the origin of mass, no matter how important the person is, do beware… (s)he is not right. The story is far more fascinating. The deeply quantum properties of pure, confined glue, is what makes most of your mass.

Discrete symmetries

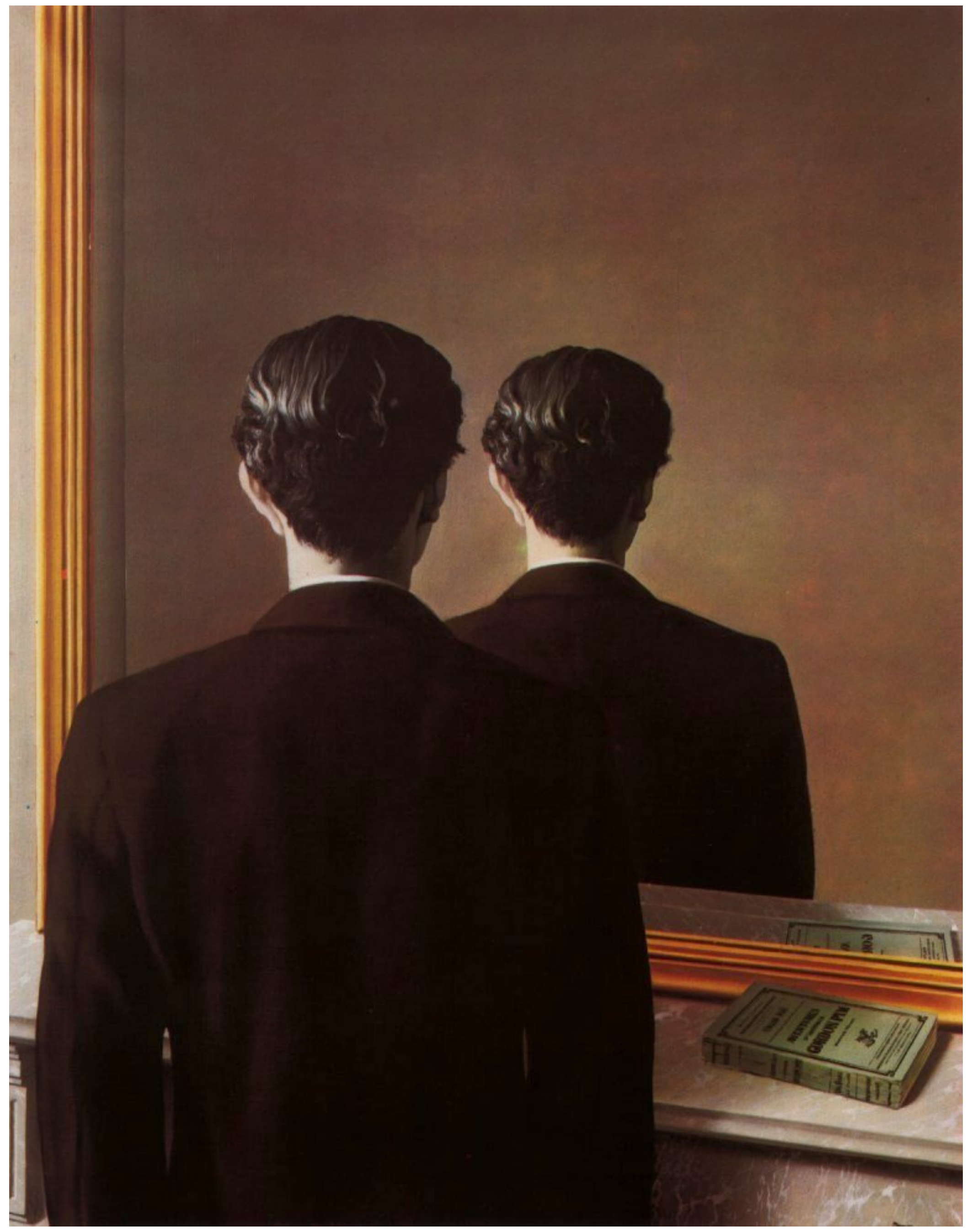
In the classical world, we have invariance uncer P,C,T. All we had was E&M and gravity.

In QFT they are not guaranteed in fact P,C,T, CP are broken symmetries. The only one that survives so far is CPT. It has several important consequences. CP violation is fundamental in the generation of matter. In the SM we need at least three families. It could be the neutrinos! That is really weird and extraordinary… We do not really know what we will find through the looking glass

The existence of antiparticles with the same mass and decay rate

The connection between spin and statistics

T-reversal and CPT are the only ones implemented by anti-unitary operators



A parity invariant mirror…

$$\begin{array}{ccc} \mathbf{q}_0, \mathbf{p}_0 & \xrightarrow{\ \mathrm{T}\ } & \mathbf{q}_0, -\mathbf{p}_0 \\ \downarrow t & & \uparrow t \\ \mathbf{q}(t), \mathbf{p}(t) & \xrightarrow{\ \mathrm{T}\ } & \mathbf{q}(t), -\mathbf{p}(t) \end{array}$$

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Deep Properties of QFT

It is a remarkable fact that CPT and Spin-Statistics follow from rather deep consequences of relativistic invariance.

More precisely:

Relativistic Invariance and locality

Unitarity, the existence of a lowest energy state (the vacuum) and the positivity of the energy

Automatically lead to the CPT and spin-statistics, in fact both are essentially the same theorem. This follows from the beautiful magic of analytic continuation. Recall that the Lorentz group has four sheets. Related by P, T and PT. We only expect the SM to be invariant under the sheet connected to the identity. All other discrete symmetries being violated.

- $\mathfrak{L}_+^\uparrow$: proper, orthochronous transformations with $\det\Lambda = 1, \Lambda^0{}_0 \geqslant 1$.
- $\mathfrak{L}_-^\uparrow$: improper, orthochronous transformations with $\det\Lambda = -1, \Lambda^0{}_0 \geqslant 1$.
- $\mathfrak{L}_-^\downarrow$: improper, non-orthochronous transformations with $\det\Lambda = -1, \Lambda^0{}_0 \leqslant -1$.
- $\mathfrak{L}_+^\downarrow$: proper, non-orthochronous transformations with $\det\Lambda = 1, \Lambda^0{}_0 \leqslant -1$.

$$\mathfrak{L}_+^\uparrow \xrightarrow{\mathscr{P}} \mathfrak{L}_-^\uparrow, \qquad \mathfrak{L}_+^\uparrow \xrightarrow{\mathscr{T}} \mathfrak{L}_-^\downarrow, \qquad \mathfrak{L}_+^\uparrow \xrightarrow{\mathscr{P}\mathscr{T}} \mathfrak{L}_+^\downarrow$$

$$x = x^0 + \sigma \cdot \mathbf{x}, \qquad \det(x) = (x^0)^2 - \mathbf{x}^2 \qquad x \to AxA^\dagger, \quad \det A = 1$$

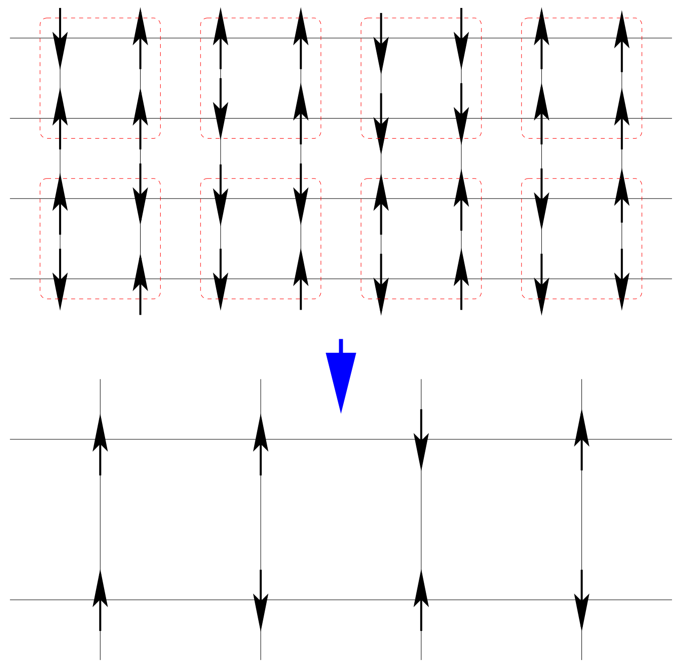
An equivalent way to write Lorentz transformations, A is a matrix in SL(2,C), complex 2x2 matrices with unit determinant. Using the spectral properties of the momentum vector, that must be inside the forward light-cone, and properties of Fourier transforms, it can be shown that the physical correlation functions can be continued analytically to complex values of momenta and coordinates, and they are also invariant under the complexified Lorentz group, two copies of SL(2,C).

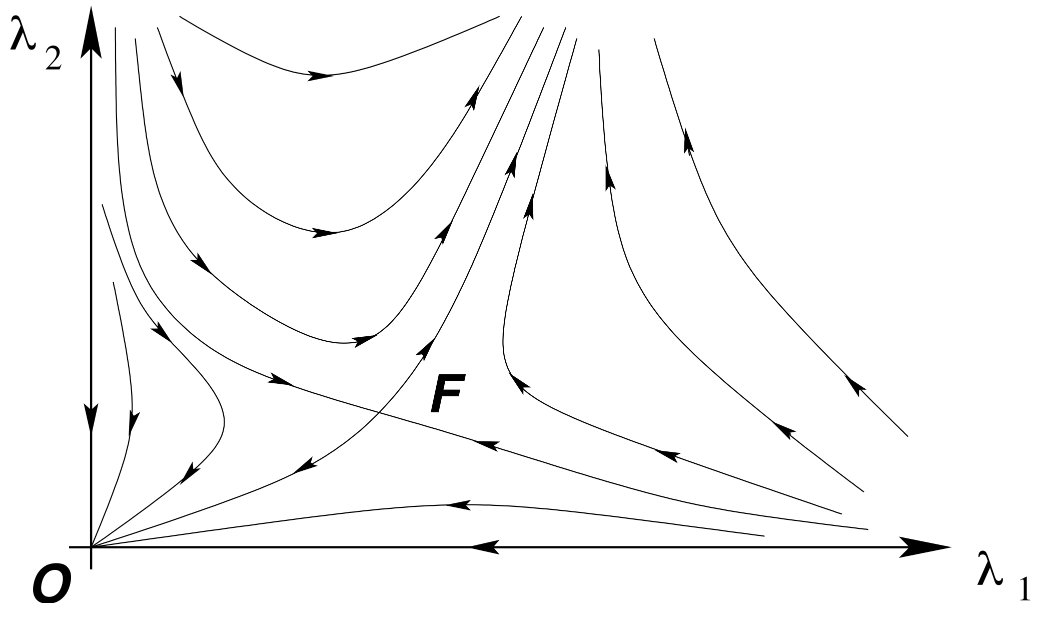
$$x \to AxB^T, \quad A, B \in SL(2, C)$$

Note that SL(2,C) is simply connected, if we choose $B = -1, \; x \to -x$ $\quad \mathfrak{L}_+^\uparrow \cup \mathfrak{L}_+^\downarrow$ Are then connected, and using the anti linearity of T, eventually leads to CPT

Scale invariance, renormalization

$\times \lambda^{-1}$





Renormalisation deals with the scale dependence of the the physics even if the original theory is scale invariant.

