QCD and jet physics

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2^{nd} Lecture

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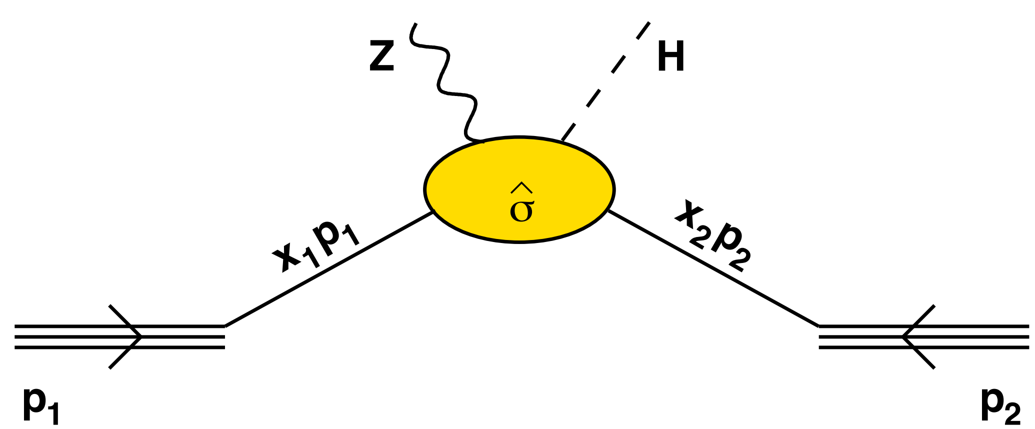
Partons in the initial state

Next: processes with partons in the initial state

- We talked a lot about final state QCD effects
- This is the only thing to worry about at e^+e^- colliders (LEP)
- Hera/Tevatron/LHC involve protons in the initial state
- Proton are made of QCD constituents

Next we will focus mainly on aspects related to initial state effects



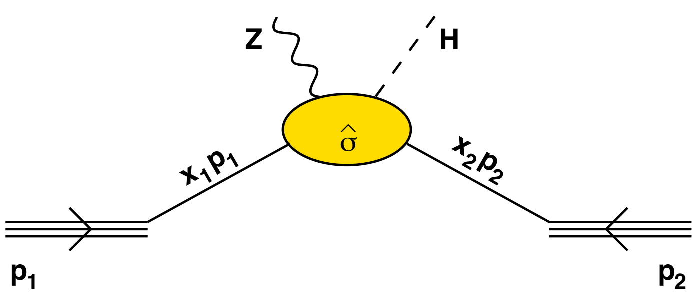
The parton model

Basic idea of the parton model: intuitive picture where in a high transverse momentum scattering partons behave as quasi free in the collision

$\Longrightarrow$ cross section is the incoherent sum of all partonic cross-sections

$$\sigma = \int dx_1 dx_2 f_1^{(P_1)}(x_1) f_2^{(P_2)}(x_2) \hat{\sigma}(x_1 x_2 s) \qquad \hat{s} = x_1 x_2 s$$

NB: This formula is wrong/incomplete (see later)



$f_i^{(P_j)}(x_i)$: parton distribution function (PDF) is the probability to find parton i in hadron j with a fraction x_i of the longitudinal momentum (transverse momentum neglected), extracted from data

$\hat{\sigma}(x_1 x_2 s)$: partonic cross-section for a given scattering process, computed in perturbative QCD

Sum rules

Momentum sum rule: conservation of incoming total momentum

$$\int_0^1 dx \sum_i x f_i^{(p)}(x) = 1$$

Conservation of flavour: e.g. for a proton

$$\int_0^1 dx \left(f_u^{(p)}(x) - f_{\bar{u}}^{(p)}(x) \right) = 2$$

$$\int_0^1 dx \left(f_d^{(p)}(x) - f_{\bar{d}}^{(p)}(x) \right) = 1$$

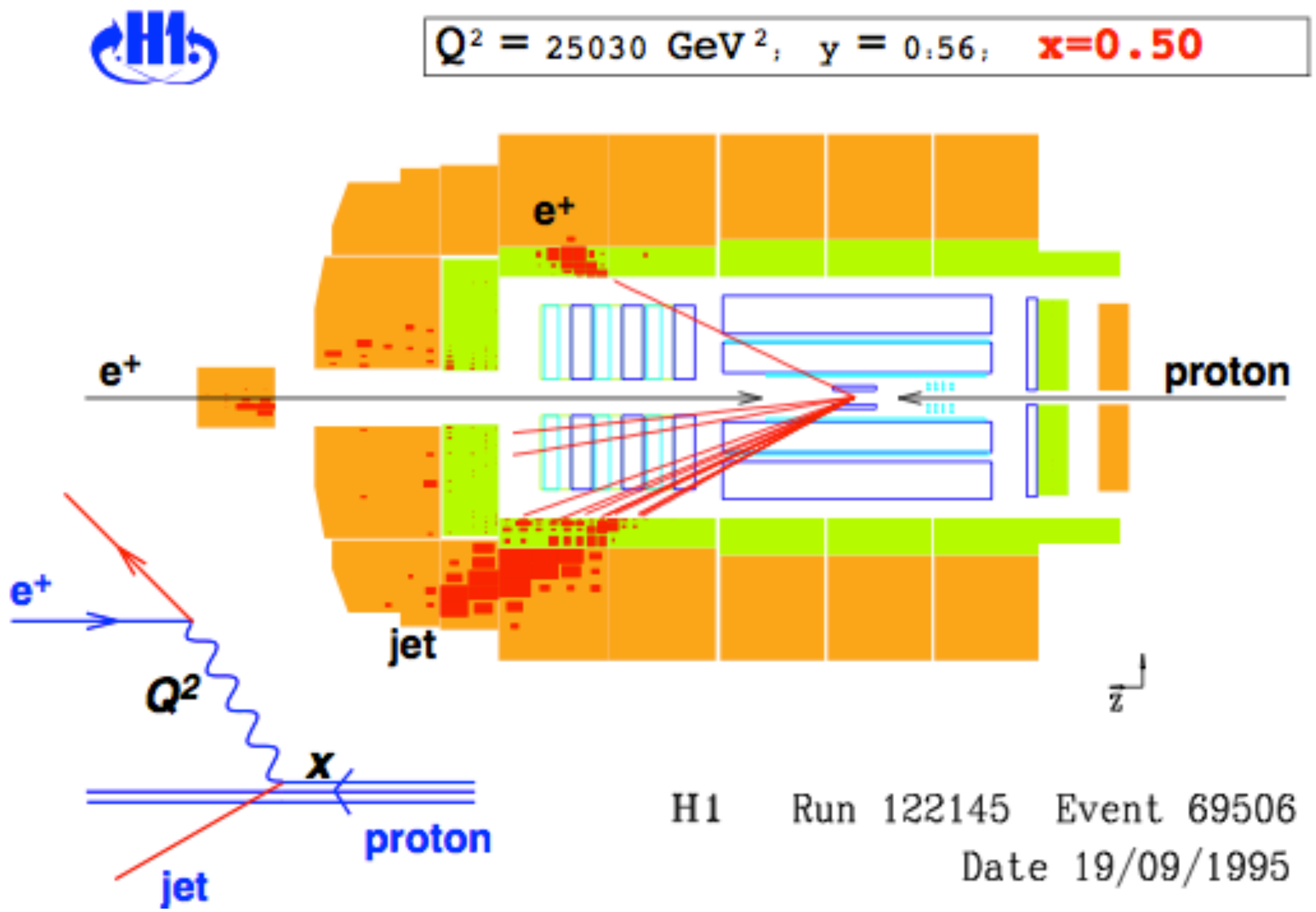
$$\int_0^1 dx \left(f_s^{(p)}(x) - f_{\bar{s}}^{(p)}(x) \right) = 0$$

In the proton: u, d valence quarks, all other quarks are called sea-quarks

How can parton densities be extracted from data?

Deep inelastic scattering

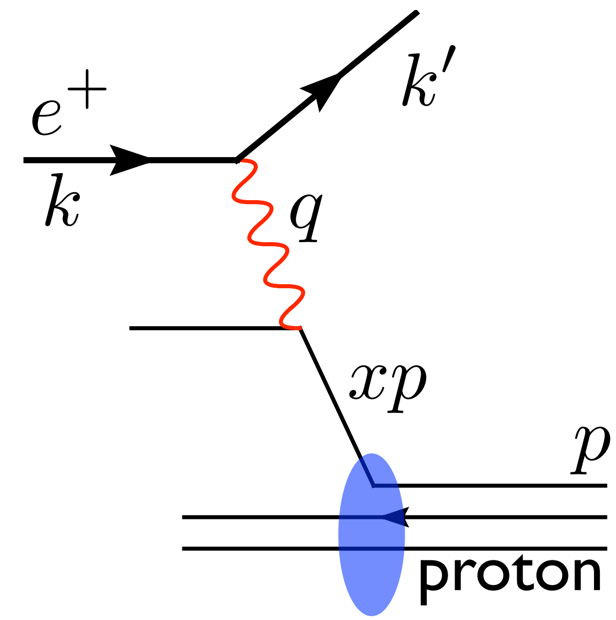
Easier than processes with two incoming hadrons is the scattering of a lepton on a (anti)-proton



Deep inelastic scattering

Protons made up of point-like quarks.
Different momentum scales involved:

- hard photon virtuality (sets the resolution scale) Q
- hard photon-quark interaction Q
- soft interaction between partons in the proton $m_p \ll Q$



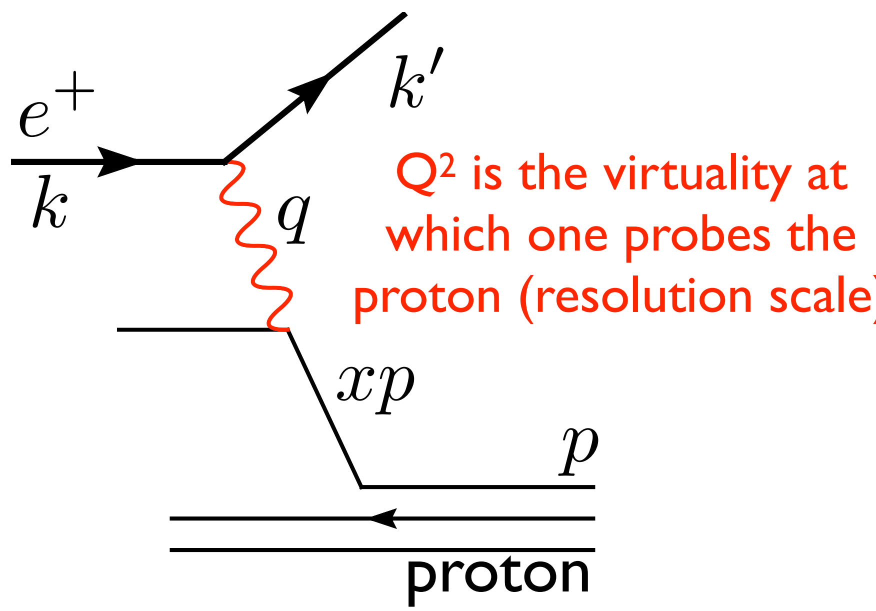
During the hard interaction, partons do not have time to interact among them, they behave as if they were free

$\Rightarrow$ approximate as incoherent scattering on single partons

Deep inelastic scattering

Kinematics:

$$Q^2 = -q^2 \quad s = (k+p)^2 \quad x_{Bj} = \frac{Q^2}{2p\cdot q} \quad y = \frac{p\cdot q}{k\cdot p}$$



Partonic variables:

$$\hat{p} = xp \quad \hat{s} = (k+\hat{p})^2 = 2k\cdot\hat{p} \quad \hat{y} = \frac{\hat{p}\cdot q}{k\cdot\hat{p}} = y \quad (\hat{p}+q)^2 = 2\hat{p}\cdot q - Q^2 = 0$$

$$\Rightarrow x = x_{Bj}$$

Hence at leading order, the experimentally accessible x_{Bj} coincides with the momentum fraction carried by the quark in the proton

Partonic cross section:

(apply QED Feynman rules and add phase space)

$$\frac{d\hat{\sigma}}{d\hat{y}} = q_l^2 \frac{\hat{s}}{Q^4} 2\pi\,\alpha_{em}\left(1+(1-\hat{y})^2\right)$$

Exercise: show that in the CM frame of the electron-quark system y is given by $(1-\cos\theta_{\rm el})/2$, with $\theta_{\rm el}$ the scattering angle of the electron in this frame

Exercise:

- show that the two particle phase space is $\frac{d\phi}{16\pi}$
- show that the squared matrix element is $\frac{16\pi\alpha q_l^2}{Q^4}\hat{s}xpk\left(1+(1-y)^2\right)$
- show that the flux factor is $\frac{1}{4xpk}$

Hence derive that

$$\frac{d\hat{\sigma}}{d\hat{y}} = q_l^2\frac{\hat{s}}{Q^4}2\,\pi\,\alpha_{em}\left(1+(1-\hat{y})^2\right)$$

Deep inelastic scattering

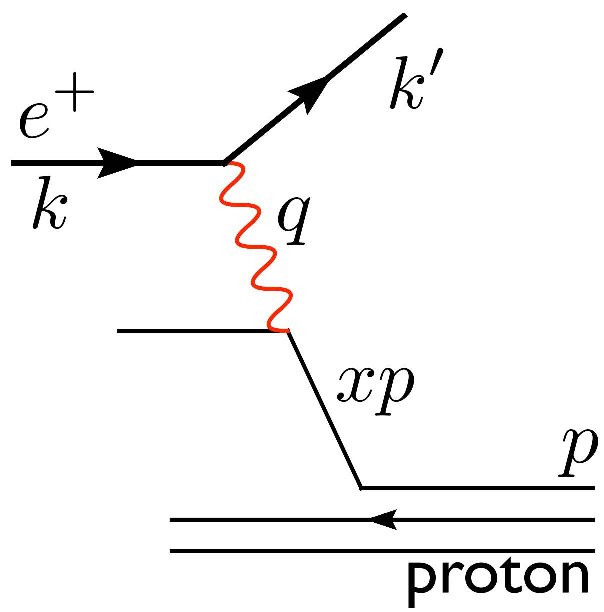
Hadronic cross section (factorization):

$$\frac{d\sigma}{dy} = \int dx \sum_l f_l^{(p)}(x) \frac{d\hat{\sigma}}{d\hat{y}}$$

Using x = x_{BJ}

$$\frac{d\sigma}{dy\, dx_{Bj}} = \sum_l f_l^{(p)}(x) \frac{d\hat{\sigma}}{d\hat{y}}$$

$$= \frac{2\pi\, \alpha_{em}^2 s x_{Bj}}{Q^4} \left(1 + (1-y)^2\right) \sum_l q_l^2 f_l^{(p)}(x_{Bj})$$



1. at fixed x_{Bj} and y the cross-section scales with s
2. the y-dependence of the cross-section is fully predicted and is typical of vector interaction with fermions ⇒ Callan-Gross relation
3. can access (sums of) parton distribution functions
4. Bjorken scaling: pdfs depend on x and not on Q^2 (violated by logarithmic radiative corrections, see later)

The structure function F_2

$$\frac{d\sigma}{dydx} = \frac{2\pi\alpha_{em}^2 s}{Q^4}\left(1+(1-y^2\right)F_2(x) \qquad F_2(x) = \sum_l x q_l^2 f_l^{(p)}(x)$$

F_2 is called structure function (describes structure/constituents of nucleus)

For electron scattering on proton

$$F_2(x) = x\left(\frac{4}{9}u(x) + \frac{1}{9}d(x)\right)$$

NB: use perturbative language of quarks and gluons despite the fact that parton distribution are non-perturbative

Bjorken scaling: the fact the structure functions are independent of Q is a direct evidence for the existence of point-like quarks in the proton (violated by logarithmic corrections)

The structure function F_2

$$\frac{d\sigma}{dydx} = \frac{2\pi\alpha_{em}^2 s}{Q^4}\left(1 + (1 - y^2\right) F_2(x) \qquad F_2(x) = \sum_l x q_l^2 f_l^{(p)}(x)$$

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NB: use perturbative language of quarks and gluons despite the fact that parton distribution are non-perturbative

Question: F_2 gives only a linear combination of u and d. How can they be extracted separately?

Isospin

Neutron is like a proton with u & d exchanged

For electron scattering on a proton

$$F_2^p(x) = x\left(\frac{4}{9}u_p(x) + \frac{1}{9}d_p(x)\right)$$

For electron scattering on a neutron

$$F_2^n(x) = x\left(\frac{1}{9}d_n(x) + \frac{4}{9}u_n(x)\right) = x\left(\frac{4}{9}d_p(x) + \frac{1}{9}u_p(x)\right)$$

F_2^n and F_2^p allow determination of u_p and d_p separately

NB: experimentally get F_2^n from deuteron: $F_2^d(x) = F_2^p(x) + F_2^n(x)$

Sea quark distributions

Inside the proton there are fluctuations, and pairs of $u\bar{u}$,$d\bar{d}$,$c\bar{c}$,$s\bar{s}$ ... can be created

An infinite number of pairs can be created as long as they have very low momentum, because of the momentum sum rules.