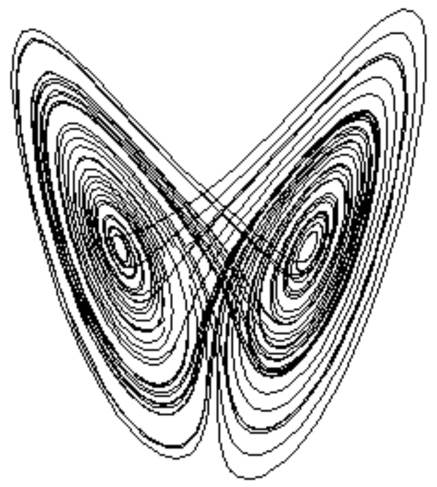
Virtual phenomena can get more complicated or simplify as we move to larger or shorter distances

$$x^\mu \longrightarrow \lambda x^\mu, \qquad \phi(x) \longrightarrow \lambda^{-\Delta}\phi(\lambda^{-1}x),$$

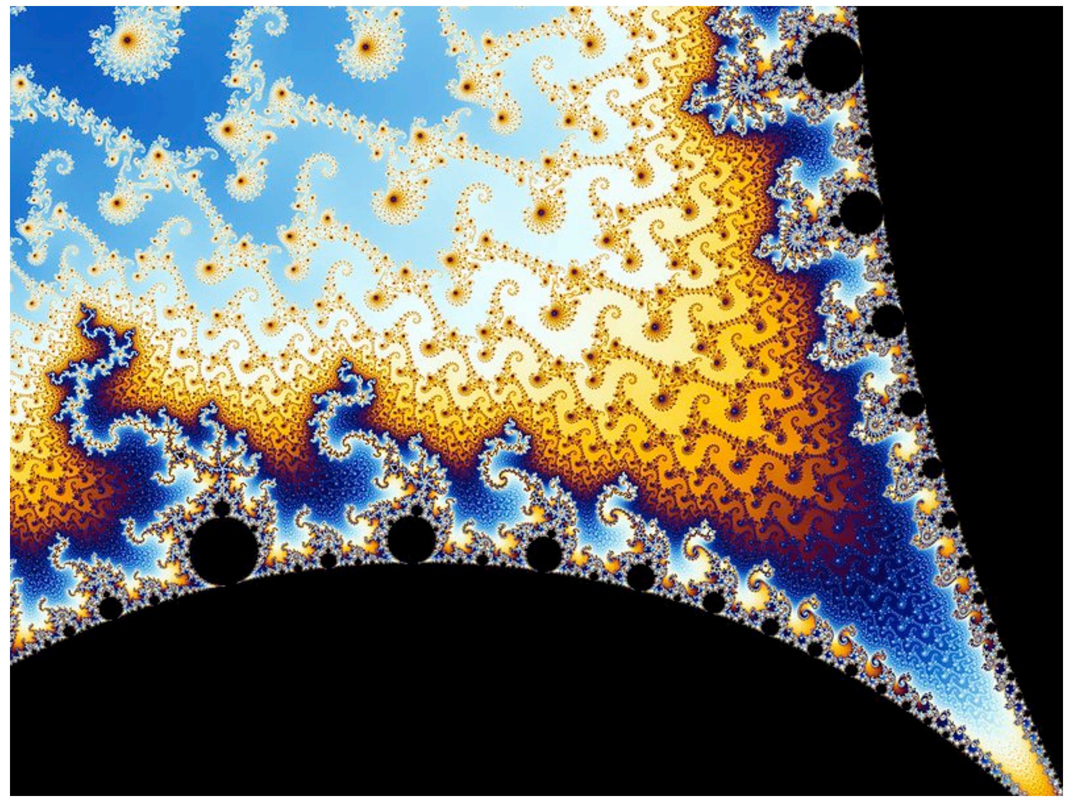
$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{g}{4!}\phi^4, \qquad \mathscr{L} \longrightarrow \lambda^{-4}\mathscr{L}[\phi]$$

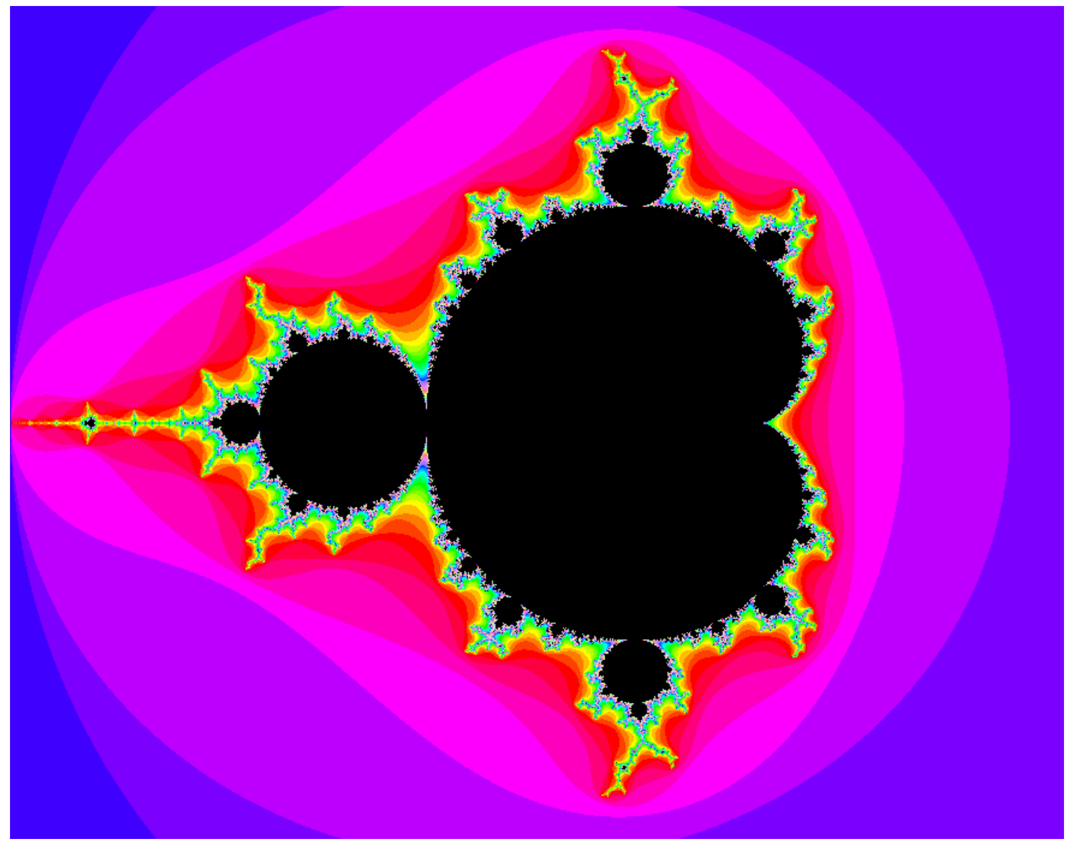
$$H \xrightarrow{\mathscr{R}} H^{(1)} \xrightarrow{\mathscr{R}} H^{(2)} \xrightarrow{\mathscr{R}} \dots \xrightarrow{\mathscr{R}} H_\star.$$

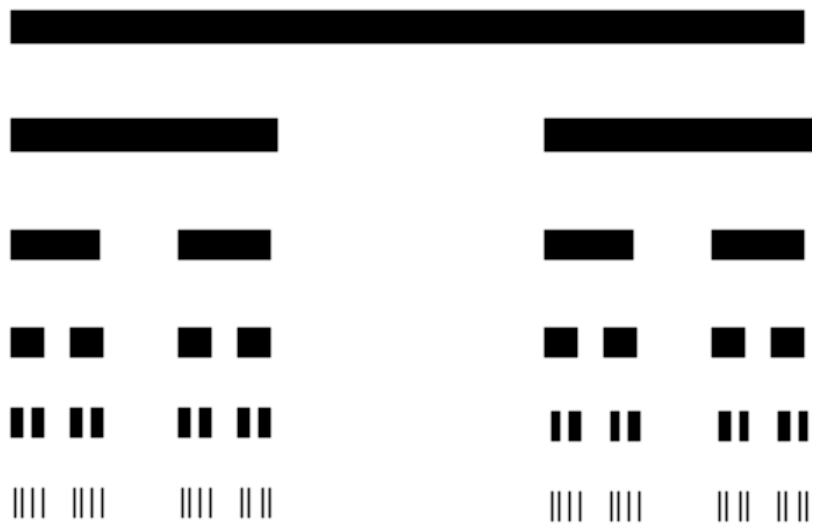
In relativistic QFT we seem to get only fixed points, no limit cycles nor strange attractors, near fixed point all is FAPP like renormalizable theories

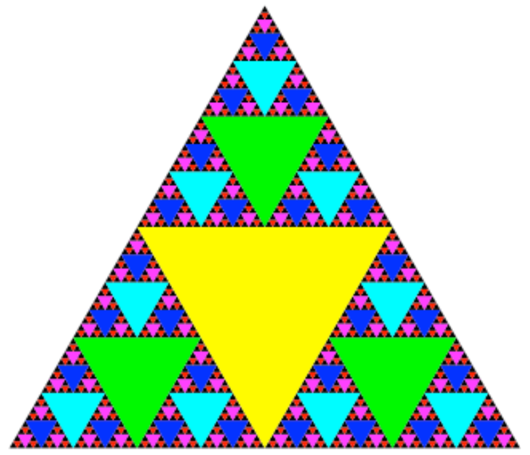


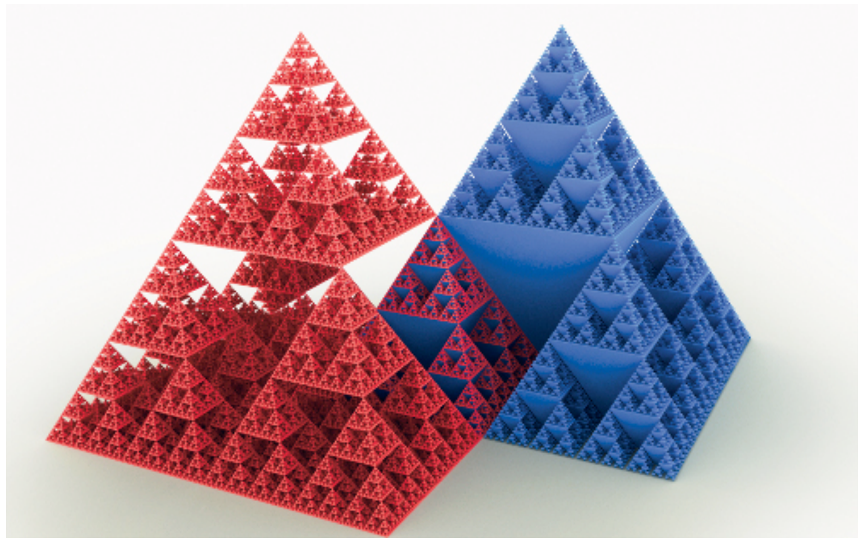
Scale invariance, fractals







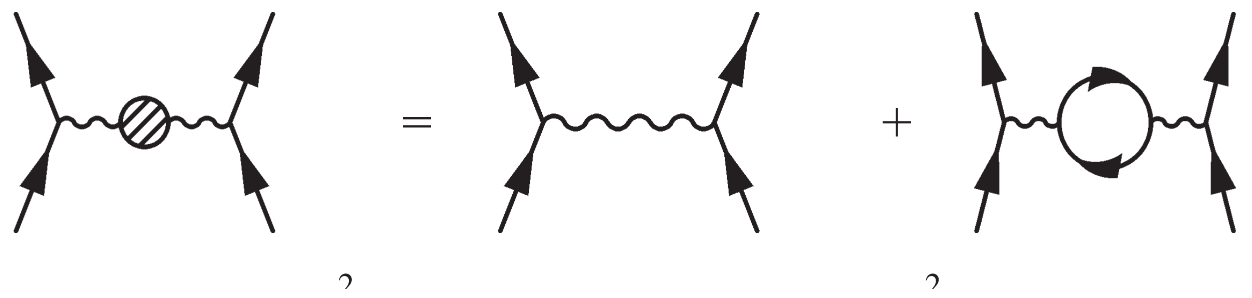




Why not in QFT? It would be rather remarkable if in a field theory we found strange attractors at high or low energies.

Lorentz or Poincaré invariance play an important role in determining the possible limit structures. Only fixed points…

Fixed points beta functions



$$\eta_{\alpha\beta}\left(\bar{v}_e\gamma^\alpha u_e\right)\left\{\frac{e^2}{4\pi q^2}\left[1+\frac{e^2}{12\pi^2}\log\left(\frac{q^2}{\Lambda^2}\right)\right]\right\}\left(\bar{v}_\mu\gamma^\beta u_\mu\right)$$

$$e(\mu)^2 = e(\Lambda)^2_{\text{bare}}\left[1+\frac{e(\Lambda)^2_{\text{bare}}}{12\pi^2}\log\left(\frac{\mu^2}{\Lambda^2}\right)\right]$$

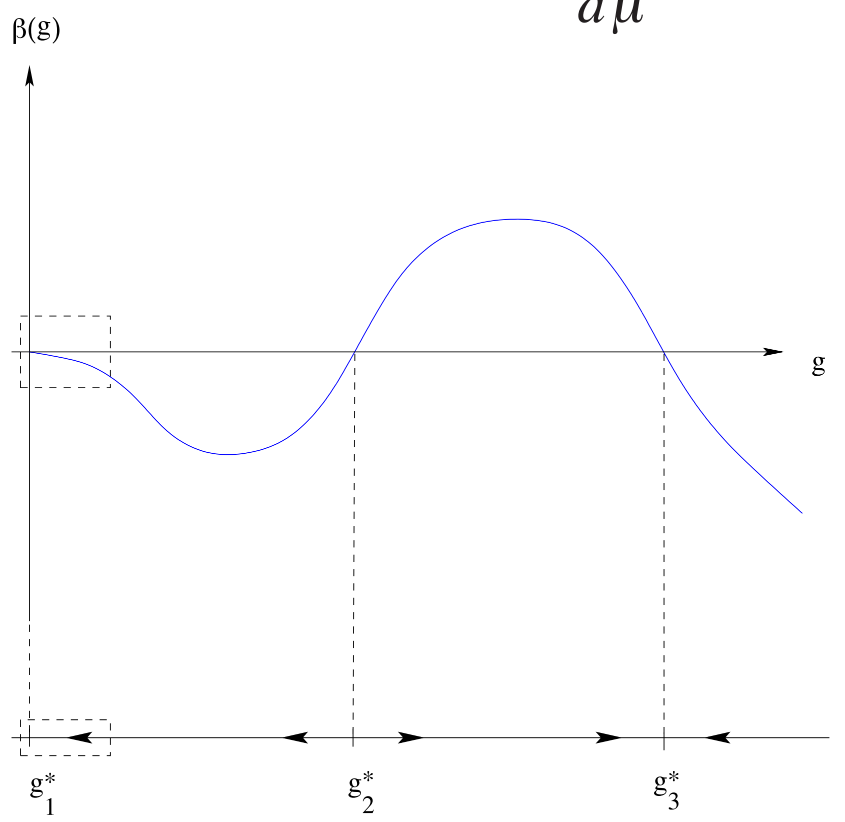
$$\beta(g) = \mu\frac{dg}{d\mu}.$$

$$\beta(e)_{\text{QED}} = \frac{e^3}{12\pi^2}.$$

At one loop   IR free (QED)

$$\beta(g) = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right).$$

UV free (QCD)



$$\beta'(g)|_{g^*} > 0 \quad , \qquad \mu\frac{dg}{d\mu} = \beta'(g-g^*)+\ldots$$

$$\mu\uparrow \quad , \qquad g\uparrow$$

$$\beta'(g)|_{g^*} < 0 \quad , \qquad \mu\frac{dg}{d\mu} = \beta'(g-g^*)+\ldots$$

$$\mu\uparrow \quad , \qquad g\downarrow$$

$$\beta(g^*) = 0.$$

There is a dynamically generated scale responsible for most of the mass of the nucleons. Gluon condensation and color confinement

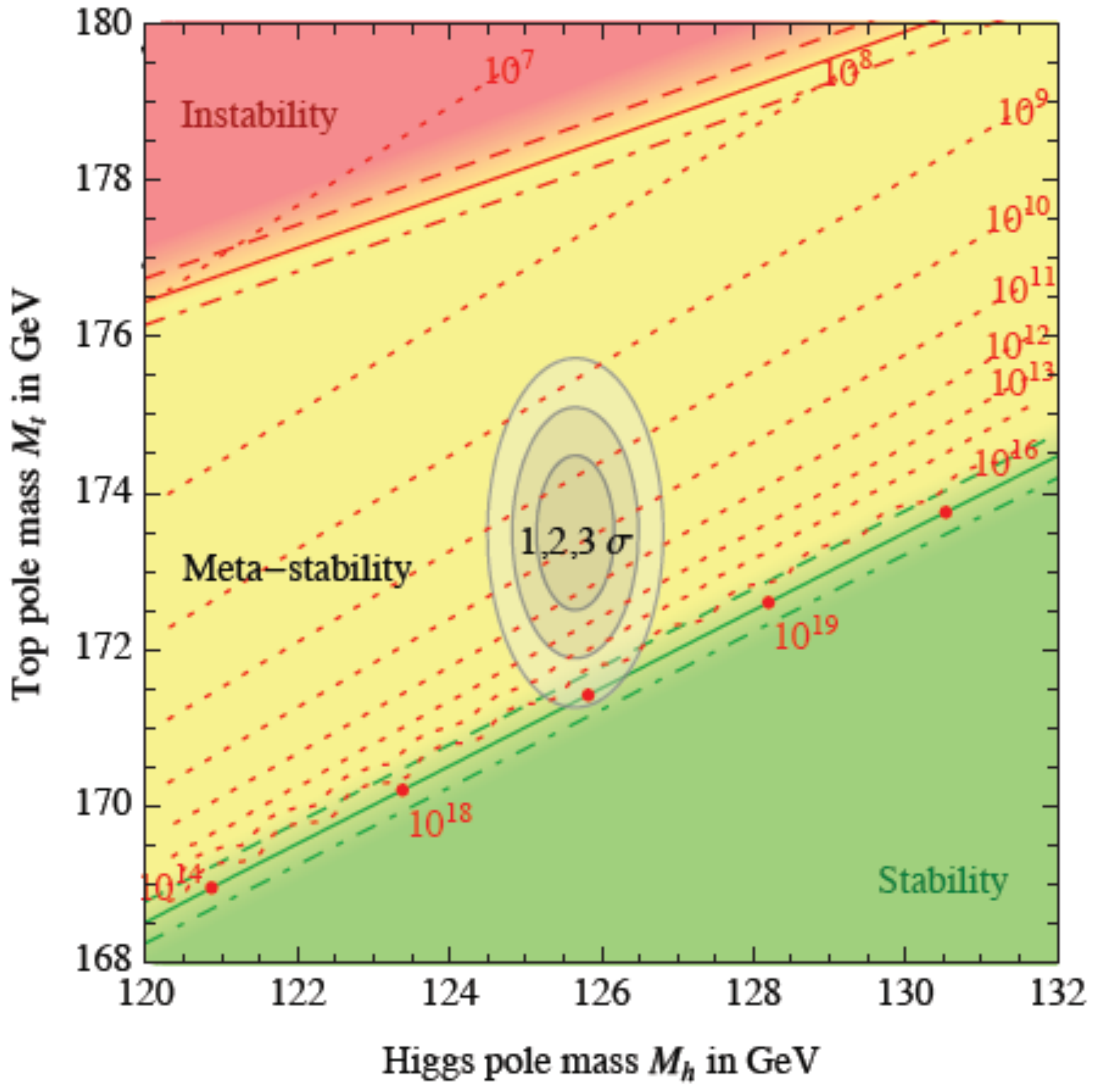
$$\langle \mathbf{p}^2\rangle = \Lambda^2_{\text{QCD}}. \quad \Lambda_{\text{QCD}} \gg m_u, m_d$$