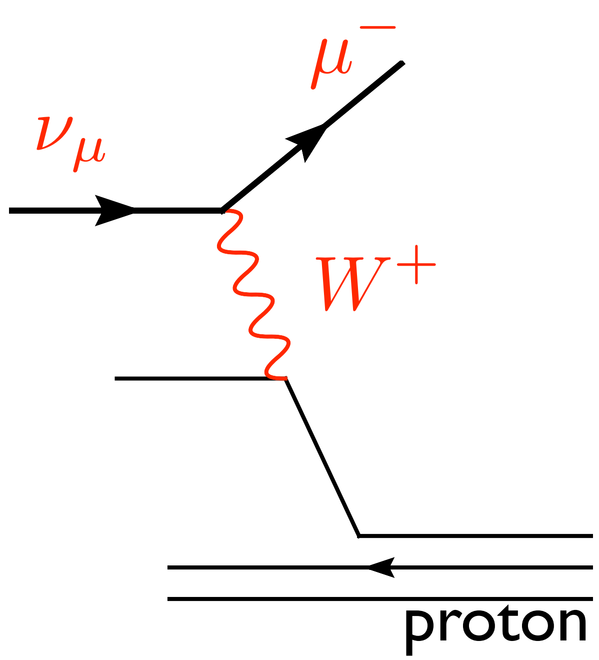
We saw before that when we say that the proton is made of uud what we mean is

$$\int_0^1 dx\,(u_p(x) - \bar{u}_p(x)) = 2 \qquad \int_0^1 dx\,\left(d_p(x) - \bar{d}_p(x)\right) = 1$$

Photons interact in the same way with u(d) and $\bar{u}$($\bar{d}$)

How can one measure the difference?

Question: What interacts differently with particle and antiparticle? W^+/W^- from neutrino scattering



Check of the momentum sum rule

$$\int_0^1 dx \sum_i x f_i^{(p)}(x) = 1$$

u_v	0,267
d_v	0,111
u_s	0,066
d_s	0,053
s_s	0,033
c_c	0,016
total	0,546

➠ *half of the longitudinal momentum carried by gluons*

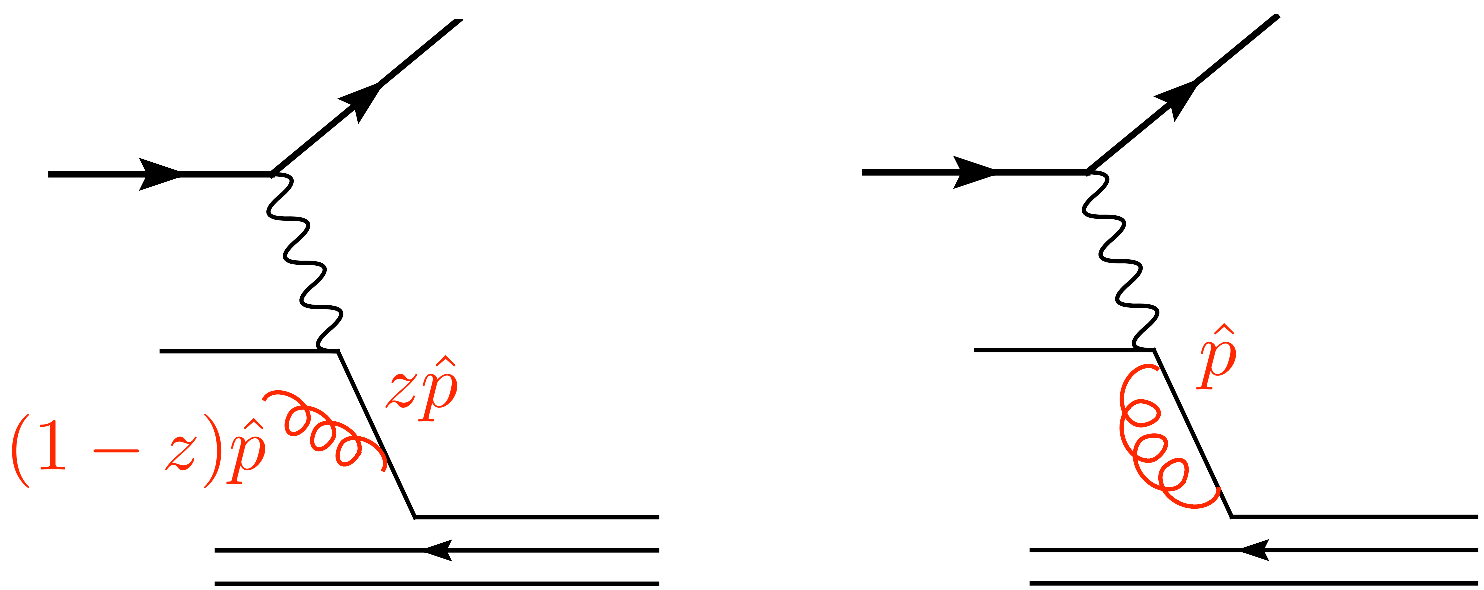
$\gamma/W^{+/-}$ don't interact with gluons

How can one measure gluon parton densities?

We need to discuss radiative effects first

Radiative corrections

To first order in the coupling:
need to consider the emission of one real gluon and a virtual one



Adding real and virtual contributions, the partonic cross-section reads

$$\sigma^{(1)} = \frac{C_F \alpha_s}{2\pi} \int dz \frac{dk_\perp^2}{k_\perp^2} \frac{1+z^2}{1-z} \left(\sigma^{(0)}(z\hat{p}) - \sigma^{(0)}(\hat{p}) \right)$$

Partial cancellation between real (positive), virtual (negative), but real gluon changes the energy entering the scattering, the virtual does not

Radiative corrections

Partonic cross-section:

$$\sigma^{(1)} = \frac{\alpha_s}{2\pi} \int dz \int_{\lambda^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2} P(z) \left(\sigma^{(0)}(z\hat{p}) - \sigma^{(0)}(\hat{p}) \right), \quad P(z) = C_F \frac{1+z^2}{1-z}$$

Soft limit: singularity at z=1 cancels between real and virtual terms

Collinear singularity: $k_\perp \to 0$ with finite z. Collinear singularity does not cancel because partonic scatterings occur at different energies

$\Rightarrow$ naive parton model does not survive radiative corrections

Similarly to what is done when renormalizing UV divergences, collinear divergences from initial state emissions are absorbed into parton distribution functions

The plus prescription

Partonic cross-section:

$$\sigma^{(1)} = \frac{\alpha_s}{2\pi} \int_{\lambda^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2} \int_0^1 dz\, P(z) \left(\sigma^{(0)}(z\hat{p}) - \sigma^{(0)}(\hat{p})\right)$$

Plus prescription makes the universal cancelation of singularities explicit

$$\int_0^1 dz f_+(z) g(z) \equiv \int_0^1 f(z)\,(g(z) - g(1))$$

The partonic cross section becomes

$$\sigma^{(1)} = \frac{\alpha_s}{2\pi} \int dz \int_{\lambda^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2} P_+(z) \sigma^{(0)}(z\hat{p})\,, \quad P(z) = C_F \left(\frac{1+z^2}{1-z}\right)$$

Collinear singularities still there, but they factorize.

Factorization scale

Schematically use
$$\ln\frac{Q^2}{\lambda^2} = \ln\frac{Q^2}{\mu_F^2} + \ln\frac{\mu_F^2}{\lambda^2}$$

$$\sigma = \sigma^{(0)} + \sigma^{(1)} = \left(1 + \frac{\alpha_s}{2\pi}\ln\frac{\mu_F^2}{\lambda^2}P_+\right) \times \left(1 + \frac{\alpha_s}{2\pi}\ln\frac{Q^2}{\mu_F^2}P_+\right)\sigma^{(0)}$$

So we define

$$f_q(x,\mu_F) = f_q(x) \times \left(1 + \frac{\alpha_s}{2\pi}\ln\frac{\mu_F^2}{\lambda^2}P_{qq}^{(0)}\right) \qquad \hat{\sigma}(p,\mu_F) = \left(1 + \frac{\alpha_s}{2\pi}\ln\frac{Q^2}{\mu_F^2}P_{qq}^{(0)}\right)\sigma^{(0)}(p)$$

NB:

- universality, i.e. the PDF redefinition does not depend on the process
- choice of $\mu_F \sim Q$ avoids large logarithms in partonic cross-sections
- PDFs and hard cross-sections don't evolve independently
- the factorization scale acts as a cut-off, it allows to move the divergent contribution into non-perturbative parton distribution functions

Improved parton model

Naive parton model:

$$\sigma = \int dx_1 dx_2 f_1^{(P_1)}(x_1) f_2^{(P_2)}(x_2) \hat{\sigma}(x_1 x_2 s) \qquad \hat{s} = x_1 x_2 s$$

After radiative corrections:

$$\sigma = \int dx_1 dx_2 f_1^{(P_1)}(x_1, \mu^2) f_2^{(P_2)}(x_2, \mu^2) \hat{\sigma}(x_1 x_2 s, \mu^2)$$

Intermediate recap

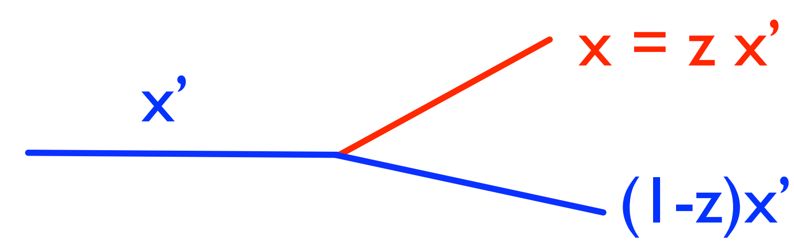
- With initial state parton collinear singularities don't cancel
- Initial state emissions with $k_\perp$ below a given scale are included in PDFs
- This procedure introduces a scale μ_F, the so-called factorization scale which factorizes the low energy (non-perturbative) dynamics from the perturbative hard cross-section
- As for the renormalization scale, the dependence of cross-sections on μ_F is due to the fact that the perturbative expansion has been truncated
- The dependence on μ_F becomes milder when including higher orders
- The redefinition of PDFs is universal and process-independent

Two incoming hard partons: $\sigma = \int dx_1 dx_2 f_1^{(P_1)}(x_1, \mu^2) f_2^{(P_2)}(x_2, \mu^2) \hat{\sigma}(x_1 x_2 s, \mu^2)$

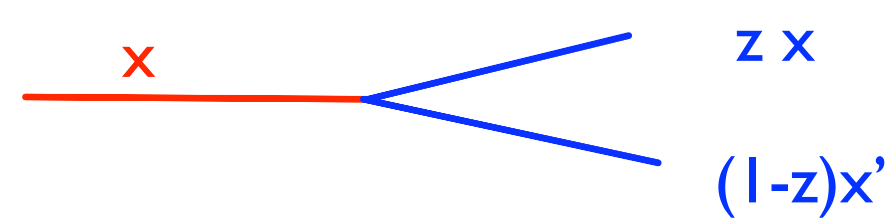
Evolution of PDFs

A parton distribution changes when

- a different parton splits and produces it



- the parton itself splits



$$\mu^2 \frac{\partial f(x,\mu^2)}{\partial \mu^2} = \int_0^1 dx' \int_x^1 dz \frac{\alpha_s}{2\pi} P(z) f(x',\mu^2) \delta(zx'-x) - \int_0^1 dz \frac{\alpha_s}{2\pi} P(z) f(x,\mu^2)$$

$$= \int_x^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} P(z) f\left(\frac{x}{z},\mu^2\right) - \int_0^1 dz \frac{\alpha_s}{2\pi} P(z) f\left(x,\mu^2\right)$$

$$= \int_x^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} P_+(z) f\left(\frac{x}{z},\mu^2\right)$$

The plus prescription $\int_0^1 dz f_+(z) g(z) \equiv \int_0^1 dz f(z) \left(g(z) - g(1)\right)$

DGLAP equation

$$\mu^2 \frac{\partial f(x,\mu^2)}{\partial \mu^2} = \int_x^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} P(z) f\left(\frac{x}{z},\mu^2\right)$$

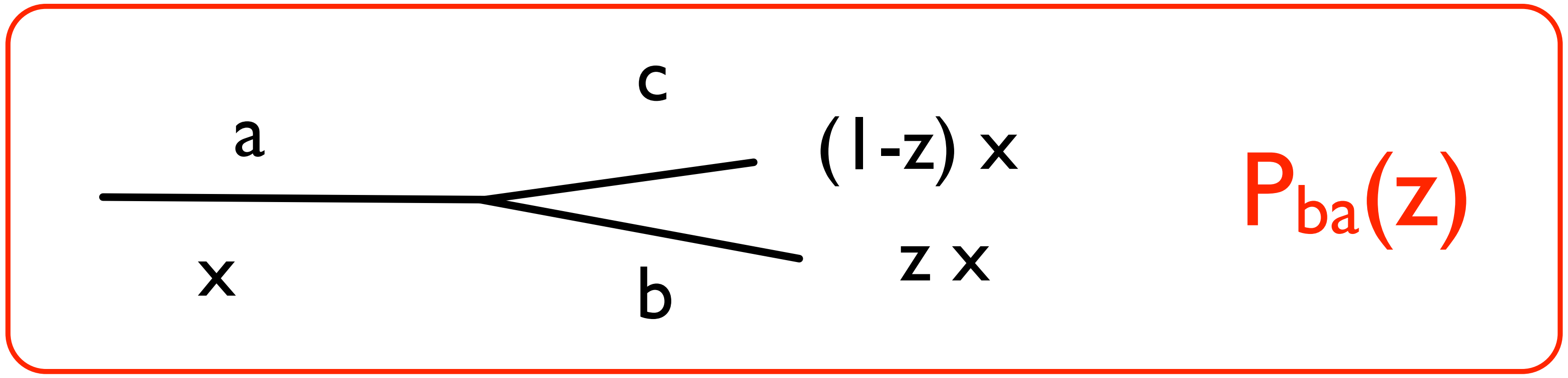
Altarelli, Parisi; Gribov-Lipatov; Dokshitzer '77

Master equation of QCD: we can not compute parton densities, but we can predict how they evolve from one scale to another

Universality of splitting functions: we can measure pdfs in one process and use them as an input for another process

Conventions for splitting functions

There are various partons types. Standard notation:



Accounting for the different species of partons the DGLAP equations become:

$$\mu^2 \frac{\partial f_i(x,\mu^2)}{\partial \mu^2} = \sum_j \int_x^1 \frac{dz}{z} P_{ij}(z) f_j\left(\frac{x}{z},\mu^2\right)$$

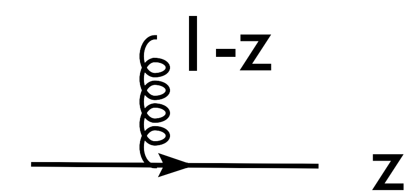
This is a system of coupled integro/differential equations

The above convolution in compact notation:

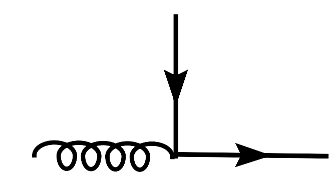
$$\mu^2 \frac{\partial f_i(x,\mu^2)}{\partial \mu^2} = \sum_j P_{ij} \otimes f_j(\mu^2)$$

Properties of splitting functions

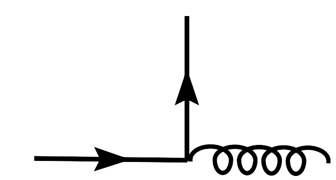
$$P_{qq}^{(0)} = P_{\bar{q}\bar{q}}^{(0)} = C_F\left[\left(\frac{1+z^2}{1-z}\right)_+\right]$$



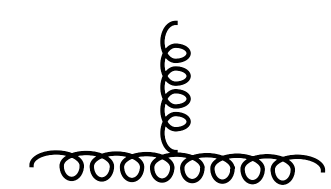
$$P_{qg}^{(0)} = P_{\bar{q}g}^{(0)} = T_R\left(z^2+(1-z)\right)$$



$$P_{gq}^{(0)} = P_{g\bar{q}}^{(0)} = C_F\frac{1+(1-z)^2}{z}$$



$$P_{gg}^{(0)} = 2C_A\left[z\frac{1}{(1-z)}_+ + \frac{1-z}{z} + z(1-z) + b_0\delta(1-z)\right]$$



- P_{qg} anf P_{gg} symmetric under z (1-z)
- P_{qq} and P_{gg} divergence for z=1 (soft gluon)
- P_{gq} and P_{gg} divergenge for z=0 (soft gluon)
- P_{qg} no soft divergence for gluon splitting to quarks

➠ gluon PDF grows at small x

Sum rules in pQCD

Beyond the naive parton model the probabilistic picture does not hold anymore. What about basic conservation principles (e.g. sum rules)?