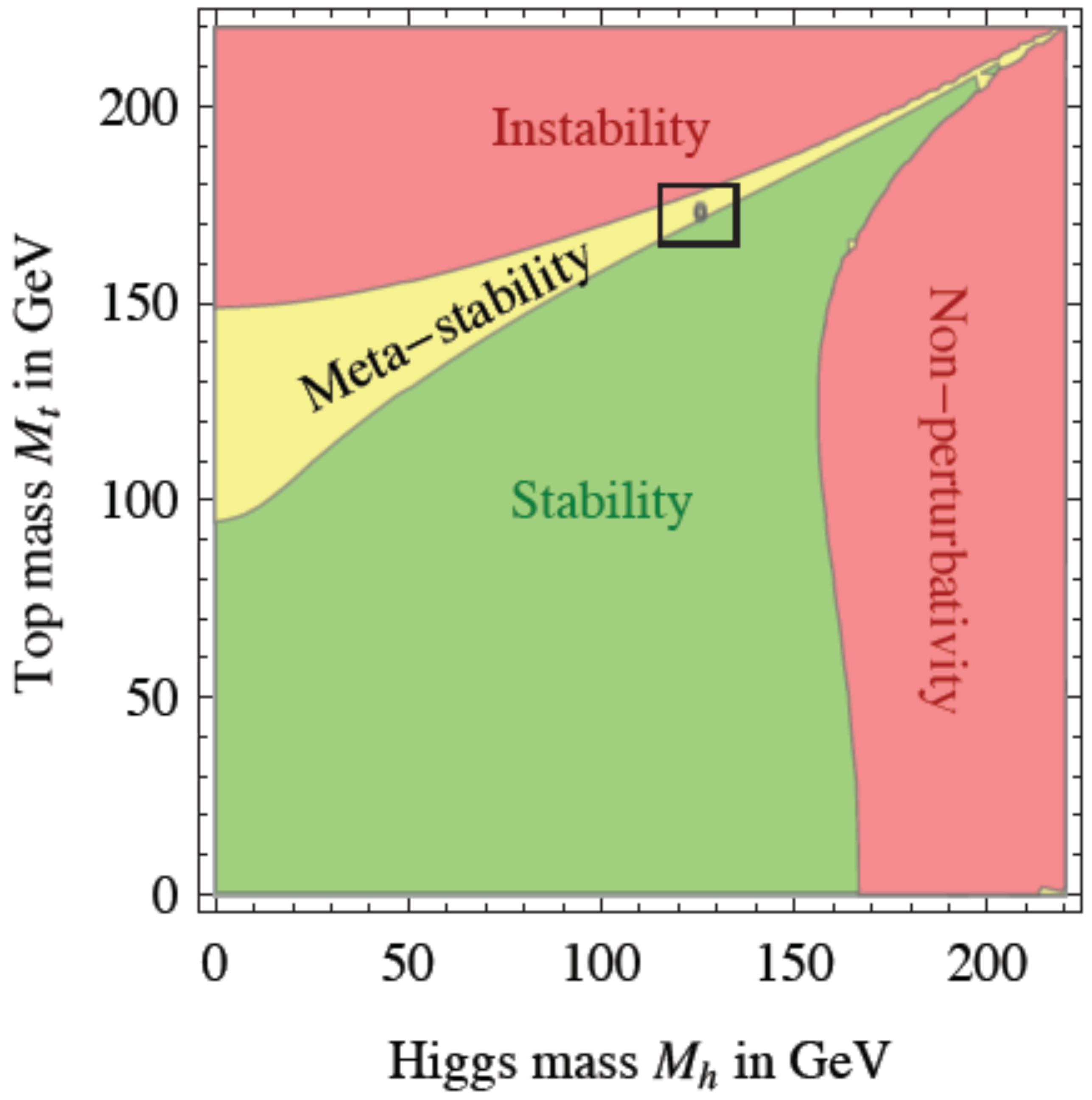
An unexpected result of the LHC

The Planck Chimney

Use the renormalization group evolution associated to the heaviest masses: top and Higgs, and we explore the possibility of the vanishing of the Higgs quartic self-coupling, hence the current SM vacuum is unstable. If there are no additional contributions, at an intermediate scale the Mexican hat flap drops…

$$\sim \sqrt{M_W M_{Pl}}$$





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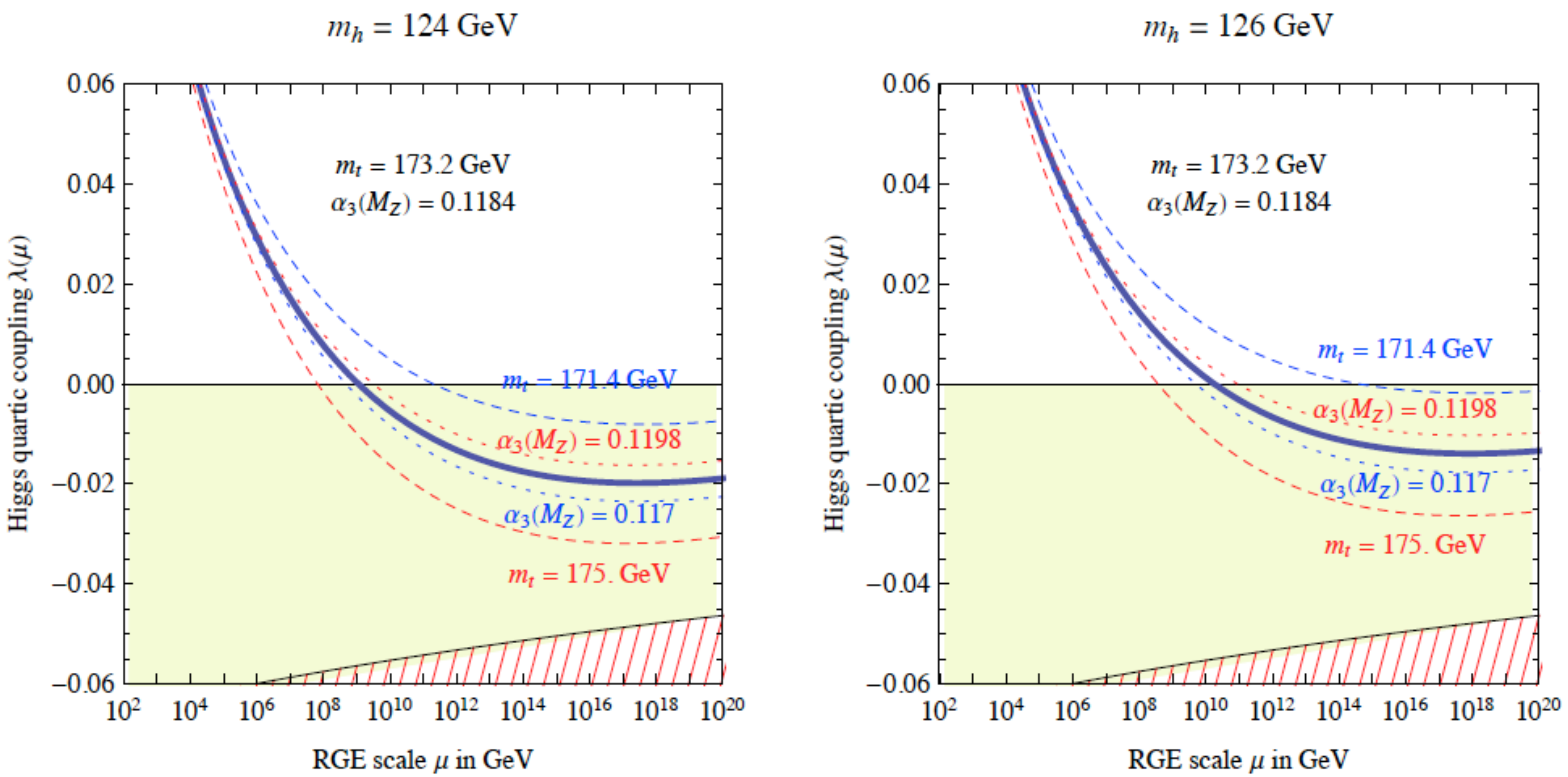


CERN HEP 12/2022

SIMONS CENTER FOR GEOMETRY AND PHYSICS Stony Brook University

QFT is a vast and complex subject

SM is a major collective achievement stretching over more than a century

It summarizes our knowledge of the fundamental laws of Nature

And also our ignorance

Many puzzles and unanswered questions remain:

Dark matter, dark energy, flavor dynamics, where is the inflation, if any

We may be at the end of a cycle. Perhaps the symmetry paradigm has been exhausted.

Naturalness, a red herring?

Gravity into the picture finally?

Anthropism, landscapes, multiverses?

Hopefully we are entering a golden decade…

Are there many elephants in the room?

Thank you!!

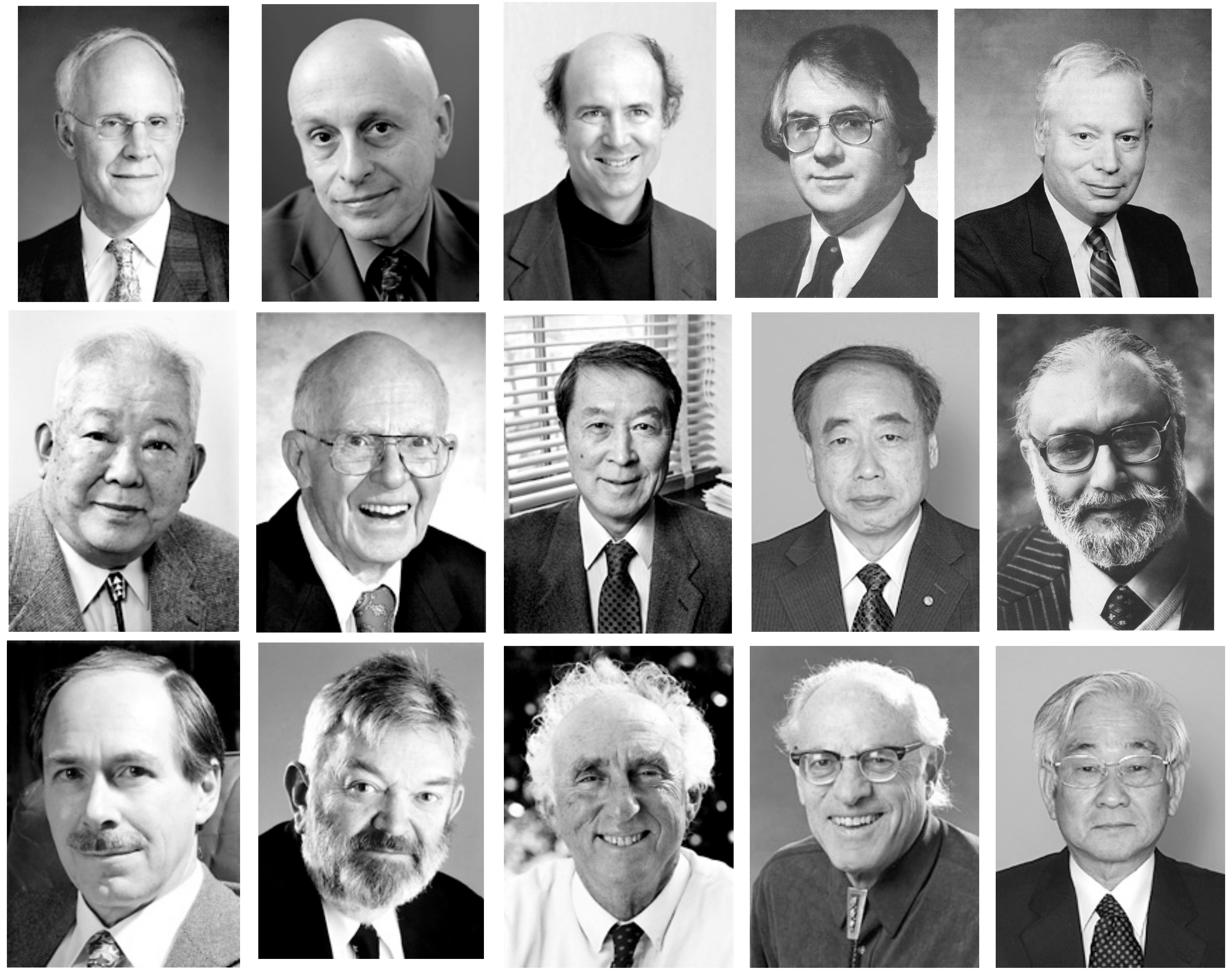
Multiverse landscape:

Did the Almighty have any choice in creating the Universe? (Einstein)

Einstein's view of unification may be fundamentally flawed…

Some parameters maybe fundamental, but some maybe purely environmental… which is which?

Who is Who in the Standard Model (just Nobels)

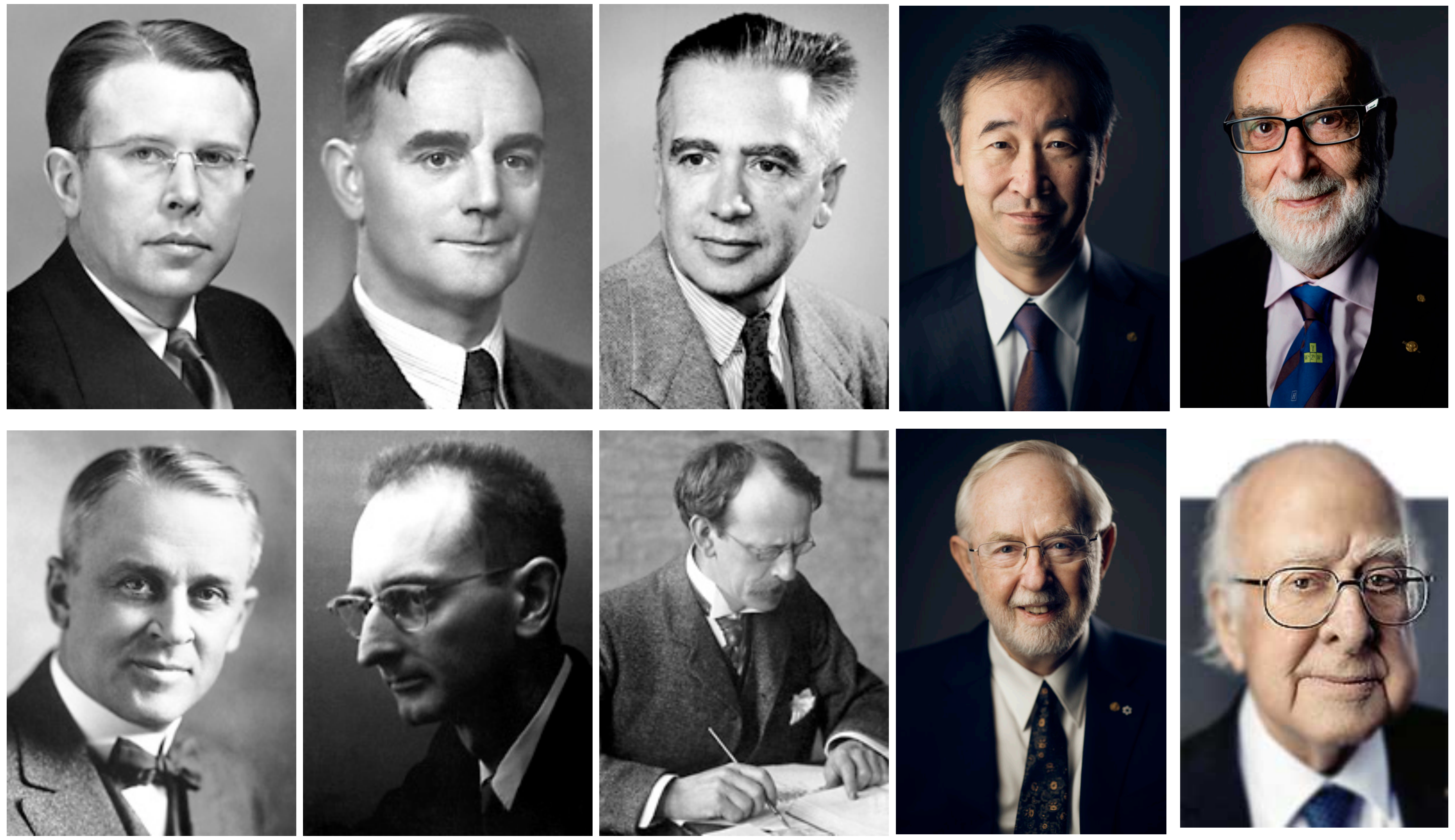


Prepared with Werner Riegler (CERN EP)

Continued…

Continued…

Continued…



87 Nobel Prices related to the Development of the Standard Model

31 for Standard Model Experiments
13 for Standard Model Instrumentation and Experiments
3 for Standard Model Instrumentation
21 for Standard Model Theory
9 for Quantum Mechanics Theory
9 for Quantum Mechanics Experiments
1 for Relativity

Standard Model Experimental (31)


SM exp
2002, THE NOBEL PRIZE IN PHYSICS
Raymond Davis Jr. "for pioneering contributions to astrophysics, in particular for the detection of cosmic neutrinos"


SM exp
2002, THE NOBEL PRIZE IN PHYSICS
Masatoshi Koshiba "for pioneering contributions to astrophysics, in particular for the detection of cosmic neutrinos"


SM exp
1995, THE NOBEL PRIZE IN PHYSICS
Martin L. Perl "for the discovery of the tau lepton"


SM exp
1995, THE NOBEL PRIZE IN PHYSICS
Frederick Reines "for the detection of the neutrino"


SM exp
1990, THE NOBEL PRIZE IN PHYSICS
Jerome I. Friedman "for their pioneering investigations concerning deep inelastic scattering of electrons on protons and bound neutrons, which have been of essential importance for the development of the quark model in particle physics"


SM exp
1990, THE NOBEL PRIZE IN PHYSICS
Henry W. Kendall "for their pioneering investigations concerning deep inelastic scattering of electrons on protons and bound neutrons, which have been of essential importance for the development of the quark model in particle physics"


SM exp
1990, THE NOBEL PRIZE IN PHYSICS
Richard E. Taylor "for their pioneering investigations concerning deep inelastic scattering of electrons on protons and bound neutrons, which have been of essential importance for the development of the quark model in particle physics"


SM exp
1980, THE NOBEL PRIZE IN PHYSICS
James Watson Cronin "for the discovery of violations of fundamental symmetry principles in the decay of neutral K-mesons"


SM exp
1980, THE NOBEL PRIZE IN PHYSICS
Val Logsdon Fitch "for the discovery of violations of fundamental symmetry principles in the decay of neutral K-mesons"


SM exp
1976, THE NOBEL PRIZE IN PHYSICS
Burton Richter "for their pioneering work in the discovery of a heavy elementary particle of a new kind"


SM exp
1976, THE NOBEL PRIZE IN PHYSICS
Samuel Chao Chung Ting "for their pioneering work in the discovery of a heavy elementary particle of a new kind"


SM exp
1961, THE NOBEL PRIZE IN PHYSICS
Robert Hofstadter "for his pioneering studies of electron scattering in atomic nuclei and for his thereby achieved discoveries concerning the structure of the nucleons"


SM exp
1959, THE NOBEL PRIZE IN PHYSICS
Emilio Gino Segrè "for their discovery of the antiproton"


SM exp
1959, THE NOBEL PRIZE IN PHYSICS
Owen Chamberlain "for their discovery of the antiproton"


SM exp

1955, THE NOBEL PRIZE IN PHYSICS
Willis Eugene Lamb "for his discoveries concerning the fine structure of the hydrogen spectrum"


SM exp

1955, THE NOBEL PRIZE IN PHYSICS
Polykarp Kusch "for his precision determination of the magnetic moment of the electron"


SM exp th

1938, THE NOBEL PRIZE IN PHYSICS
Enrico Fermi "for his demonstrations of the existence of new radioactive elements produced by neutron irradiation, and for his related discovery of nuclear reactions brought about by slow neutrons"


SM exp

1936, THE NOBEL PRIZE IN PHYSICS
Victor Franz Hess "for his discovery of cosmic radiation"


SM exp

1936, THE NOBEL PRIZE IN PHYSICS
Carl David Anderson "for his discovery of the positron"

1935, THE NOBEL PRIZE IN CHEMISTRY
Frédéric Joliot "in recognition of their synthesis of new radioactive elements"

1935, THE NOBEL PRIZE IN CHEMISTRY
Irène Joliot-Curie "in recognition of their synthesis of new radioactive elements"


SM exp

1935, THE NOBEL PRIZE IN PHYSICS
James Chadwick "for the discovery of the neutron"


SM exp

1927, THE NOBEL PRIZE IN PHYSICS
Arthur Holly Compton "for his discovery of the effect named after him"


SM exp

1923, THE NOBEL PRIZE IN PHYSICS
Robert Andrews Millikan "for his work on the elementary charge of electricity and on the photoelectric effect"


SM exp

1922, THE NOBEL PRIZE IN CHEMISTRY
Francis William Aston "for his discovery, by means of his mass spectrograph, of isotopes, in a large number of non-radioactive elements, and for his enunciation of the whole-number rule"


SM exp

1908, THE NOBEL PRIZE IN CHEMISTRY
Ernest Rutherford "for his investigations into the disintegration of the elements, and the chemistry of radioactive substances"


SM exp

1905, THE NOBEL PRIZE IN PHYSICS
Philipp Eduard Anton von Lenard "for his work on cathode rays"