Exercise: show that e.g.

$$\int_0^1 dx\,\left(f_u(x,\mu^2) - f_{\bar{u}}(x,\mu^2)\right) = \text{constant} \quad \text{if and only if} \quad \int_0^1 dz P_{qq}(z) = 0$$

Solution:

1. Start from DGLAP for u

$$\mu^2 \frac{\partial f_u(x,\mu^2)}{\partial \mu^2} = \frac{\alpha_s(\mu^2)}{2\pi} \int_x^1 \frac{dz}{z} \left(P_{uu}(z) f_u\left(\frac{x}{z},\mu^2\right) + P_{ug}(z) f_g\left(\frac{x}{z},\mu^2\right)\right)$$

2. Subtract the same equation for $\bar{u}$ and integrate over x

Sum rules in pQCD

2. Subtract the same equation for ū and integrate over x

$$\mu^2 \frac{\partial}{\partial \mu^2} \int_0^1 dx \left(f_u(x,\mu^2) - f_{\bar{u}}(x,\mu^2)\right) = \frac{\alpha_s(\mu^2)}{2\pi} \int_0^1 dx \int_x^1 \frac{dz}{z} P_{qq}(z) \left(f_u\left(\frac{x}{z},\mu^2\right) - f_{\bar{u}}\left(\frac{x}{z},\mu^2\right)\right)$$

3. Swap x and z integration, replace x with y = x/z

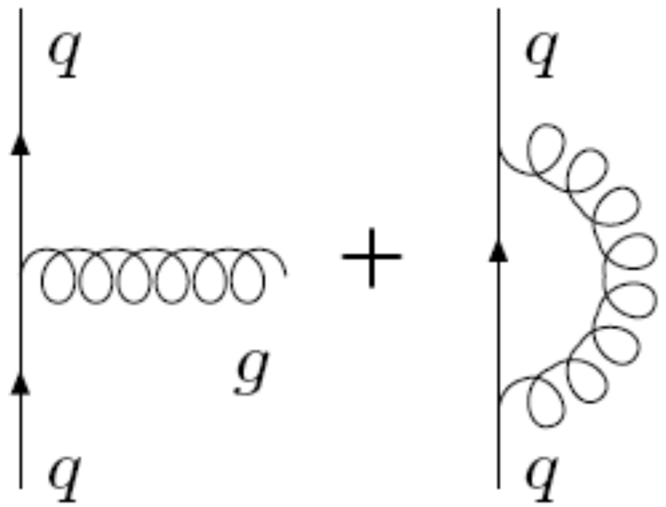
$$\mu^2 \frac{\partial}{\partial \mu^2} \int_0^1 dx \left(f_u(x,\mu^2) - f_{\bar{u}}(x,\mu^2)\right) = \frac{\alpha_s(\mu^2)}{2\pi} \int_0^1 dz P_{qq}(z) \int_0^1 dy \left(f_u\left(y,\mu^2\right) - f_{\bar{u}}\left(y,\mu^2\right)\right)$$

Conclusion: the integral $\int_0^1 dx \left(f_u(x,\mu^2) - f_{\bar{u}}(x,\mu^2)\right)$

does not depend on the scale if, and only if $\int_0^1 dz P_{qq}(z) = 0$

Properties of splitting functions

$$P_{qq}^{(0)} = P_{\bar{q}\bar{q}}^{(0)} = C_F \left[\left(\frac{1+z^2}{1-z} \right)_+ + \frac{3}{2}\delta(1-z) \right]$$



☛ the delta-term is the virtual correction (present only when the flavour does not change)

We have just seen that in order to conserve quark (baryon) number, the integral of the quark distribution can not vary with Q^2, hence, the splitting functions must integrate to zero

Exercise: use this fact to compute the coefficients of the pure delta terms in P_{qq} and P_{gg} without performing the loop integral!

History of splitting functions

☑ $P_{ab}^{(0)}$: Altarelly, Parisi; Gribov-Lipatov; Dokshitzer (1977)

☑ $P_{ab}^{(1)}$: Curci, Furmanski, Petronzio (1980)

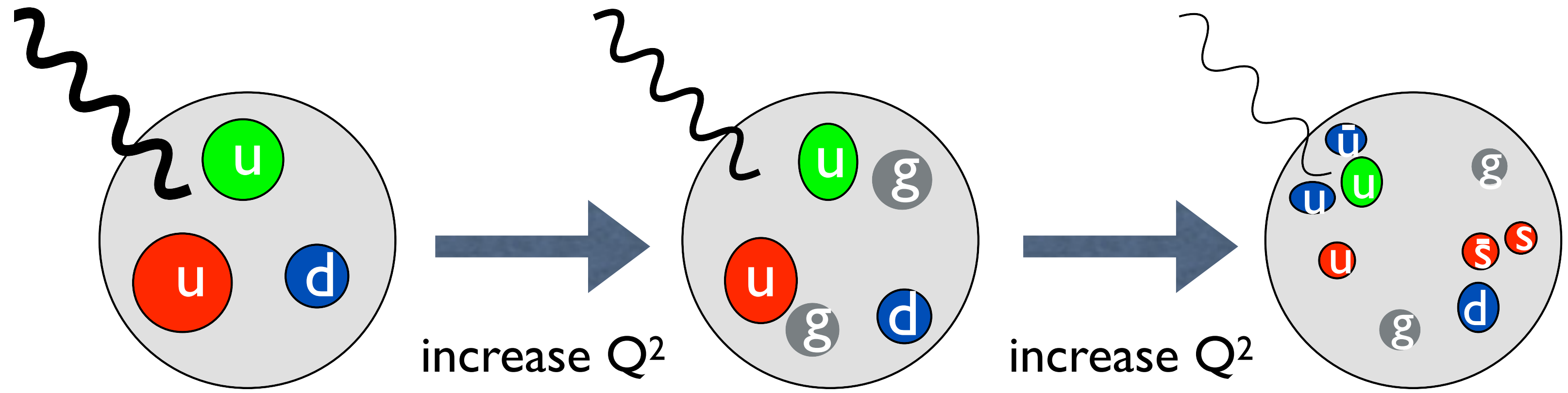
☑ $P_{ab}^{(2)}$: Moch, Vermaseren, Vogt (2004)

☛ Essential input for NNLO pdfs determination (state of the art today)

Evolution

So, in perturbative QCD we can not predict values for

- the coupling
- the masses
- the parton densities
- ...



What we can predict is the evolution with the Q^2 of those quantities.
These quantities must be extracted at some scale from data.

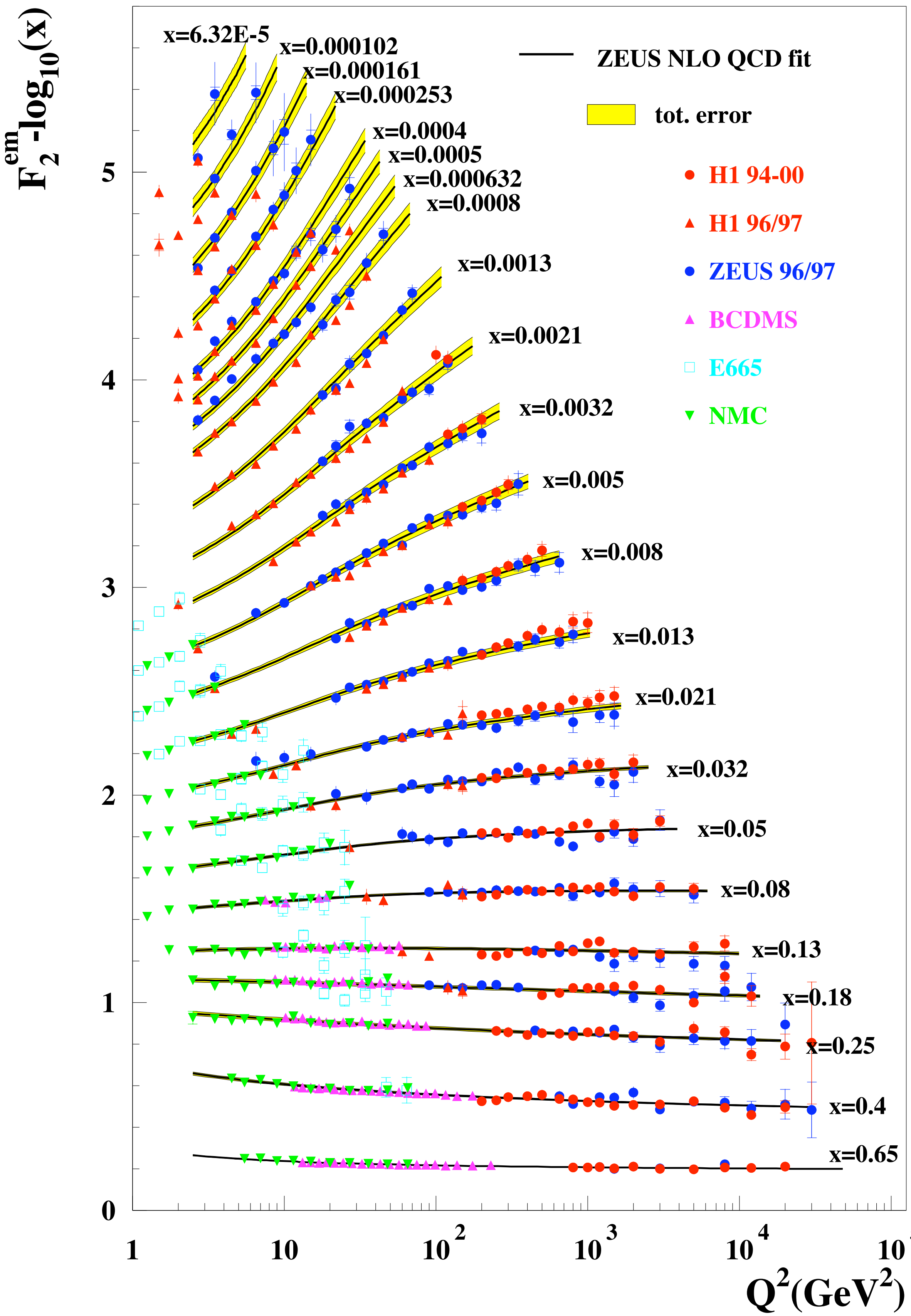
- not only is the coupling scale-dependent, but partons have a scale dependent sub-structure
- we started with the question of how one can access the gluon pdf: Because of the DGLAP evolution, we can access the gluon pdf indirectly, through the way it changes the evolution of quark pdfs. Today also direct measurements using Tevatron jet data and LHC tt and jet data

Recap.

- Parton model: incoherent sum of all partonic cross-sections
- Sum rules (momentum, charge, flavor conservation)
- Determination of parton densities (electron & neutrino scattering)
- Radiative corrections: failure of parton model
- Factorization of initial state divergences into scale dependent parton densities
- DGLAP evolution of parton densities $\Rightarrow$ measure gluon PDF
- While PDFs loose the naive probabilistic interpretation basic conservation principle still hold (momentum sum rules, energy, flavour conservation)

Data: F_2

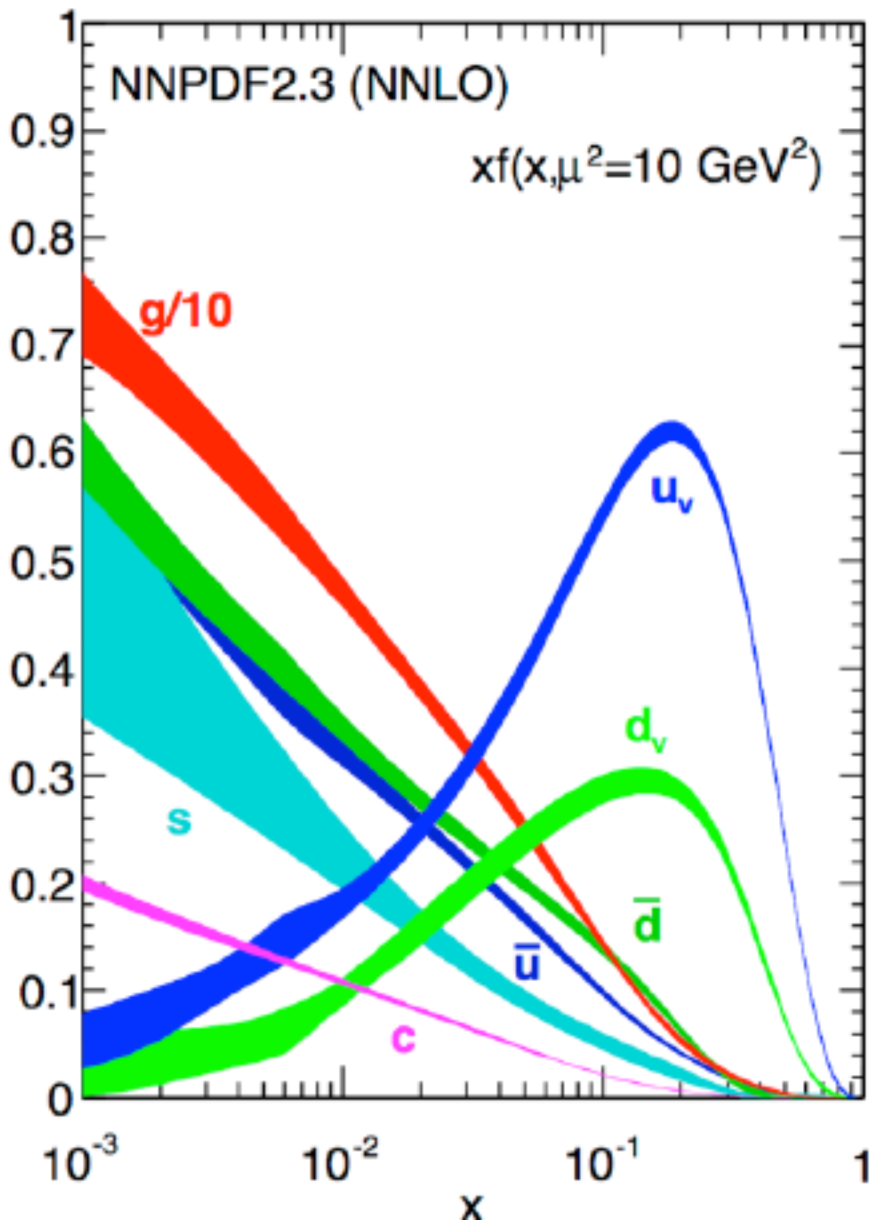
- DGLAP evolution equations allow to predict the Q^2 dependence of DIS data
- gluons crucial in driving the evolution



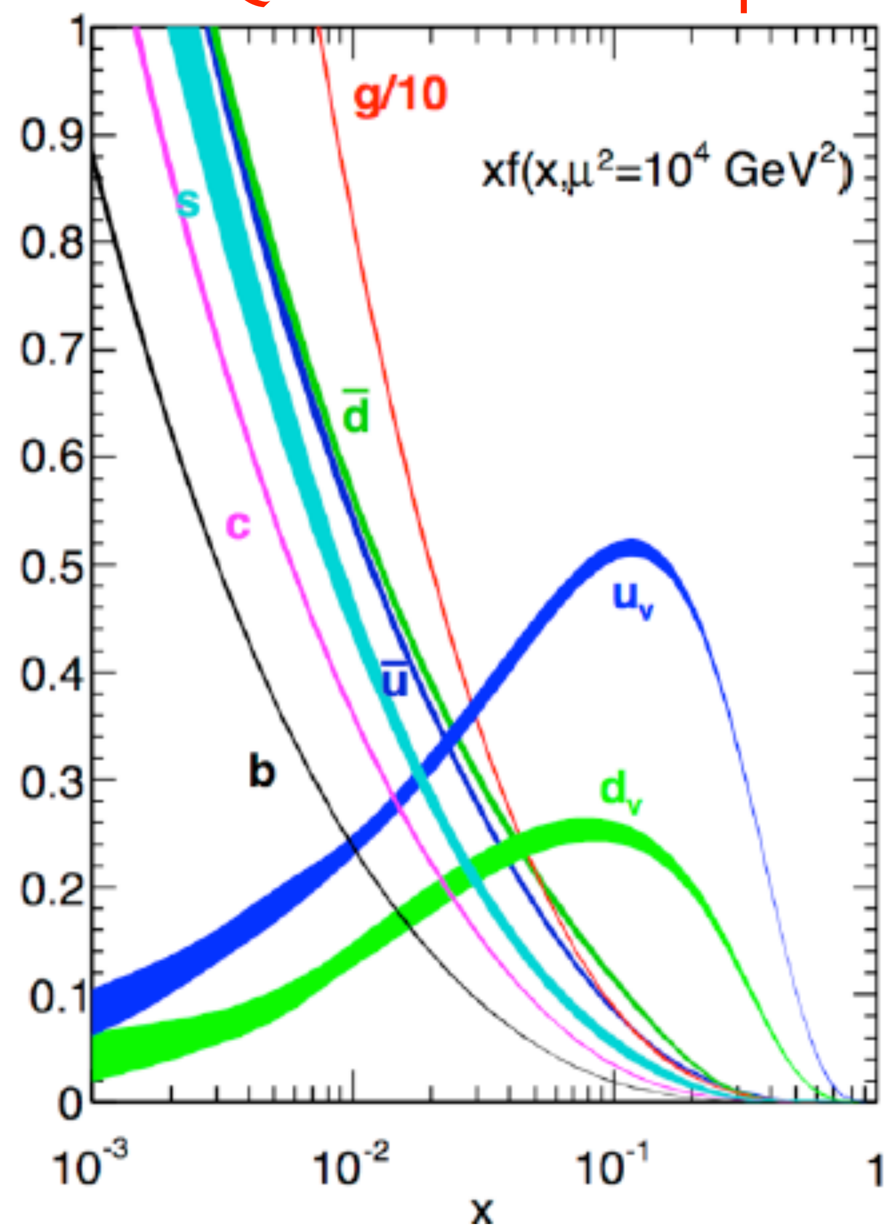
DGLAP Evolution

The DGLAP evolution is a key to precision LHC phenomenology: it allows to measure PDFs at some scale (say in DIS) and evolve upwards to make LHC (7, 8, 13, 14, 33, 100.... TeV) predictions