SM exp

1903, THE NOBEL PRIZE IN PHYSICS
Antoine Henri Becquerel "in recognition of the extraordinary services he has rendered by his discovery of spontaneous radioactivity"


SM exp

1903, THE NOBEL PRIZE IN PHYSICS
Pierre Curie "in recognition of the extraordinary services they have rendered by their joint researches on the radiation phenomena discovered by Professor Henri Becquerel"


SM exp

1903, THE NOBEL PRIZE IN PHYSICS
Marie Curie, née Sklodowska "in recognition of the extraordinary services they have rendered by their joint researches on the radiation phenomena discovered by Professor Henri Becquerel"


SM exp

1906, THE NOBEL PRIZE IN PHYSICS
Joseph John Thomson "in recognition of the great merits of his theoretical and experimental investigations on the conduction of electricity by gases"

Standard Model Instrumentation and Experimental (13)


SM instr exp
1988, THE NOBEL PRIZE IN PHYSICS
Leon M. Lederman "for the neutrino beam method and the demonstration of the doublet structure of the leptons through the discovery of the muon neutrino"


SM instr exp
1988, THE NOBEL PRIZE IN PHYSICS
Melvin Schwartz "for the neutrino beam method and the demonstration of the doublet structure of the leptons through the discovery of the muon neutrino"


SM instr exp
1988, THE NOBEL PRIZE IN PHYSICS
Jack Steinberger "for the neutrino beam method and the demonstration of the doublet structure of the leptons through the discovery of the muon neutrino"


SM instr exp
1984, THE NOBEL PRIZE IN PHYSICS
Carlo Rubbia "for their decisive contributions to the large project, which led to the discovery of the field particles W and Z, communicators of weak interaction"


SM instr exp
1984, THE NOBEL PRIZE IN PHYSICS
Simon van der Meer "for their decisive contributions to the large project, which led to the discovery of the field particles W and Z, communicators of weak interaction"


SM instr exp
1968, THE NOBEL PRIZE IN PHYSICS
Luis Walter Alvarez "for his decisive contributions to elementary particle physics, in particular the discovery of a large number of resonance states, made possible through his development of the technique of using hydrogen bubble chamber and data analysis"


SM instr exp
1954, THE NOBEL PRIZE IN PHYSICS
Walther Bothe "for the coincidence method and his discoveries made therewith"


SM intrs exp

1951, THE NOBEL PRIZE IN PHYSICS
Sir John Douglas Cockcroft "for their pioneer work on the transmutation of atomic nuclei by artificially accelerated atomic particles"


SM instr exp
1951, THE NOBEL PRIZE IN PHYSICS
Ernest Thomas Sinton Walton "for their pioneer work on the transmutation of atomic nuclei by artificially accelerated atomic particles"


SM instr exp
1950, THE NOBEL PRIZE IN PHYSICS
Cecil Frank Powell "for his development of the photographic method of studying nuclear processes and his discoveries regarding mesons made with this method"


SM instr exp
1948, THE NOBEL PRIZE IN PHYSICS
Patrick Maynard Stuart Blackett "for his development of the Wilson cloud chamber method, and his discoveries therewith in the fields of nuclear physics and cosmic radiation"


SM instr exp
1943, THE NOBEL PRIZE IN PHYSICS
Otto Stern "for his contribution to the development of the molecular ray method and his discovery of the magnetic moment of the proton"


SM intstr exp
1939, THE NOBEL PRIZE IN PHYSICS
Ernest Orlando Lawrence "for the invention and development of the cyclotron and for results obtained with it, especially with regard to artificial radioactive elements"

Standard Model Instrumentation (3)


SM instr
1992, THE NOBEL PRIZE IN PHYSICS
Georges Charpak "for his invention and development of particle detectors, in particular the multiwire proportional chamber"


SM instr
1960, THE NOBEL PRIZE IN PHYSICS
Donald Arthur Glaser "for the invention of the bubble chamber"


SM instr
1927, THE NOBEL PRIZE IN PHYSICS
Charles Thomson Rees Wilson "for his method of making the paths of electrically charged particles visible by condensation of vapour"

Standard Model Theory (21)


SM th
2008, THE NOBEL PRIZE IN PHYSICS
Yoichiro Nambu "for the discovery of the mechanism of spontaneous broken symmetry in subatomic physics"


SM th
2008, THE NOBEL PRIZE IN PHYSICS
Makoto Kobayashi "for the discovery of the origin of the broken symmetry which predicts the existence of at least three families of quarks in nature"


SM th
2008, THE NOBEL PRIZE IN PHYSICS
Toshihide Maskawa "for the discovery of the origin of the broken symmetry which predicts the existence of at least three families of quarks in nature"


SM th
2004, THE NOBEL PRIZE IN PHYSICS
David J. Gross "for the discovery of asymptotic freedom in the theory of the strong interaction"


SM th
2004, THE NOBEL PRIZE IN PHYSICS
H. David Politzer "for the discovery of asymptotic freedom in the theory of the strong interaction"


SM th
2004, THE NOBEL PRIZE IN PHYSICS
Frank Wilczek "for the discovery of asymptotic freedom in the theory of the strong interaction"


SM th
1999, THE NOBEL PRIZE IN PHYSICS
Gerardus 't Hooft "for elucidating the quantum structure of electroweak interactions in physics"


SM th
1999, THE NOBEL PRIZE IN PHYSICS
Martinus J.G. Veltman "for elucidating the quantum structure of electroweak interactions in physics"


SM th
1982, THE NOBEL PRIZE IN PHYSICS
Kenneth G. Wilson "for his theory for critical phenomena in connection with phase transitions"


SM th
1979, THE NOBEL PRIZE IN PHYSICS
Sheldon Lee Glashow "for their contributions to the theory of the unified weak and electromagnetic interaction between elementary particles, including, inter alia, the prediction of the weak neutral current"


SM th
1979, THE NOBEL PRIZE IN PHYSICS
Abdus Salam "for their contributions to the theory of the unified weak and electromagnetic interaction between elementary particles, including, inter alia, the prediction of the weak neutral current"


SM th
1979, THE NOBEL PRIZE IN PHYSICS
Steven Weinberg "for their contributions to the theory of the unified weak and electromagnetic interaction between elementary particles, including, inter alia, the prediction of the weak neutral current"


SM th
1969, THE NOBEL PRIZE IN PHYSICS
Murray Gell-Mann "for his contributions and discoveries concerning the classification of elementary particles and their interactions"


SM th
1965, THE NOBEL PRIZE IN PHYSICS
Sin-Itiro Tomonaga "for their fundamental work in quantum electrodynamics, with deep-ploughing consequences for the physics of elementary particles"


SM th
1965, THE NOBEL PRIZE IN PHYSICS
Julian Schwinger "for their fundamental work in quantum electrodynamics, with deep-ploughing consequences for the physics of elementary particles"


SM th
1965, THE NOBEL PRIZE IN PHYSICS
Richard P. Feynman "for their fundamental work in quantum electrodynamics, with deep-ploughing consequences for the physics of elementary particles"


SM th

1963, THE NOBEL PRIZE IN PHYSICS
Eugene Paul Wigner "for his contributions to the theory of the atomic nucleus and the elementary particles, particularly through the discovery and application of fundamental symmetry principles"


SM th

1957, THE NOBEL PRIZE IN PHYSICS
Chen Ning Yang "for their penetrating investigation of the so-called parity laws which has led to important discoveries regarding the elementary particles"


SM th

1957, THE NOBEL PRIZE IN PHYSICS
Tsung-Dao (T.D.) Lee "for their penetrating investigation of the so-called parity laws which has led to important discoveries regarding the elementary particles"


SM th

1949, THE NOBEL PRIZE IN PHYSICS
Hideki Yukawa "for his prediction of the existence of mesons on the basis of theoretical work on nuclear forces"


QM SR

1921, THE NOBEL PRIZE IN PHYSICS
Albert Einstein "for his services to Theoretical Physics, and especially for his discovery of the law of the photoelectric effect"

Quantum Mechanics Theory (9)


QM th

1954, THE NOBEL PRIZE IN PHYSICS
Max Born "for his fundamental research in quantum mechanics, especially for his statistical interpretation of the wavefunction"


QM th

1945, THE NOBEL PRIZE IN PHYSICS
Wolfgang Pauli "for the discovery of the Exclusion Principle, also called the Pauli Principle"


QM th

1933, THE NOBEL PRIZE IN PHYSICS
Erwin Schrödinger "for the discovery of new productive forms of atomic theory"


QM th

1933, THE NOBEL PRIZE IN PHYSICS
Paul Adrien Maurice Dirac "for the discovery of new productive forms of atomic theory"


QM th

1932, THE NOBEL PRIZE IN PHYSICS
Werner Karl Heisenberg "for the creation of quantum mechanics, the application of which has, inter alia, led to the discovery of the allotropic forms of hydrogen"


QM th

1929, THE NOBEL PRIZE IN PHYSICS
Prince Louis-Victor Pierre Raymond de Broglie "for his discovery of the wave nature of electrons"


QM th

1922, THE NOBEL PRIZE IN PHYSICS
Niels Henrik David Bohr "for his services in the investigation of the structure of atoms and of the radiation emanating from them"


QM th

1918, THE NOBEL PRIZE IN PHYSICS
Max Karl Ernst Ludwig Planck "in recognition of the services he rendered to the advancement of Physics by his discovery of energy quanta"


QM th

1902, THE NOBEL PRIZE IN PHYSICS
Hendrik Antoon Lorentz "in recognition of the extraordinary service they rendered by their researches into the influence of magnetism upon radiation phenomena"

Quantum Mechanics Experimental (9)


QM exp

1925, THE NOBEL PRIZE IN PHYSICS
James Franck "for their discovery of the laws governing the impact of an electron upon an atom"


QM exp

1925, THE NOBEL PRIZE IN PHYSICS
Gustav Ludwig Hertz "for their discovery of the laws governing the impact of an electron upon an atom"

 QM exp

1924, THE NOBEL PRIZE IN PHYSICS

Karl Manne Georg Siegbahn "for his discoveries and research in the field of X-ray spectroscopy"

 QM exp

1919, THE NOBEL PRIZE IN PHYSICS

Johannes Stark "for his discovery of the Doppler effect in canal rays and the splitting of spectral lines in electric fields"

 QM exp

1917, THE NOBEL PRIZE IN PHYSICS

Charles Glover Barkla "for his discovery of the characteristic Röntgen radiation of the elements"

 QM exp

1914, THE NOBEL PRIZE IN PHYSICS

Max von Laue "for his discovery of the diffraction of X-rays by crystals"

 QM exp

1911, THE NOBEL PRIZE IN PHYSICS

Wilhelm Wien "for his discoveries regarding the laws governing the radiation of heat"

 QM exp

1902, THE NOBEL PRIZE IN PHYSICS

Pieter Zeeman "in recognition of the extraordinary service they rendered by their researches into the influence of magnetism upon radiation phenomena"

 QM/SM exp

1901, THE NOBEL PRIZE IN PHYSICS

Wilhelm Conrad Röntgen "in recognition of the extraordinary services he has rendered by the discovery of the remarkable rays subsequently named after him"

Relativity (1)

 SR exp

1907, THE NOBEL PRIZE IN PHYSICS

Albert Abraham Michelson "for his optical precision instruments and the spectroscopic and metrological investigations carried out with their aid"

Appendix
Summary of elementary properties of QFT

From classical to quantum fields

In scattering experiments we observe asymptotic free particles characterised by their energy-momentum charge and other quantum numbers. Consider just E,p. In the NR-case we describe the one-particle states by kets carrying a unitary rep. of the rotation group.

$$|\mathbf{p}\rangle \in \mathcal{H}_1, \qquad \langle \mathbf{p}|\mathbf{p}'\rangle = \delta(\mathbf{p}-\mathbf{p}') \qquad \int d^3p\, |\mathbf{p}\rangle\langle \mathbf{p}| = \mathbf{1}. \qquad \mathcal{U}(R)|\mathbf{p}\rangle = |R\mathbf{p}\rangle \qquad \hat{P}^i = \int d^3p\, |\mathbf{p}\rangle p^i \langle \mathbf{p}|$$

To deal with multi-particle states it is convenient to introduce creation and annihilation operators, this leads to the Fock space of states, built out of the vacuum by acting with creation operators:

$$|\mathbf{p}\rangle = a^\dagger(\mathbf{p})|0\rangle, \qquad a(\mathbf{p})|0\rangle = 0 \qquad \langle 0|0\rangle = 1$$

$$[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = \delta(\mathbf{p}-\mathbf{p}'), \qquad [a(\mathbf{p}), a(\mathbf{p}')] = [a^\dagger(\mathbf{p}), a^\dagger(\mathbf{p}')] = 0,$$

We need relativistic invariance, hence we need to find ways to count states in an invariant way. This is necessary also when we deal with decay rates and cross sections. We need to count final states in a way consistent with Lorentz invariance. We can easily construct such an invariant phase space volume:

$$\int \frac{d^4p}{(2\pi)^4} (2\pi)\delta(p^2-m^2)\,\theta(p^0)\, f(p)$$ to integrate over p0, we use a nice identity:

$$\delta[g(x)] = \sum_{x_i = \text{zeros of } g} \frac{1}{|g'(x_i)|}\delta(x-x_i) \qquad \delta(p^2-m^2) = \frac{1}{2p^0}\delta\left(p^0 - \sqrt{\mathbf{p}^2+m^2}\right) + \frac{1}{2p^0}\delta\left(p^0 + \sqrt{\mathbf{p}^2+m^2}\right)$$

$$\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}$$ with $E_\mathbf{p} \equiv \sqrt{\mathbf{p}^2+m^2}$ and $(2E_\mathbf{p})\delta(\mathbf{p}-\mathbf{p}')$ are invariant

...continued

Now proceed by imitation of the NR case, with the non-trivial result that we have a unitary representation of the Lorentz group

$|p\rangle = (2\pi)^{\frac{3}{2}}\sqrt{2E_{\mathbf{p}}}|\mathbf{p}\rangle$ $\quad\langle p|p'\rangle = (2\pi)^3(2E_{\mathbf{p}})\delta(\mathbf{p}-\mathbf{p}')$ $\quad\hat{P}^\mu = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}|p\rangle p^\mu\langle p|$ $\quad\mathscr{U}(\Lambda)|p\rangle = |\Lambda^\mu{}_\nu p^\nu\rangle \equiv |\Lambda p\rangle$

$$\langle 0|0\rangle = 1$$

$$\alpha(\mathbf{p}) \equiv (2\pi)^{\frac{3}{2}}\sqrt{2E_{\mathbf{p}}}a(\mathbf{p}) \qquad [\alpha(\mathbf{p}),\alpha^\dagger(\mathbf{p}')] = (2\pi)^3(2E_{\mathbf{p}})\delta(\mathbf{p}-\mathbf{p}'),$$
$$\alpha^\dagger(\mathbf{p}) \equiv (2\pi)^{\frac{3}{2}}\sqrt{2E_{\mathbf{p}}}a^\dagger(\mathbf{p}) \qquad [\alpha(\mathbf{p}),\alpha(\mathbf{p}')] = [\alpha^\dagger(\mathbf{p}),\alpha^\dagger(\mathbf{p}')] = 0.$$
$$|f\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}f(\mathbf{p})\alpha^\dagger(\mathbf{p})|0\rangle$$

Let us construct some observable in this theory. It will be an operator depending on space time, and satisfying some simple conditions:

Hermiticity $\qquad \phi(x)^\dagger = \phi(x).$

Microcausality $\qquad [\phi(x),\phi(y)] = 0, \qquad (x-y)^2 < 0.$

Translational invariance $\qquad e^{i\hat{P}\cdot a}\phi(x)e^{-i\hat{P}\cdot a} = \phi(x-a)$

Lorentz invariance $\qquad \mathscr{U}(\Lambda)^\dagger\phi(x)\mathscr{U}(\Lambda) = \phi(\Lambda^{-1}x).$

Linearity $\qquad \phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left[f(\mathbf{p},x)\alpha(\mathbf{p}) + g(\mathbf{p},x)\alpha^\dagger(\mathbf{p})\right].$

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left[e^{-iE_{\mathbf{p}}t+i\mathbf{p}\cdot\mathbf{x}}\alpha(\mathbf{p}) + e^{iE_{\mathbf{p}}t-i\mathbf{p}\cdot\mathbf{x}}\alpha^\dagger(\mathbf{p})\right]$$

+ve energy $\qquad$ -ve energy

We have obtained from first principles the quantisation of the Klein-Gordon field. There are more straightforward ways, but the procedure shows how to implement the basis principles of the theory, Lorentz invariance, locality and positivity of the spectrum

Some important properties

$$[\phi(t,\mathbf{x}), \partial_t \phi(t,\mathbf{y})] = i\delta(\mathbf{x}-\mathbf{y}).$$

$$[\phi(x), \phi(x')] = i\Delta(x-x')$$

$$(\partial_\mu \partial^\mu + m^2)\phi(x) = 0$$

$$i\Delta(x-y) = -\mathrm{Im}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}} e^{-iE_{\mathbf{p}}(t-t')+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}$$

$$= \int \frac{d^4p}{(2\pi)^4}(2\pi)\delta(p^2-m^2)\varepsilon(p^0)e^{-ip\cdot(x-x')}$$

$$\Delta(x-y) = 0 \qquad \text{for } (x-y)^2 < 0$$

The construction is free of paradoxes. It satisfies the KG equation because the +ve and -ve energy plane waves satisfy it. Of course with a free field we do not go very far…

We should design more powerful techniques leading to similar properties by for more general theories where interactions can take place.

There are two general approaches: the canonical-formalism, and the Feynman path integral. We will briefly introduce the first, just as a reminder.

Quantization

We repeat the bosonic arguments, except for the fact that we have now anti-commutation relations between electron and positron creation-annihilation operators

$$\widehat{\psi}_\alpha(t,\vec{x}) = \sum_{s=\pm\frac{1}{2}} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} \left[u_\alpha(\vec{k},s)\,\widehat{b}(\vec{k},s) e^{-i\omega_{\vec{k}}t + i\vec{k}\cdot\vec{x}} + v_\alpha(\vec{k},s)\,\widehat{d}^{\,\dagger}(\vec{k},s) e^{i\omega_{\vec{k}}t - i\vec{k}\cdot\vec{x}} \right].$$

$$\{\widehat{\psi}_\alpha(t,\mathbf{x}), \widehat{\psi}^\dagger_\beta(t,\mathbf{y})\} = \delta(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta}$$

$$\begin{aligned}
\{b(\mathbf{k},s), b^\dagger(\mathbf{k}',s')\} &= (2\pi)^3(2\omega_{\mathbf{k}})\delta(\mathbf{k}-\mathbf{k}')\delta_{ss'}, \\
\{b(\mathbf{k},s), b(\mathbf{k}',s')\} &= \{b^\dagger(\mathbf{k},s), b^\dagger(\mathbf{k}',s')\} = 0. \\
\{d(\mathbf{k},s), d^\dagger(\mathbf{k}',s')\} &= (2\pi)^3(2\omega_{\mathbf{k}})\delta(\mathbf{k}-\mathbf{k}')\delta_{ss'}, \\
\{d(\mathbf{k},s), d(\mathbf{k}',s')\} &= \{d^\dagger(\mathbf{k},s), d^\dagger(\mathbf{k}',s')\} = 0.
\end{aligned}$$

$$\hat{H} = \frac{1}{2}\sum_{s=\pm\frac{1}{2}} \int \frac{d^3k}{(2\pi)^3} \left[b^\dagger(\mathbf{k},s)b(\mathbf{k},s) - d(\mathbf{k},s)d^\dagger(\mathbf{k},s) \right].$$

$$\widehat{H} = \sum_{s=\pm\frac{1}{2}} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} \left[\omega_{\vec{k}} b^\dagger(\vec{k},s) b(\vec{k},s) + \omega_{\vec{k}} d^\dagger(\vec{k},s) d(\vec{k},s) \right] - 2\int d^3k\, \omega_{\vec{k}} \delta(\vec{0}).$$

We have a conserved charge and current

$$j^\mu = \overline{\psi}\gamma^\mu\psi, \qquad \partial_\mu j^\mu = 0 \qquad Q = e\int d^3x\, j^0$$

The two-point function or Feynman propagator is:

$$S_{\alpha\beta}(x_1,x_2) = \langle 0|T\left[\psi_\alpha(x_1)\overline{\psi}_\beta(x_2)\right]|0\rangle$$

$$T\left[\psi_\alpha(x)\overline{\psi}_\beta(y)\right] = \theta(x^0-y^0)\psi_\alpha(x)\overline{\psi}_\beta(y) - \theta(y^0-x^0)\overline{\psi}_\beta(y)\psi_\alpha(x).$$

Canonical quantization

Remember: PHYSICS is where the ACTION is!