Measure PDFs at 10 GeV



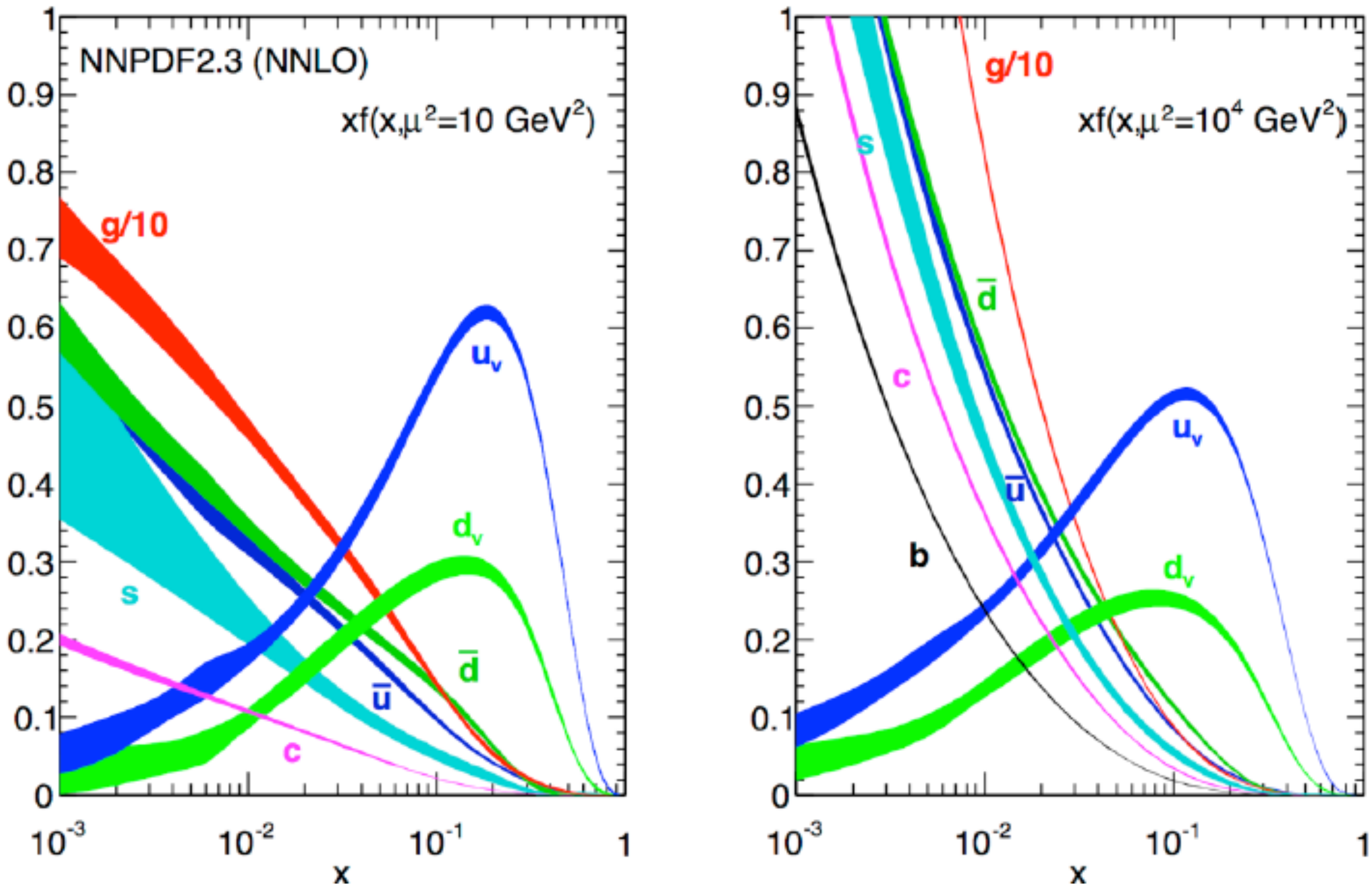
Evolve in Q^2 and make LHC predictions



Different PDFs evolve in different ways (different equations + unitarity constraint)

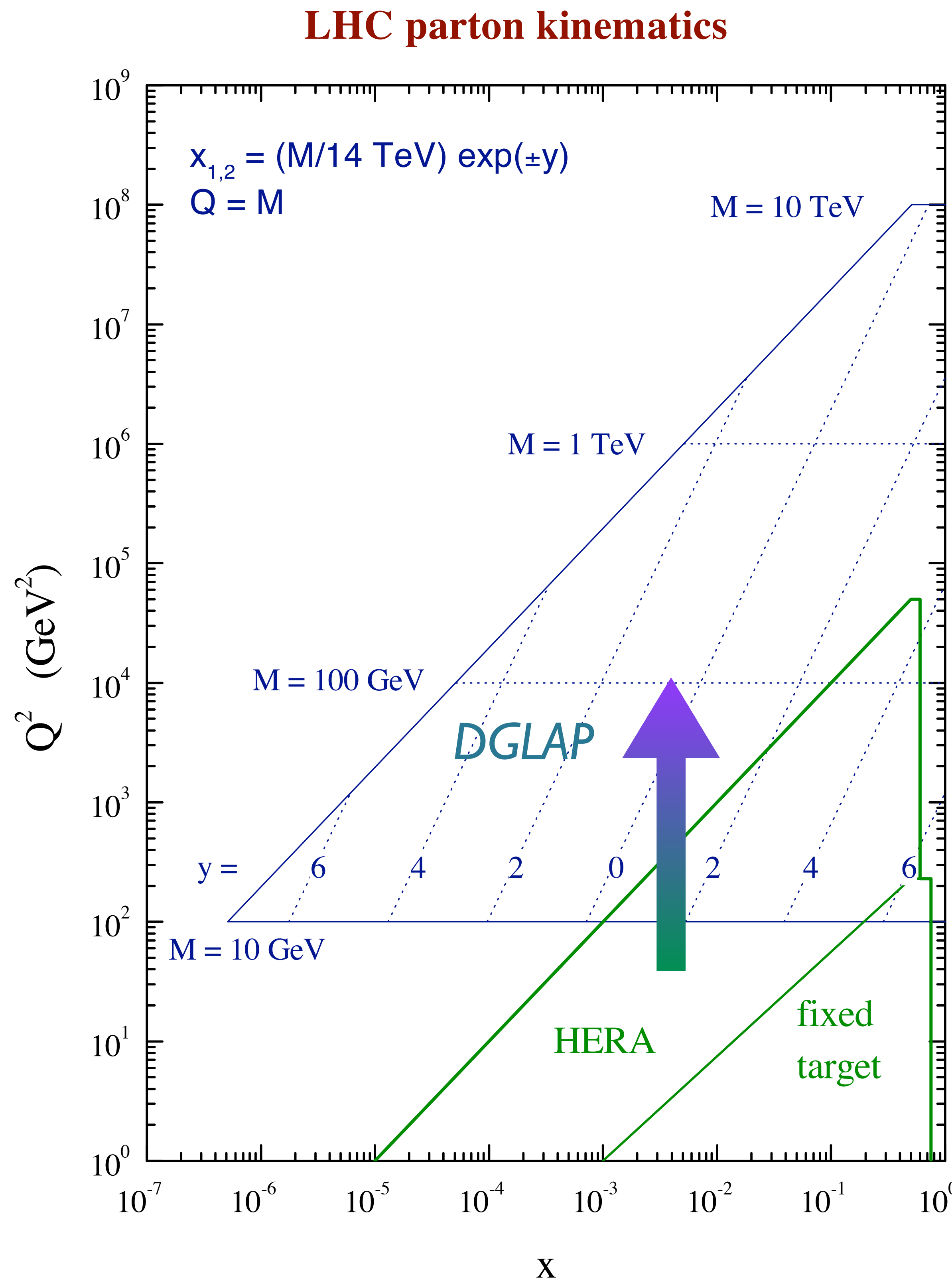
Typical features of PDFs

- vanish at $x \to 1$
- valence quarks peak at $x \simeq 1/3$
- gluon and sea distribution rise for $x \to 0$ (region dominated by gluons)



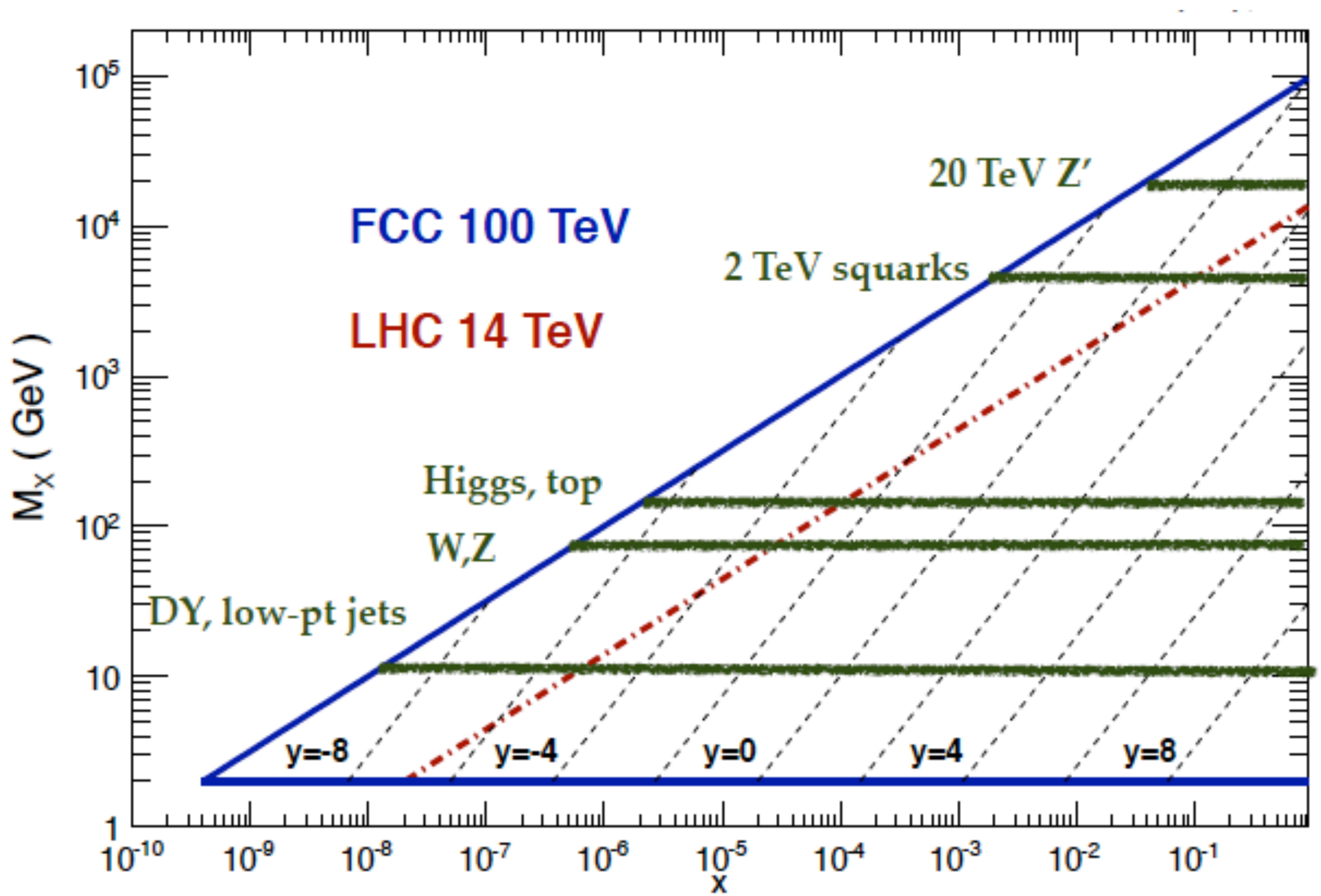
Parton density coverage

- most of the LHC x-range covered by Hera
- need 2-3 orders of magnitude Q^2-evolution
- rapidity distributions probe extreme x-values
- 100 GeV physics at LHC: small-x, sea partons
- TeV physics: large x



Parton density coverage

Coverage of 14 TeV LHC with respect to 100 TeV FCC



Progress in PDFs

PDFs are an essential ingredient for the LHC program.

Recent progress includes

- better assessment of uncertainties (e.g. different groups now agree at the 1σ level where data is available)
- exploit wealth of new information from LHC Run I and Run II measurements
- progress in tools and methods to include these data in the fits
- inclusion of PDFs for photons, leptons

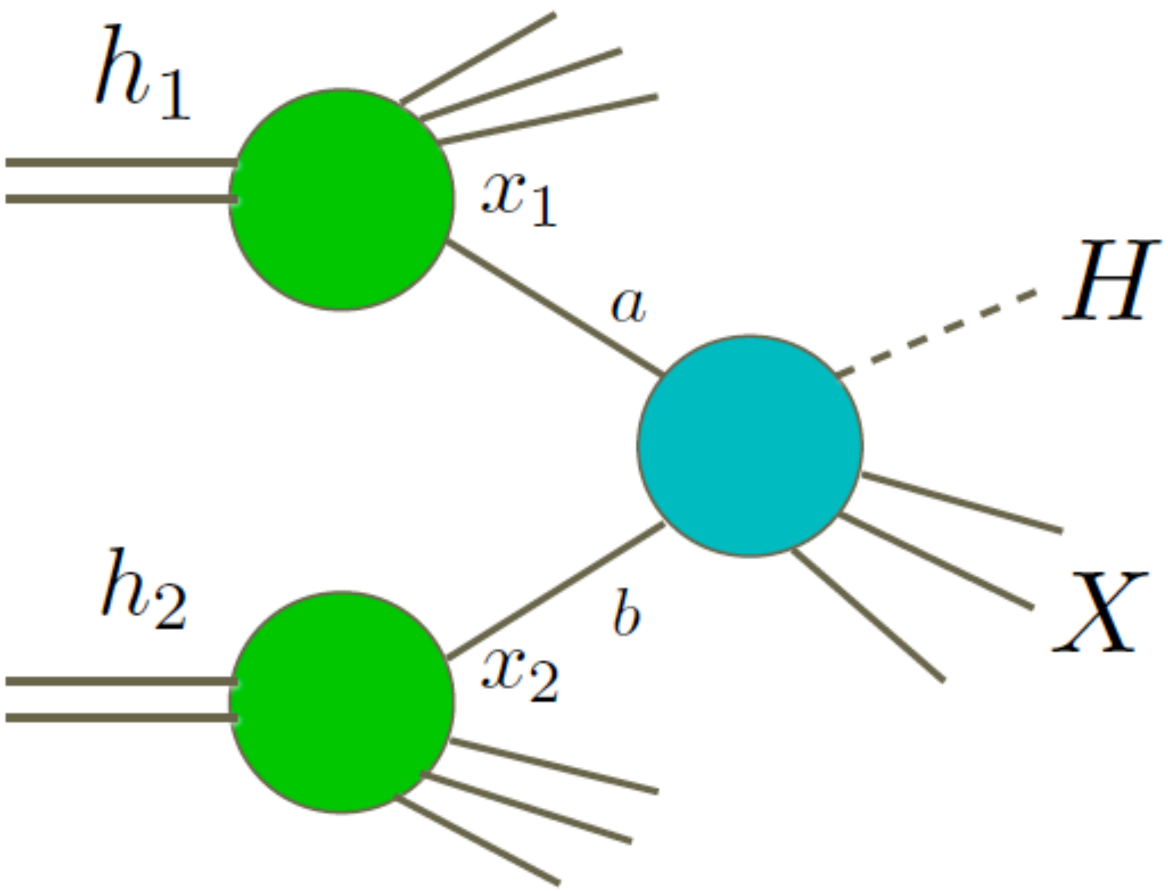
Progress in PDFs

Some issues under discussion

- which data to include in the fits (and how to deal with incompatible data)
- enhance relevance of some data (reduce effect of inconsistent data sets)
- heavy-quark treatment and masses
- parametrization for PDFs (theoretical bias, reduced in Neural Network PDFs)
- include theoretical improvement (e.g. resummation) for some observables
- unphysical behaviour close to x=0 and x=1
- meaning of uncertainties
- α_s as external input or fitted with PDFs
- how not to "fit away" New Physics effects in PDFs

Parton luminosities

Even more interesting that PDFs are parton luminosities for each production channel