Proceed by analogy with ordinary QM

$$S[x,\dot{x}] = \int dt\, L(x,\dot{x})$$

$$L = \sum_i \frac{1}{2} m_i \dot{\mathbf{x}}_i^2 - V(\mathbf{x})$$

$$S[\phi(x)] \equiv \int d^4x\, \mathscr{L}(\phi, \partial_\mu\phi) = \int d^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2\right)$$

$$\mathbf{x}_a,\ \dot{\mathbf{x}}_a \longleftrightarrow \phi(\mathbf{x},0),\ \dot{\phi}(\mathbf{x},0)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$$

$$\partial_\mu\left[\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\right] - \frac{\partial\mathscr{L}}{\partial\phi} = 0 \qquad \Longrightarrow \qquad (\partial_\mu\partial^\mu + m^2)\phi = 0$$

canonical momenta

$$p = \frac{\partial L}{\partial \dot{x}}$$

$$\pi(x) \equiv \frac{\partial\mathscr{L}}{\partial(\partial_0\phi)} = \frac{\partial\phi}{\partial t}$$

$$H = \sum_i p_i\, \dot{x}^i - L$$

$$H \equiv \int d^3x \left(\pi\frac{\partial\phi}{\partial t} - \mathscr{L}\right) = \frac{1}{2}\int d^3x \left[\pi^2 + (\nabla\phi)^2 + m^2\right].$$

$$[q^i, p_j] = i\hbar$$

$$[\phi(t,\mathbf{x}), \partial_t\phi(t,\mathbf{y})] = i\delta(\mathbf{x}-\mathbf{y}).$$

Expanding in solutions to the KG equations and performing the canonical quantisation, we recover the algebra of creation and annihilation operator we had before, but we get a surprise

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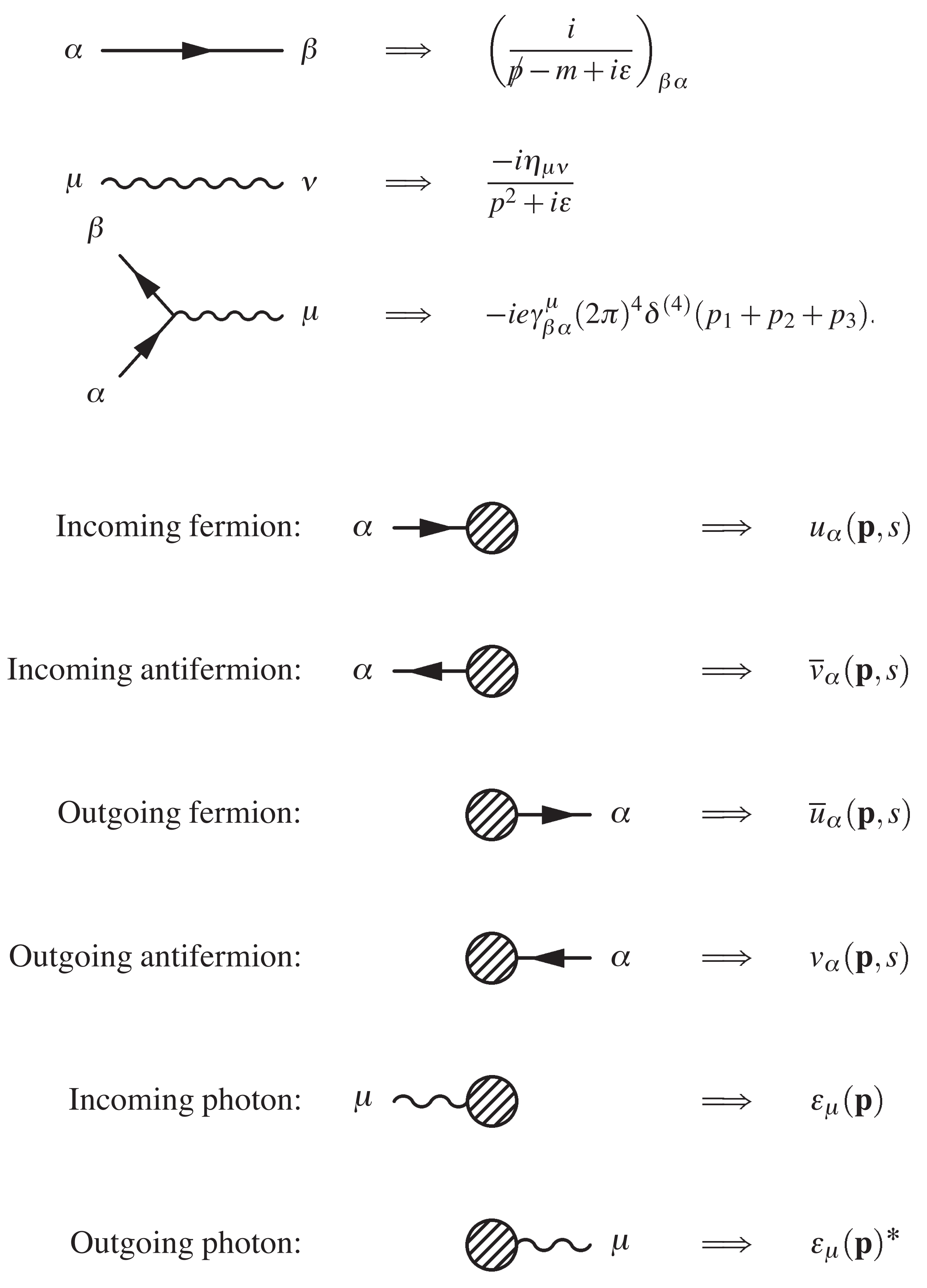
Computational tools

There are two general procedures to obtain computational rules in QFT: The canonical formalism and the Path Integral formulation.

You may recall that one used the Interaction Representation, Wick's theorem, T-products, Gaussian integrations…

In the end we get a collection of well-defined rules that allow us to compute the probability amplitude associates to a given scattering process, out of which we can evaluate the decay width, differential and total cross section and many other quantities that can be observed for instance in collider experiments. The next few pages provide simply a reminder

QED Feynman rules

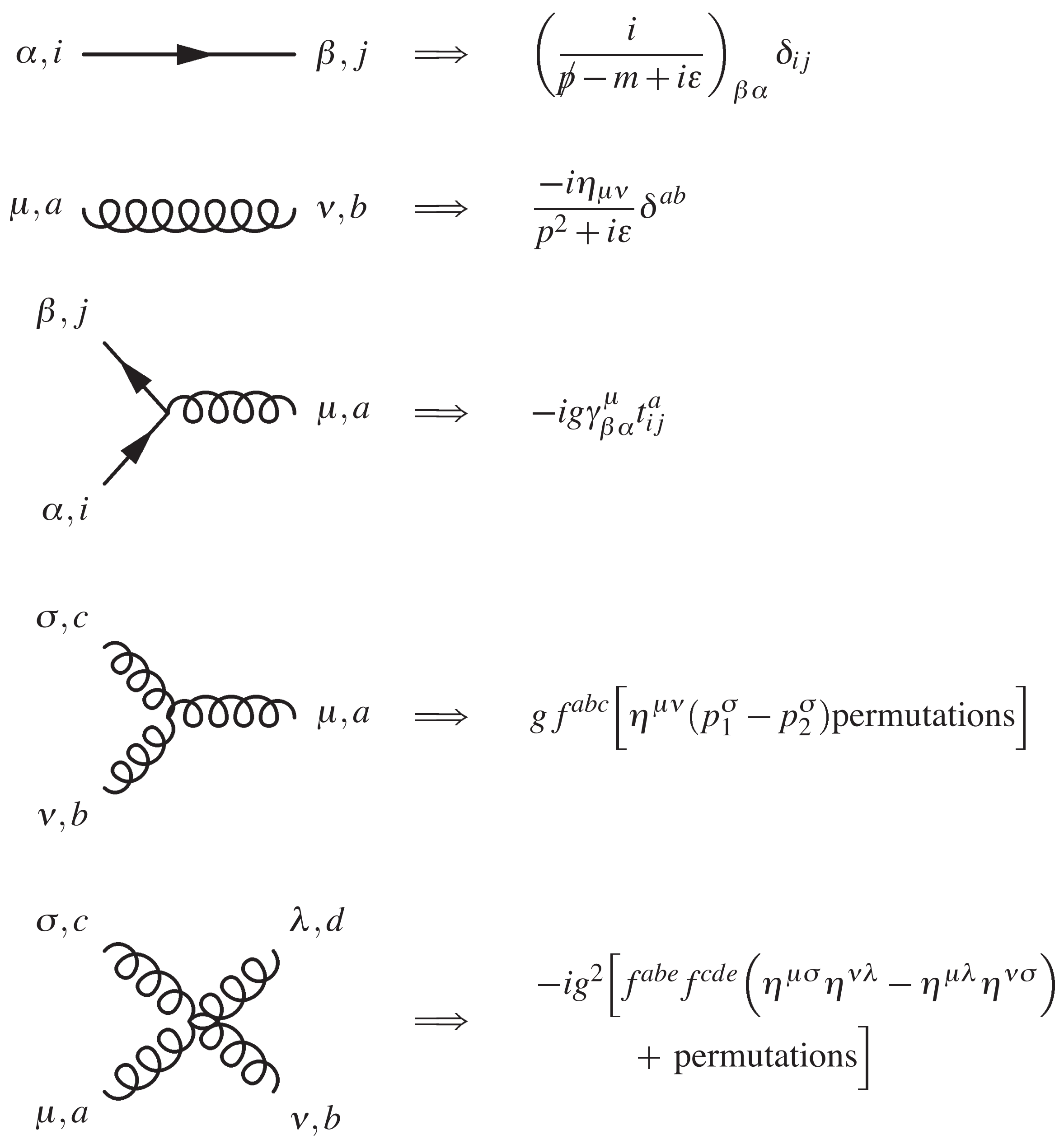


Integrate over loop momenta

$$\int \frac{d^d p}{(2\pi)^4};$$

A minus sign has to be included for every fermion loop and for every positron line that goes from the initial to the final state. With some extra effort we can derive the Feynman rules for QCD-like theories. They appear in the next page. The quark and anti-quark factors are similar to the electron positron ones, except that we need to include color quantum numbers. The real difference comes with the gluon or non-abelian vector bosons interactions, the are quite involved and contain a large amount of interesting physics perturbatively and specially non-perturbatively.

Standard Model Feynman rules



Although the rules seem to be those for QCD, notice that we could always include in the group theory factors t^a-{ij} chiral projectors and make the group not simple but semi-simple as in the case of the SM: SU(3)xSU(2)xU(1). If we work in nice renormalizable gauges, the only difference is that we have to include the Feynman rules for the couplings of the scalar sector. Something we will do later.

$$t^a_{ij} \to t^a_{ij}\,\frac{1}{2}(1 \pm \gamma_5)$$

With this simple trick the hard part, which is the coupling of the W,Z, and photons can be read simply from the rules in the LHS.

We will not engage into any simple graph computation. You certainly computed some in your particle physics class