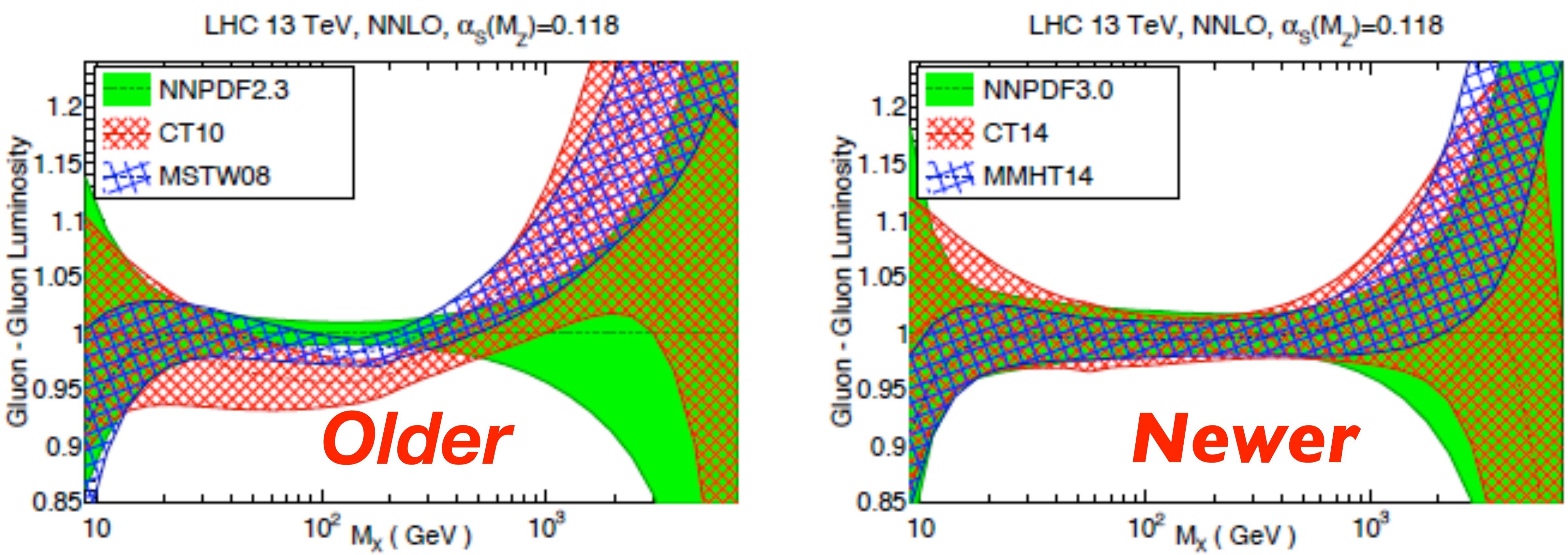


$$\sigma(S) = \sum_{i,j} \int d\tau \left[\frac{1}{S} \frac{dL_{ij}}{d\tau} \right] [\hat{s}\hat{\sigma_{ij}}]$$

$$\tau \frac{dL_{ij}}{d\tau} = \int_0^1 dx_1 dx_2 x_1 f_i(x_1, \mu_F^2) \times x_2 f_j(x_2, \mu_F^2) \delta(\tau - x_1 x_2)$$

Progress in PDFs: gluon luminosity

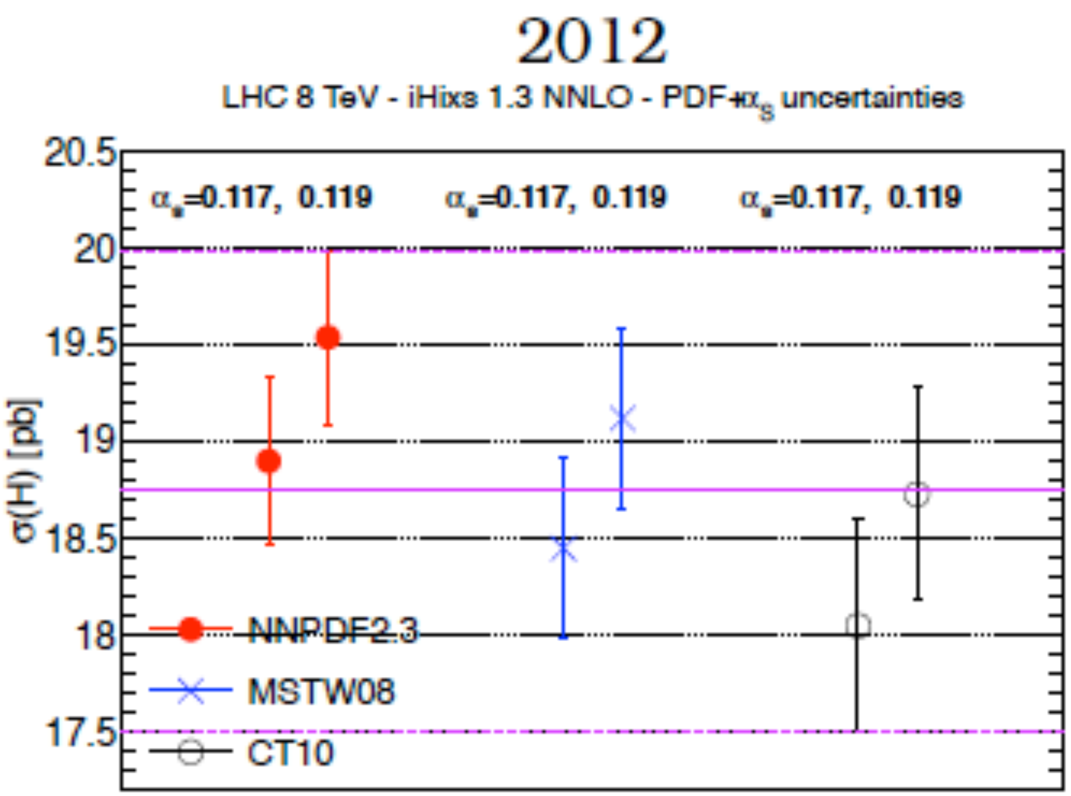
Example: gluon-gluon luminosity as needed for Higgs measurements

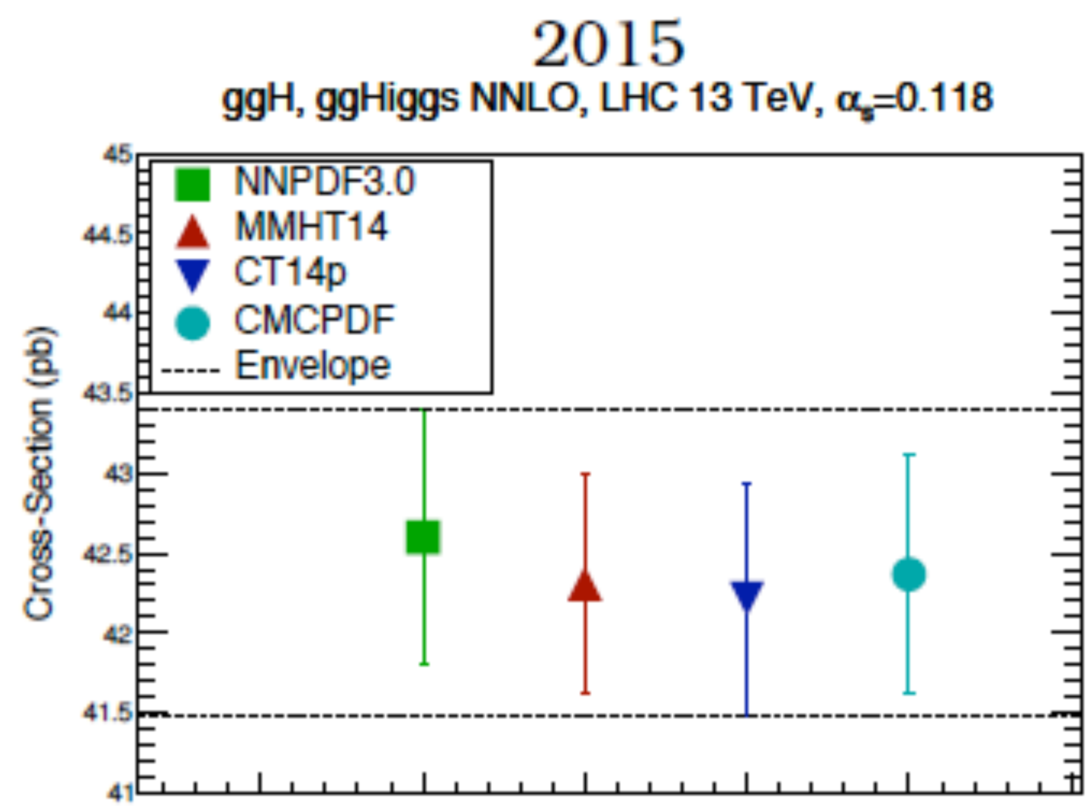


- obvious improvement from older sets to newer ones
- agreement at 1σ between different PDFs in the intermediate mass region relevant for Higgs studies (but larger differences at large M, key-region for NP searches)

Progress in PDFs: Higgs case

Improved control on gluon distributions results in more consistent Higgs production cross-sections





- PDF uncertainty in the Higgs cross-section down to about 2-3%
- envelope of 3 PDFs (previous recommendation) no longer needed

Intermediate recap.

- In the QCD parton model, hadrons are treated as bound states of quasi-fee point-like quarks is very successful to explain DIS measurements
- In this model, the probability to find a parton with a given momentum fraction is given by the (scale independent) parton distribution function
- The model breaks down once one includes initial state radiation since collinear divergences do not cancel
- This leads to scale dependent parton distribution functions
- The dependence is governed by the DGLAP evolution equations
- QCD factorisation means that PDFs are universal and process-independent quantities: they can be measured in some process, at some scale, and use in a different process at a different scale
- PDFs are today determined by global fits to data

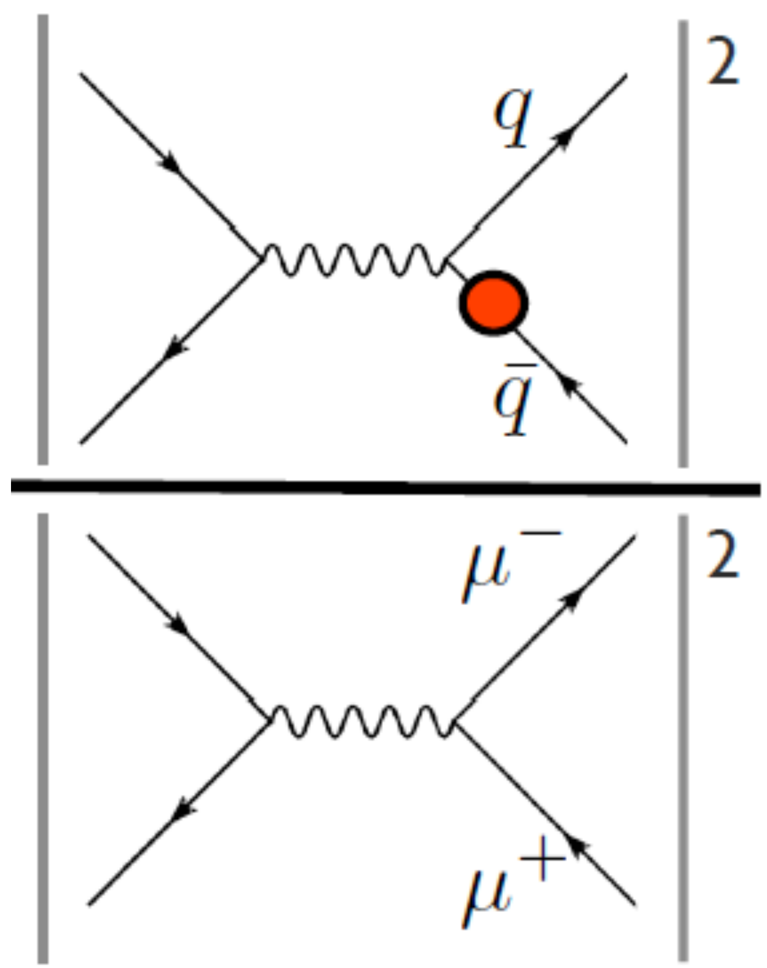
Next

Next we'll discuss generic properties of QCD amplitudes

- Soft-collinear divergences (and how they are dealt with)
- Kinoshita-Lee-Nauenberg theorem
- The concept of infrared finiteness
- Sterman Weinberg jets

R-ratio

Let's consider again the R-ratio

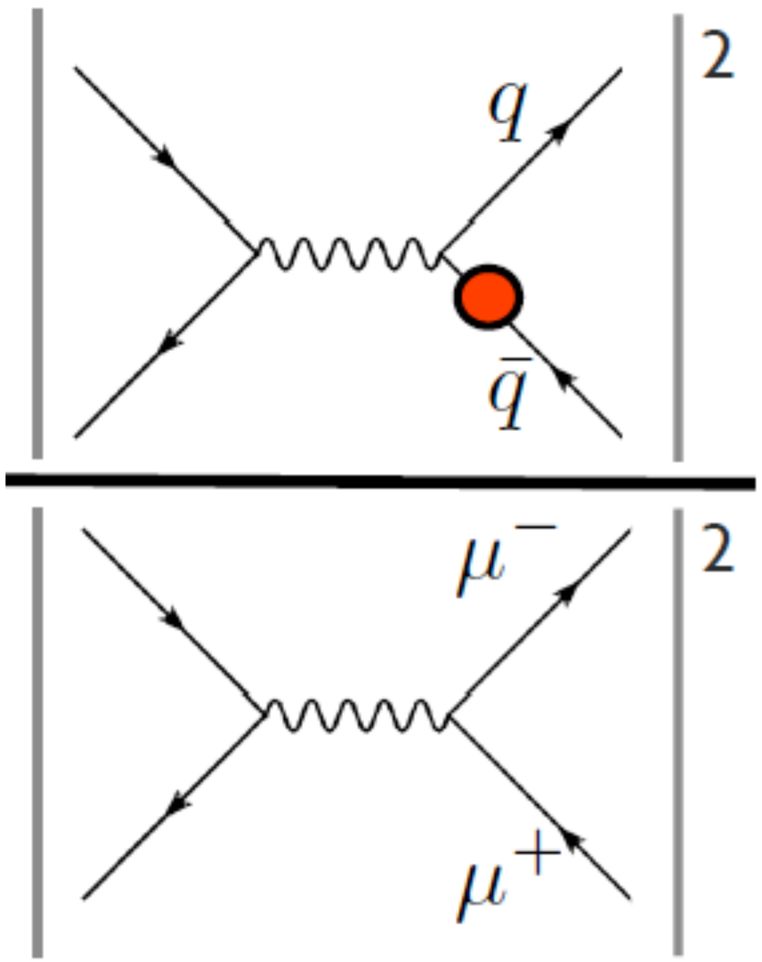


Leading order result

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \approx N_c \sum_q e_q^2$$

R-ratio

Let’s consider again the R-ratio



We have seen a good agreement between the leading order result and data, but there are various unanswered questions

- Since free quarks do not exist, why is the leading order result so good?
- In particular, why can one identify the cross-sections for the production of quarks to that of hadrons?
- Can one probe QCD further by testing more exclusive observables?

Quark-hadron duality

The reliability of parton-level calculations to describe hadron-level observables is known as quark-hadron duality.

This duality relies on the time separation between a hard scattering (partons are produced) and a soft process (quarks hadronize). Since the two processes happen at very different time-scales there is not quantum interference and the soft process does not alter the hard momentum flow "too much"

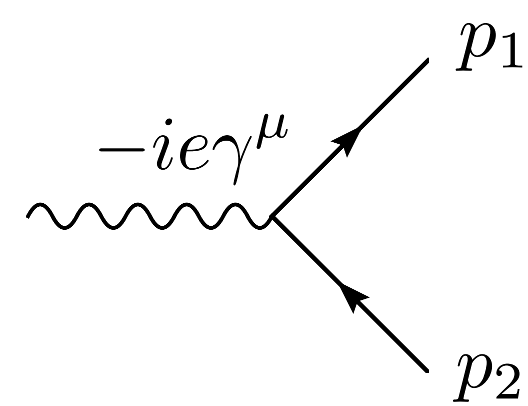
With this in mind, let's apply the parton description and look for a better approximation of R, i.e. let's compute QCD corrections, at least in some approximation

The soft approximation

QCD corrections are only in the final state, i.e. corrections to $\gamma^* \to q\bar{q}$

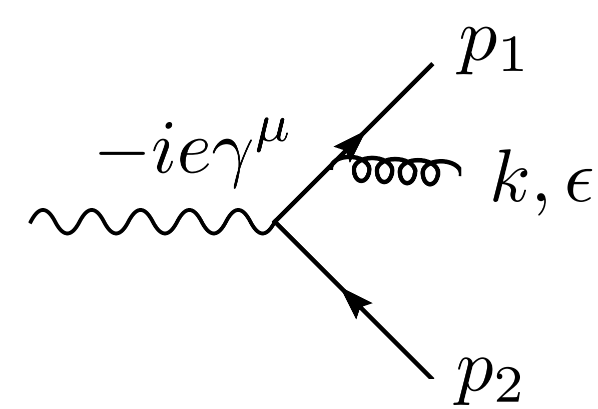
At leading order:

$$M_0^\mu = \bar{u}(p_1)(-ie\gamma^\mu)v(p_2)$$



Emit one gluon:

$$\begin{aligned} M^\mu_{q\bar{q}g} &= \bar{u}(p_1)(-ig_s t^a \not{\epsilon})\frac{i(\not{p}_1+\not{k})}{(p_1+k)^2}(-ie\gamma^\mu)v(p_2) \\ &+ \bar{u}(p_1)(-ie\gamma^\mu)\frac{i(\not{p}_2-\not{k})}{(p_2-k)^2}(-ig_s t^a \not{\epsilon})v(p_2) \end{aligned}$$



Consider the soft approximation: $k \ll p_1, p_2$

$$M^\mu_{q\bar{q}g} = \bar{u}(p_1)\left((-ie\gamma^\mu)(-ig_s t^a)v(p_2)\right)\left(\frac{p_1\epsilon}{p_1 k} - \frac{p_2\epsilon}{p_2 k}\right)$$

$\Rightarrow$ factorization of soft part (crucial for resummed calculations)

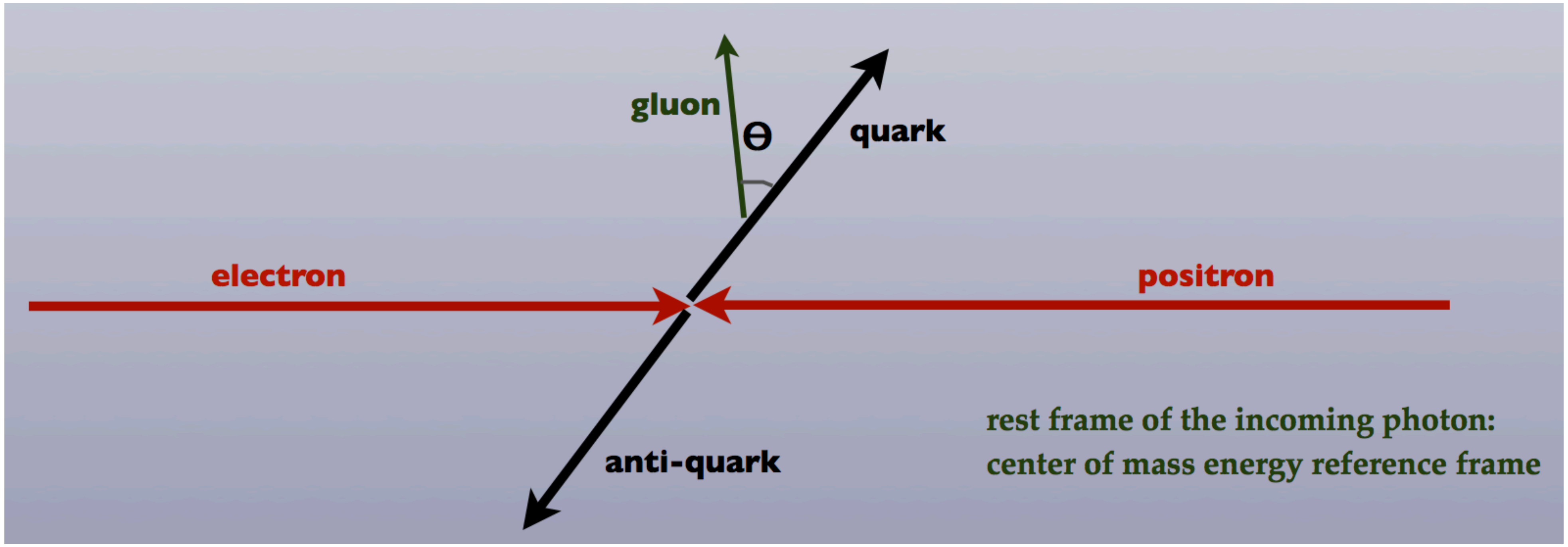
Soft divergences

The squared amplitude becomes

$$|M^{\mu}_{q\bar{q}g}|^2 = \sum_{\text{pol}} \left| \bar{u}(p_1)\left((-ie\gamma^{\mu})(-ig_s t^a)v(p_2)\right) \left(\frac{p_1\epsilon}{p_1 k} - \frac{p_2\epsilon}{p_2 k}\right)\right|^2$$

$$= \quad |M_{q\bar{q}}|^2 C_F g_s^2 \frac{2p_1p_2}{(p_1k)(p_2k)}$$

The above is a Lorentz-invariant amplitude. Go to the centre-of-mass frame:



Soft divergences

The squared amplitude becomes

$$|M^\mu_{q\bar{q}g}|^2 = \sum_{\text{pol}} \left| \bar{u}(p_1)\left((-ie\gamma^\mu)(-ig_s t^a)v(p_2)\right)\left(\frac{p_1\epsilon}{p_1 k} - \frac{p_2\epsilon}{p_2 k}\right)\right|^2$$

$$= \quad |M_{q\bar{q}}|^2 C_F g_s^2 \frac{2p_1 p_2}{(p_1 k)(p_2 k)}$$

Including phase space, in this frame, in terms of energy and angle of the gluon one contains

$$d\phi_{q\bar{q}g}|M_{q\bar{q}g}|^2 \quad = \quad d\phi_{q\bar{q}}|M_{q\bar{q}}|^2 \frac{d^3k}{2\omega(2\pi)^3} C_F g_s^2 \frac{2p_1 p_2}{(p_1 k)(p_2 k)}$$

$$= \quad d\phi_{q\bar{q}}|M_{q\bar{q}}|^2 \omega d\omega d\cos\theta \frac{d\phi}{2\pi}\frac{2\alpha_s C_F}{\pi}\frac{1}{\omega^2(1-\cos^2\theta)}$$

The differential cross section becomes

$$d\sigma_{q\bar{q}g} = d\sigma_{q\bar{q}} \frac{2\alpha_s C_F}{\pi}\frac{d\omega}{\omega}\frac{d\theta}{\sin\theta}\frac{d\phi}{2\pi}$$

Soft & collinear divergences

Cross section for producing a $q\bar{q}$-pair and a gluon is infinite (IR divergent)

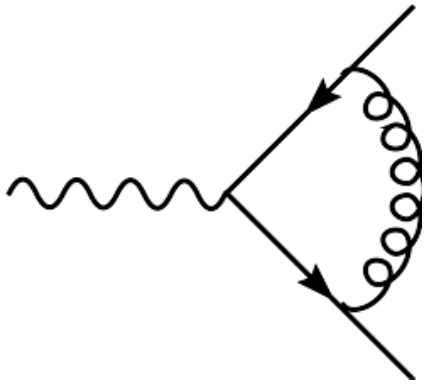
$$d\sigma_{q\bar{q}g} = d\sigma_{q\bar{q}} \frac{2\alpha_s C_F}{\pi} \frac{d\omega}{\omega} \frac{d\theta}{\sin\theta} \frac{d\phi}{2\pi}$$

$\omega \to 0$: soft divergence

$\theta \to 0$: collinear divergence

But the full $O(\alpha_s)$ correction to R is finite, because one must include a virtual correction which cancels the divergence of the real radiation

$$d\sigma_{q\bar{q},v} \sim -d\sigma_{q\bar{q}} \frac{2\alpha_s C_F}{\pi} \frac{d\omega}{\omega} \frac{d\theta}{\sin\theta} \frac{d\phi}{2\pi}$$



NB: here we kept only soft terms, if we do the full calculation one gets a finite correction of α_s/π

Soft & collinear divergences

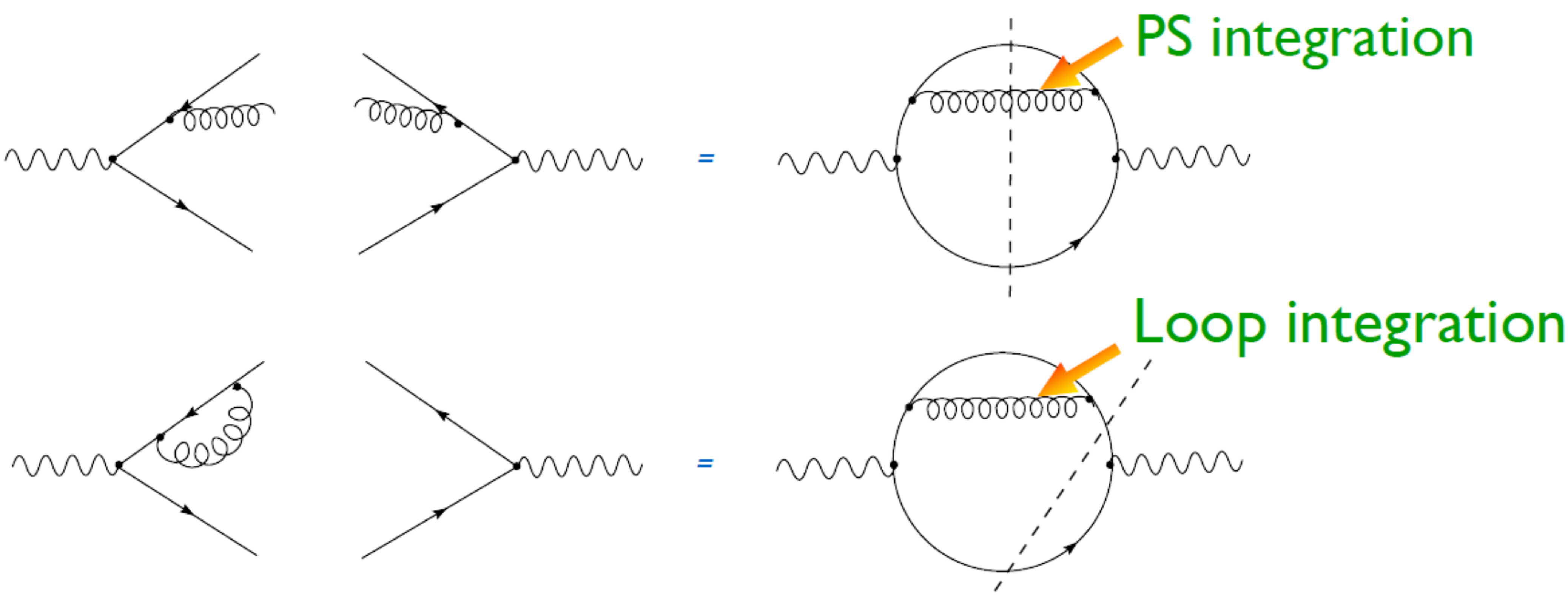
$\underline{\omega \to 0}$ soft divergence: the four-momentum of the emitted particle approaches zero, typical of gauge theories, even if matter (radiating particle) is massive

$\underline{\theta \to 0}$ collinear divergence: particle emitted collinear to emitter. Divergence present only if all particles involved are massless

NB: the appearance of soft and collinear divergences discussed in the specific context of $e^+e^- \to qq$ are a general property of QCD

Infrared finiteness

Cancellation of IR divergences in R is not a miracle. It follows directly from unitarity provided the measurement is inclusive enough

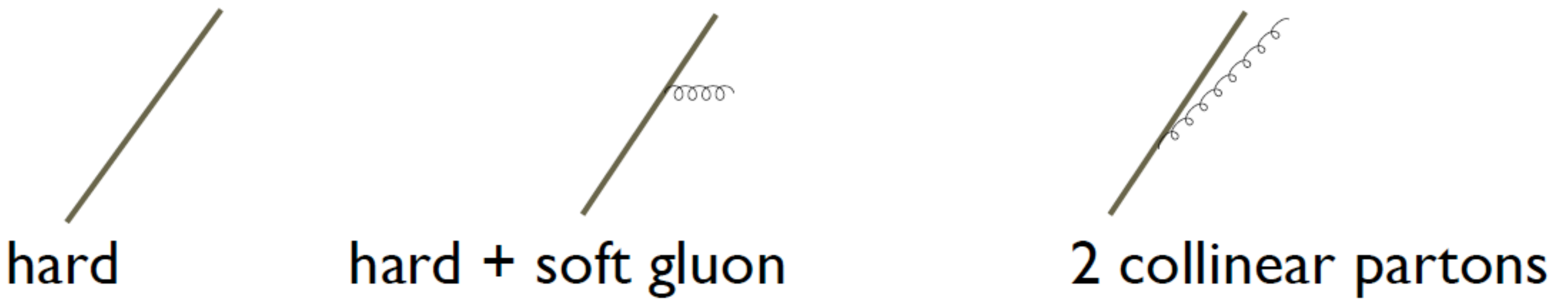


In the infrared region real and virtual are kinematically equivalent but for a (-1) from unitarity

Compute and regulate real and virtual separately, until a cancelation of divergences is achieved

KLN Theorem

Kinoshita-Lee-Nauenberg theorem: Infrared singularities in a massless theory cancel out after summing over degenerate (initial and final) states



Physically a hard parton can not be distinguished from a hard parton plus a soft gluon or from two collinear partons with the same energy. They are degenerate states.
Hence, one needs to add them to get a physically sound observable

Infrared safety (= finiteness)

So, the R-ratio is an infrared safe quantity.

In perturbation theory one can compute only IR-safe quantities, otherwise get infinities, which can not be renormalized away (why not…?)

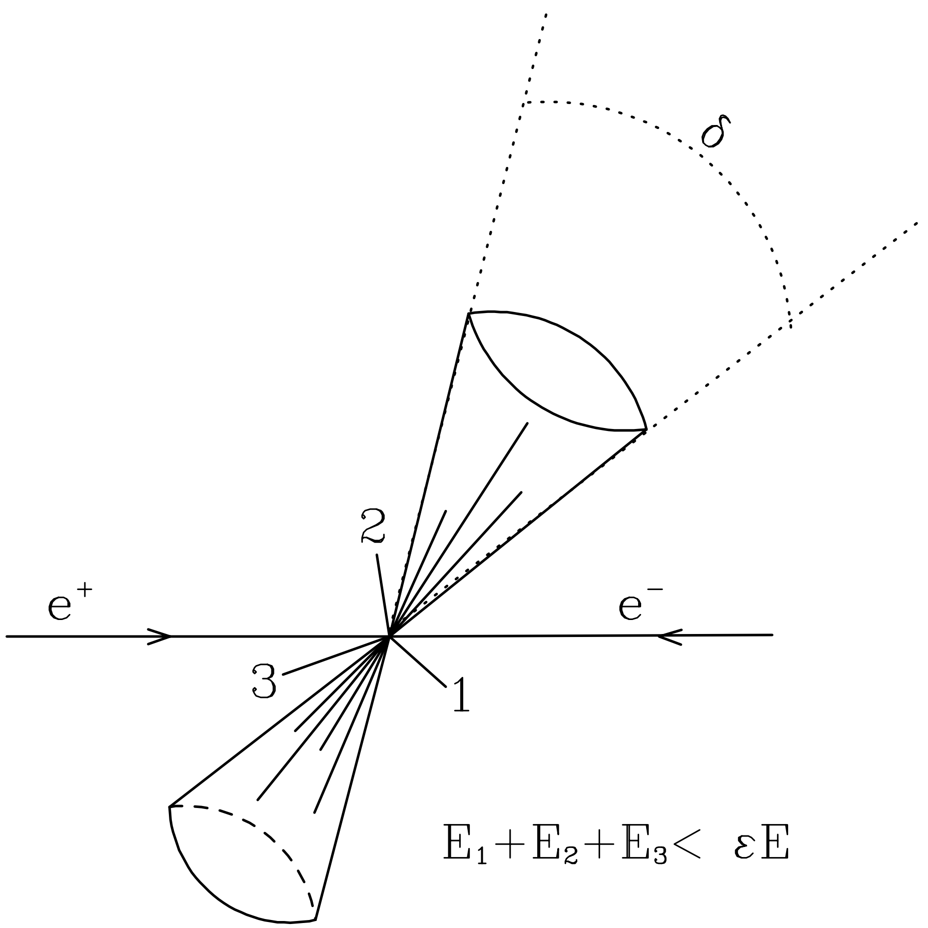
So, the natural questions are:

- are there other IR-safe quantities?
- what property of R guarantees its IR-safety?

Sterman-Weinberg jets

First formulation of cross-sections which are finite in perturbation theory and describe the hadronic final state

Introduce two parameters ε and δ:
a pair of Sterman-Weinberg jets are two cones of opening angle δ that contain all the energy of the event excluding at most a fraction ε

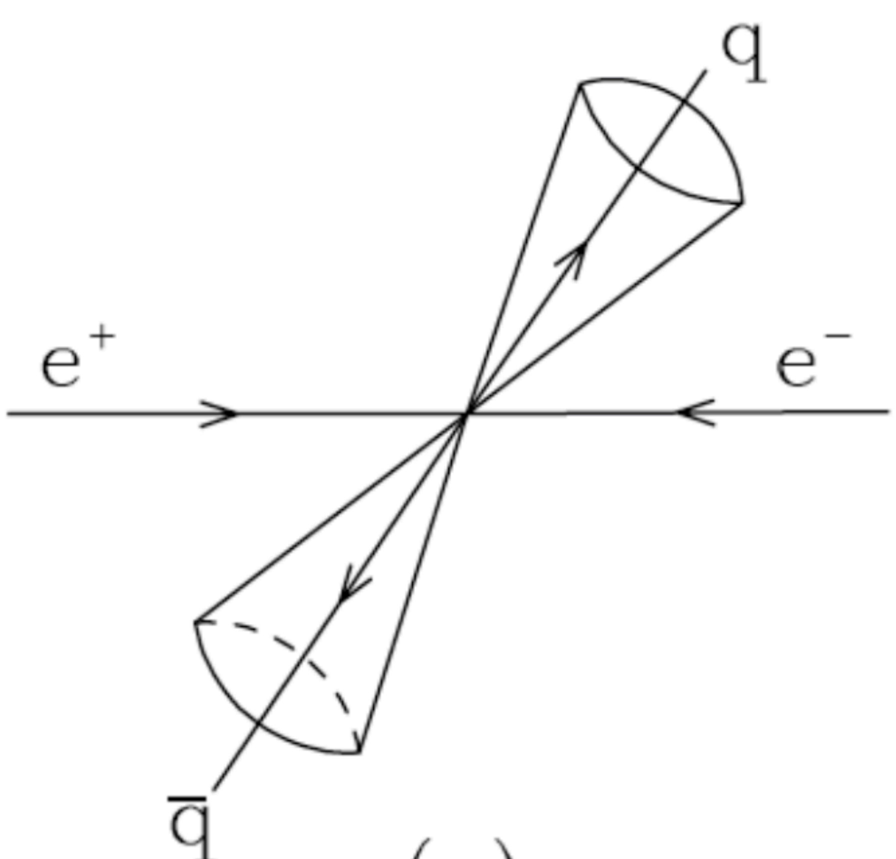


Why finite? the cancelation between real and virtual is not destroyed in the soft/collinear regions

Sterman-Weinberg jets

Let's compute the O(as) correction to the Sterman-Weinberg jet cross-section in the soft-collinear approximation

a) We have a Born term σ_B which is completely within the Sterman-Weinberg jet definition: since there are only two quarks they keep all the energy inside the cones

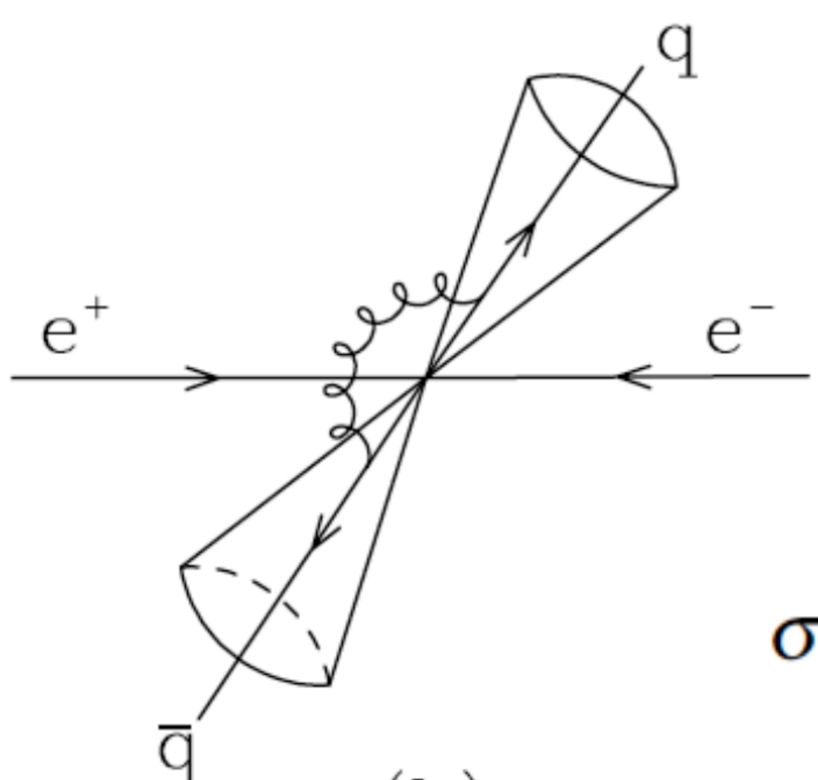


(a)

Sterman-Weinberg jets

Let's compute the O(as) correction to the Sterman-Weinberg jet cross-section in the soft-collinear approximation

b) We have a virtual term which is also completely within the Sterman-Weinberg jet definition (only two quarks)



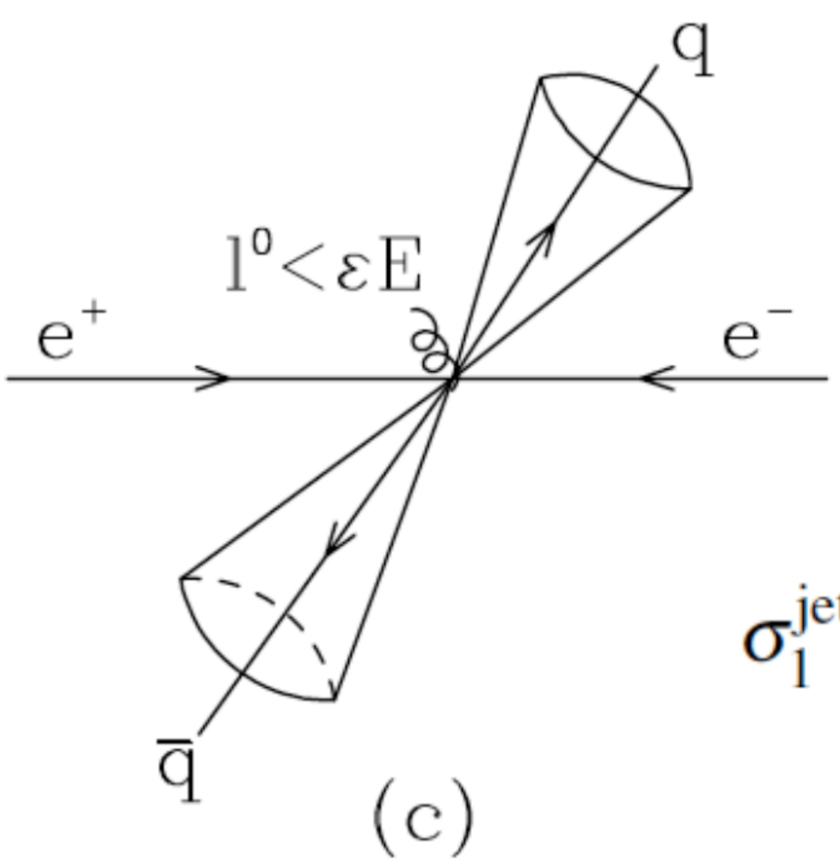
(b)

$$\sigma_1^{\mathrm{jet,virt}} = -C_F \frac{g_S^2}{\pi^2} \sigma_{\mathrm{born}} \int_0^E \frac{dl_0}{l_0} \int_0^\pi \frac{d\cos\theta}{1-\cos^2\theta}$$

Sterman-Weinberg jets

Let's compute the O(as) correction to the Sterman-Weinberg jet cross-section in the soft-collinear approximation

c) We have a real term: the emitted gluon can be emitted also outside the jet provided it carries only little energy, or..

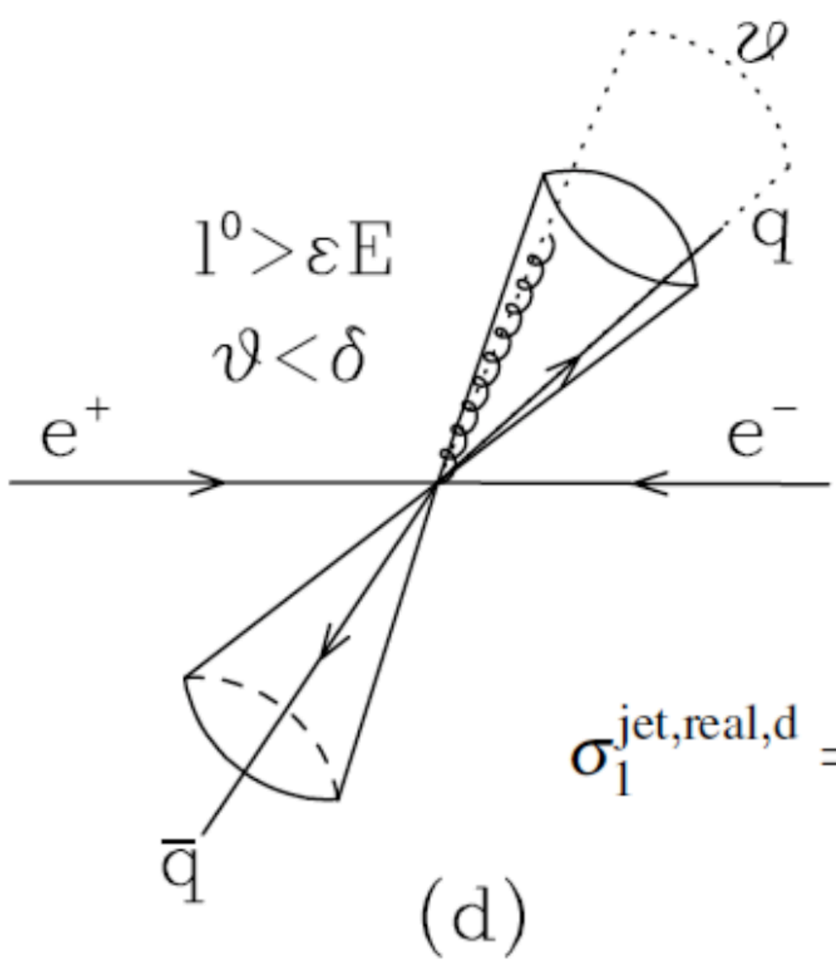


$$\sigma_1^{\text{jet,real,c}} = C_F \frac{g_S^2}{\pi^2} \sigma_{\text{born}} \int_0^{\varepsilon E} \frac{dl_0}{l_0} \left[\int_0^{\pi} \frac{d\cos\theta}{1-\cos^2\theta} \right]$$

Sterman-Weinberg jets

Let's compute the O(as) correction to the Sterman-Weinberg jet cross-section in the soft-collinear approximation

d) .. or it can carry a considerable fraction of energy provided it is emitted inside the cones



$$\sigma_1^{\text{jet,real,d}} = C_F \frac{g_S^2}{\pi^2} \sigma_{\text{born}} \int_{\varepsilon E}^{E} \frac{dl_0}{l_0} \left[\int_0^{\delta} \frac{d\cos\theta}{1-\cos^2\theta} + \int_{\pi-\delta}^{\pi} \frac{d\cos\theta}{1-\cos^2\theta} \right]$$

Sterman-Weinberg jets

Adding all the contributions, the Sterman-Weinberg jet cross-section up to $O(\alpha_s)$ in the soft-collinear approximation is given by

$$\sigma_1 = \sigma_0 \left(1 + \frac{2\alpha_s C_F}{\pi} \ln \epsilon \ln \delta^2 \right)$$

Effective expansion parameter in QCD is often $\alpha_s C_F/\pi$ not α_s

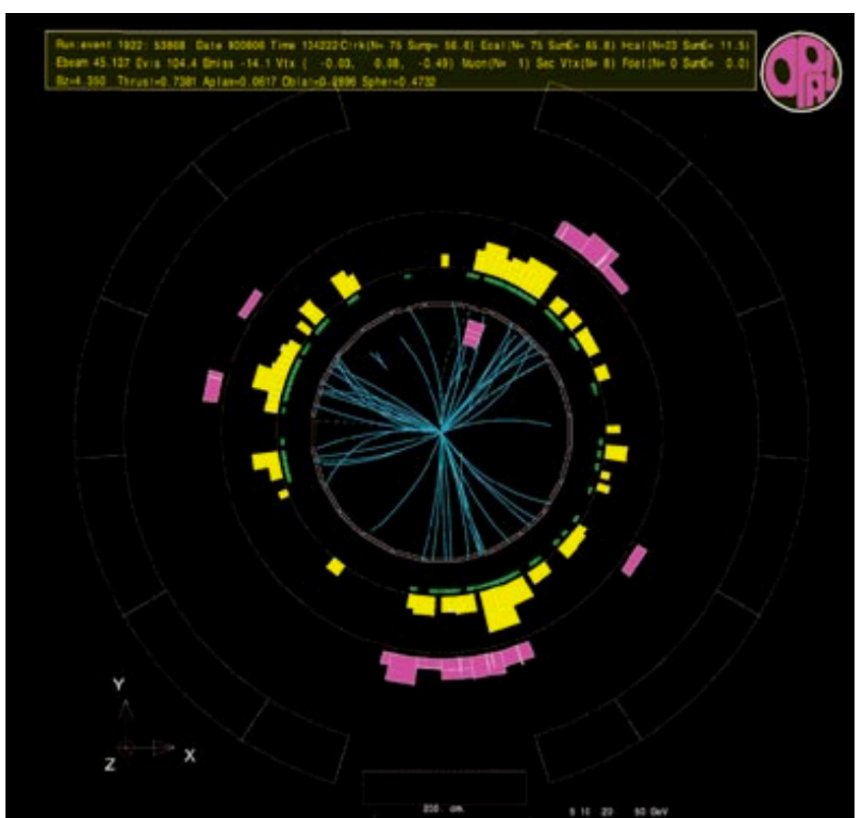
α_s-expansion enhanced by a double log: left-over from real-virtual cancellation

- if more gluons are emitted, one gets for each gluon
  - a power of $\alpha_s C_F/\pi$
  - a soft logarithm $\ln\varepsilon$
  - a collinear logarithm $\ln\delta$
- if ε and/or δ become too small the above result diverges
- if the logs are large, fixed order meaningless, one needs to resum large infrared and collinear logarithms to all orders in the coupling constant

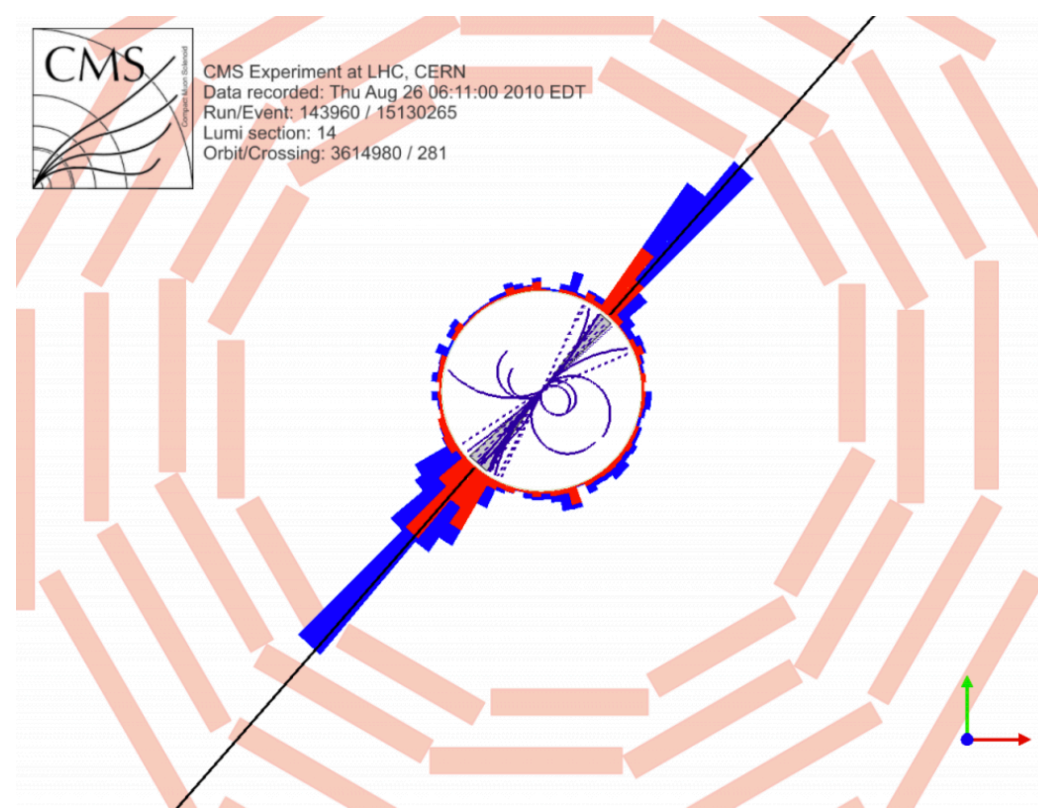
Jets

- Jets were discovered in the late 70s in electron-position collision
- They provided the first direct evidence for the gluon (we'll discuss indirect evidence later)
- In the 80s and 90s jets provided many other stringent tests of QCD at LEP
- Today jets are one of the powerful tools to look for New Physics at the LHC

Gluon discovery: 3jet event in e^+e^-



High energy di-jet event at CMS



Infrared safety: definition

An observable $\mathcal{O}$ is infrared and collinear safe if

$$\mathcal{O}_{n+1}(k_1, k_2, \ldots, k_i, k_j, \ldots k_n) \to \mathcal{O}_n(k_1, k_2, \ldots k_i + k_j, \ldots k_n)$$

whenever one of the k_i/k_j becomes soft or k_i and k_j are collinear

i.e. the observable is insensitive to emission of soft particles or to collinear splittings

Infrared safety: examples

Infrared safe ?

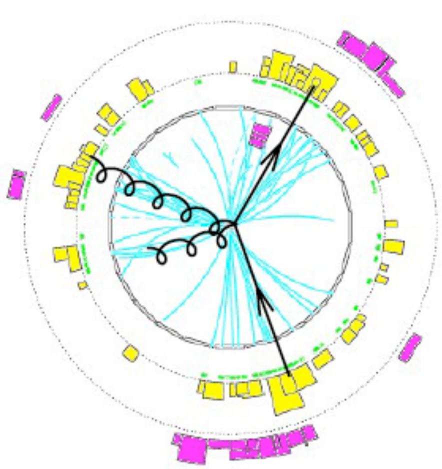
- energy of the hardest particle in the event NO
- multiplicity of gluons NO
- momentum flow into a cone in rapidity and angle YES
- cross-section for producing one gluon with $E > E_{min}$ and $\theta > \theta_{min}$ NO
- jet cross-sections DEPENDS

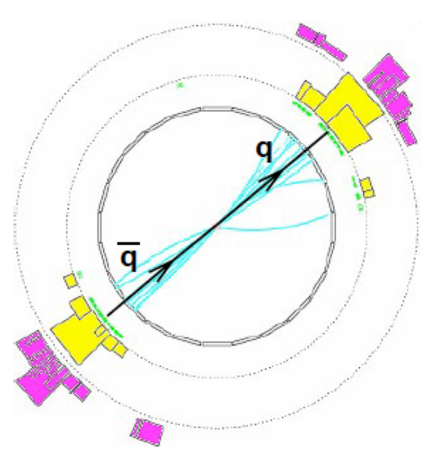
Only for infrared safe quantities is a comparison of data and theory well defined to all orders in perturbation theory

Other IR safe quantities

Event shapes: describe the shape of the event, but are largely insensitive to soft and collinear branching

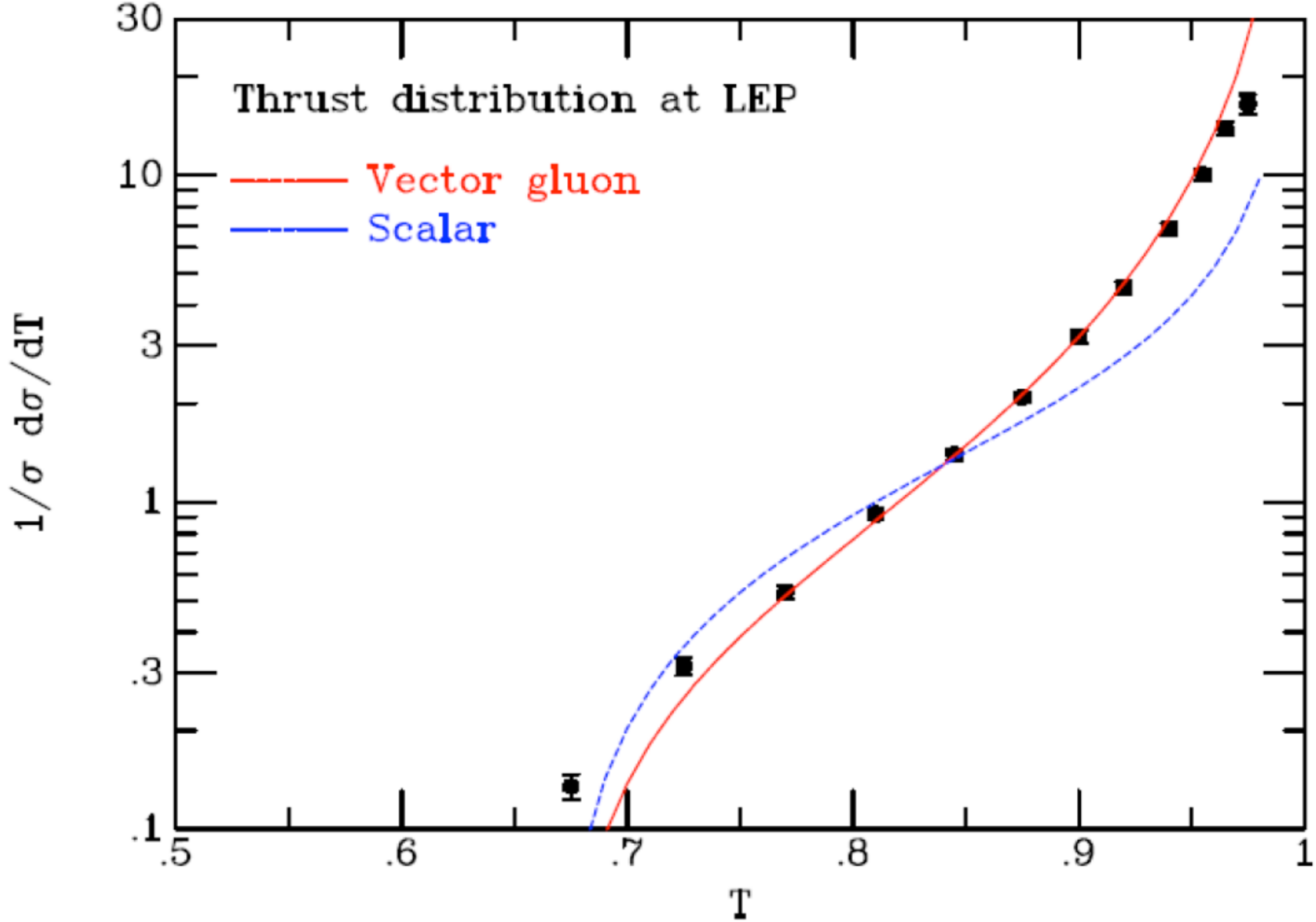
- widely used to measure α_s
- measure color factors
- test QCD
- learn about non-perturbative physics



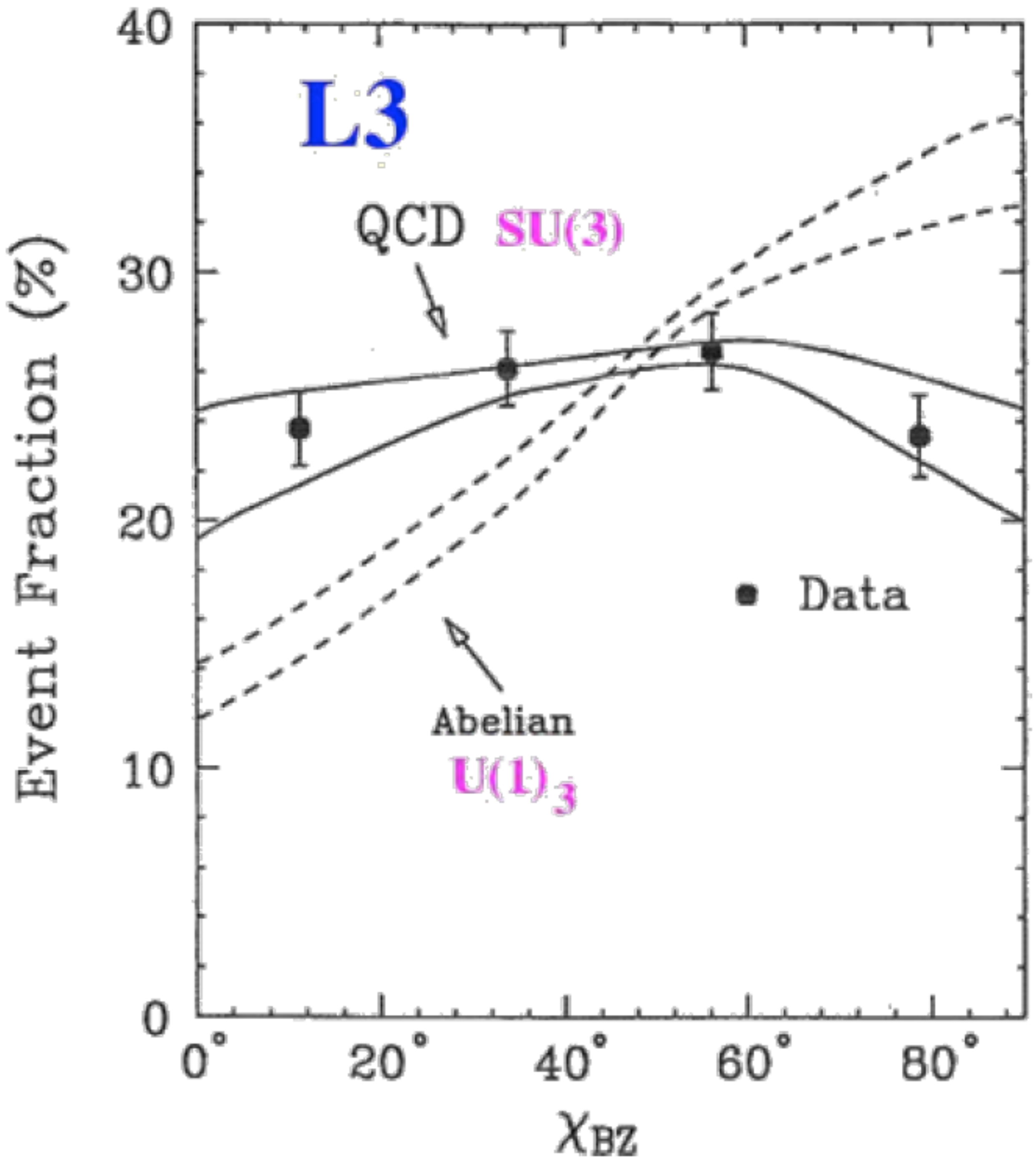


Name of Observable	Definition	Typical Value for: 			QCD calculation
Thrust	$T = \max_{\vec{n}}\left(\frac{\sum_i \lvert \vec{p}_i \vec{n} \rvert}{\sum_i \lvert \vec{p}_i \rvert}\right)$	1	≥2/3	≥1/2	(resummed) $O(\alpha_s^2)$
Thrust major	Like T, however T_{maj} and $\vec{n}_{maj}$ in plane $\perp \vec{n}_T$	0	≤1/3	≤1/√2	$O(\alpha_s^2)$
Thrust minor	Like T, however T_{min} and $\vec{n}_{min}$ in direction $\perp$ to $\vec{n}_T$ and $\vec{n}_{maj}$	0	0	≤1/2	$O(\alpha_s^2)$
Oblateness	$O = T_{maj} - T_{min}$	0	≤1/3	0	$O(\alpha_s^2)$
Sphericity	$S = 1.5\,(Q_1 + Q_2)$; $Q_1 \le ... \le Q_3$ are Eigenvalues of $S^{\alpha\beta} = \frac{\sum_i p_i^\alpha p_i^\beta}{\sum_i p_i^2}$	0	≤3/4	≤1	none (not infrared safe)
Aplanarity	$A = 1.5\, Q_1$	0	0	≤1/2	none (not infrared safe)
Jet (Hemisphere) masses	$M_\pm^2 = \left(\sum_i E_i^2 - \sum_i \vec{p}_i^{\,2}\right)_{i \in S_\pm}$ ($S_\pm$: Hemispheres $\perp$ to $\vec{n}_T$) $M_H^2 = \max(M_+^2, M_-^2)$ $M_D^2 = \lvert M_+^2 - M_-^2 \rvert$	0 0	≤1/3 ≤1/3	≤1/2 0	(resummed) $O(\alpha_s^2)$
Jet broadening	$B_\pm = \frac{\sum_{i \in S_\pm} \lvert \vec{p}_i \times \vec{n}_T \rvert}{2 \sum_i \lvert \vec{p}_i \rvert}$; $B_T = B_+ + B_-$ $B_w = \max(B_+, B_-)$	0 0	≤1/(2√3) ≤1/(2√3)	≤1/(2√2) ≤1/(2√3)	(resummed) $O(\alpha_s^2)$
Energy-Energy Correlations	$EEC(\chi) = \sum_{events} \sum_{i,j} \frac{E_i E_j}{E_{vis}^2} \int_{\chi - \frac{\Delta\chi}{2}}^{\chi + \frac{\Delta\chi}{2}} \delta(\chi - \chi_{ij})$				(resummed) $O(\alpha_s^2)$
Asymmetry of EEC	$AEEC(\chi) = EEC(\pi - \chi) - EEC(\chi)$				$O(\alpha_s^2)$
Differential 2-jet rate	$D_2(y) = \frac{R_2(y - \Delta y) - R_2(y)}{\Delta y}$				(resummed) $O(\alpha_s^2)$

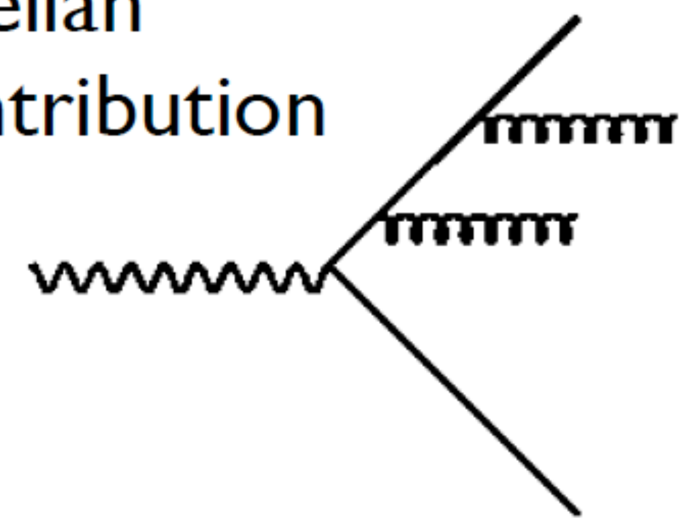
Example: spin of the gluon



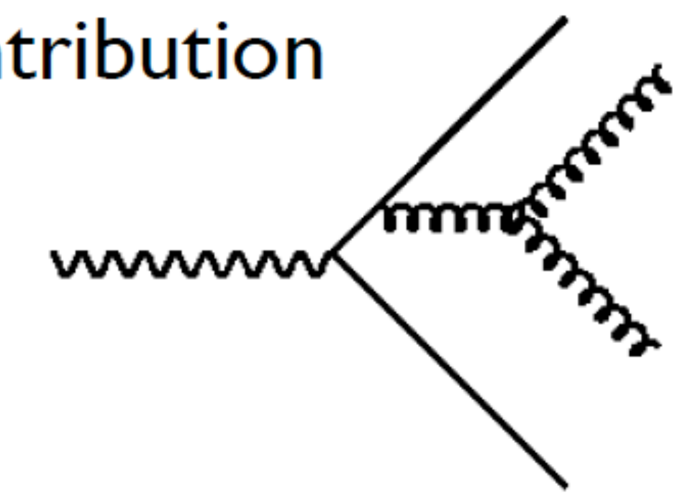
Example: non-abelian nature of QCD



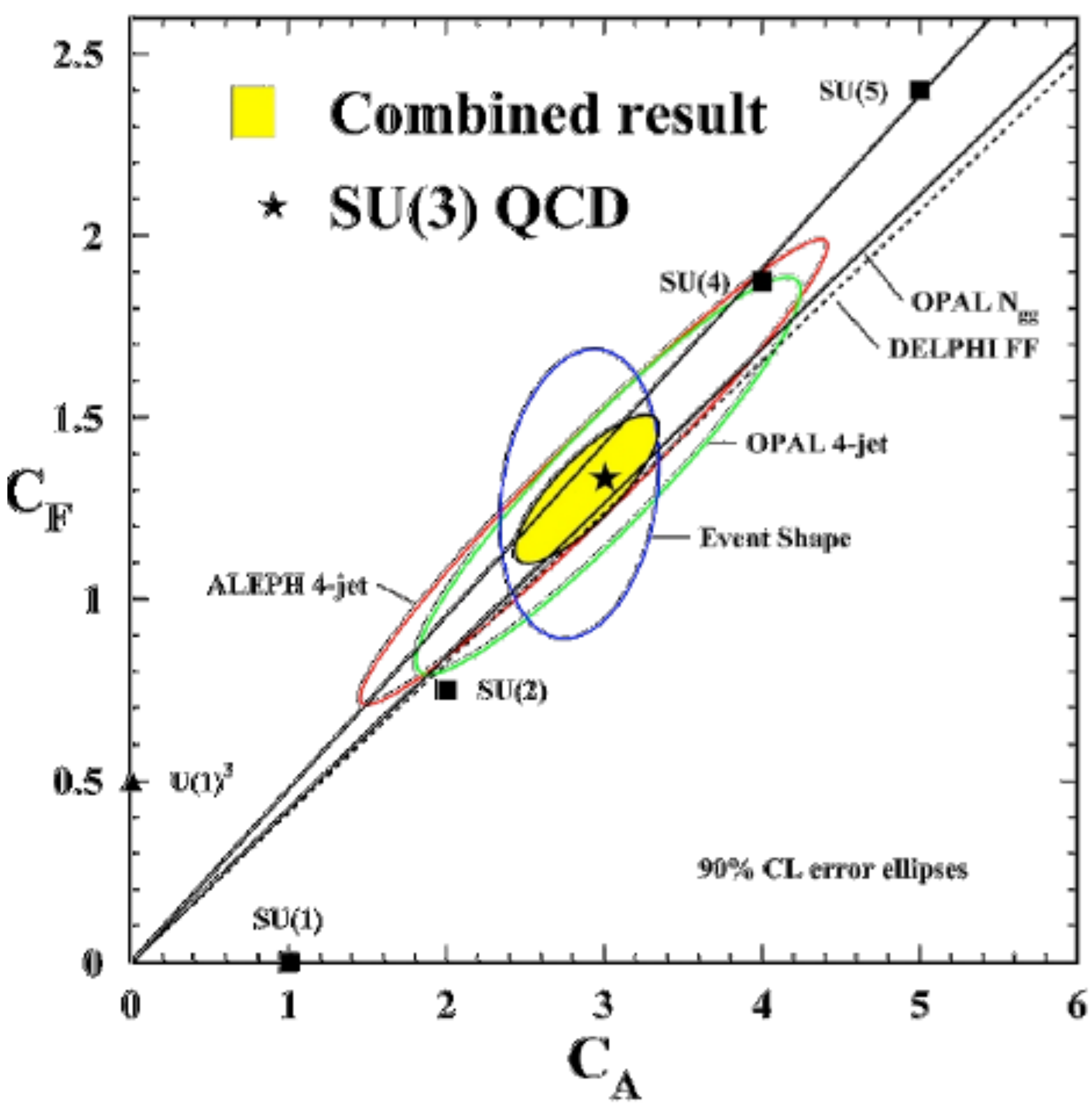
Abelian contribution



Non-Abelian contribution



Example: fits of colour fators



Fits of colour factors from 4-jet rates and event shapes

$$C_A = 2.89 \pm 0.21$$
$$C_F = 1.30 \pm 0.09$$

Well compatible with QCD:

$$C_A = 3$$
$$C_F = \frac{4}{3}$$

Recap

Brief recap on the infrared behaviour of QCD

- we have seen that soft and collinear divergences arise universally in QCD calculations
- these divergences cancel in e^+e^- observables in inclusive observables (KLN theorem)
- we have performed a first genuine QCD calculation: the cross-section for Sterman Weinberg jets in e^+e^- collisions
- perturbative QCD can be used to compute jet-cross section and other infrared-safe event shape variables
- comparison of theory and calculations provide stringent tests of QCD