Next: jets

Where do jets enter? Essentially everywhere at colliders!

Jets are an essential tool for a variety of studies:

- top reconstruction
- mass measurements
- most Higgs and NP searches
- general tool to attribute structure to an event
- instrumental for QCD studies, e.g. inclusive-jet measurements
  ⇒ important input for PDF determinations
Jets

Jets provide a way of projecting away the multiparticle dynamics of an event ⇒ leave a simple quasi-partonic picture of the hard scattering

The projection is fundamentally ambiguous ⇒ jet physics is a rich subject

Ambiguities:
1) Which particles should belong to a same jet?
2) How does recombine the particle momenta to give the jet-momentum?
Jet developments

- Sterman Weinberg
- UA1+2 cones
- Jade, seq. rec.
- Snowmass (cone)
- $k_t$
- Cambridge Aachen
- Tev Run II wkshp (midpoint cone)


- fast-$k_t$, SISCones, anti-$k_t$, jet-areas, jet-flavour, non-perturbative effects,
  quality measures, jet-substructure, grooming ...
Two broad classes of jet algorithms

Today many extensions of the original Sterman-Weinberg jets. Modern jet-algorithms divided into two broad classes:

- **Cone type** (UA1, JetCLU, Midpoint, SISCone..)
  - *top down approach*: cluster particles according to distance in coordinate-space
  - *idea*: put cones along dominant direction of energy flow

- **Sequential** (kt-type, Jade, Cambridge/Aachen..)
  - *bottom up approach*: cluster particles according to distance in momentum-space
  - *idea*: undo branchings occurred in the PT evolution
Inclusive $k_t$/Durham-algorithm

Catani et. al ’92–’93; Ellis&Soper ’93

Inclusive algorithm:

1. For any pair of final state particles $i,j$ define the distance

$$d_{ij} = \frac{\Delta y_{ij}^2 + \Delta \phi_{ij}^2}{R^2} \min\{k_{ti}^2, k_{tj}^2\}$$

2. For each particle $i$ define a distance with respect to the beam

$$d_{iB} = k_{ti}^2$$

3. Find the smallest distance. If it is a $d_{ij}$ recombine $i$ and $j$ into a new particle ($\Rightarrow$ recombination scheme); if it is $d_{iB}$ declare $i$ to be a jet and remove it from the list of particles

NB: if $\Delta R_{ij} \equiv \Delta y_{ij}^2 + \Delta \phi_{ij}^2 < R$ then partons $(ij)$ are always recombined, so $R$ sets the minimal interjet angle

4. repeat the procedure until no particles are left
**Exclusive** $k_t$/Durham-algorithm

**Inclusive algorithm** gives a variable number of jets per event, according to the specific event topology.

**Exclusive version:** run the inclusive algorithm but stop when either

- all $d_{ij}, d_{iB} > d_{cut}$ or
- when reaching the desired number of jets $n$
$k_t$/Durham-algorithm in $e^+e^-$

$k_t$ originally designed in $e^+e^-$, most widely used algorithm in $e^+e^-$ (LEP)

$$y_{ij} = 2 \min \{E_{i}^2, E_{j}^2 \} \left(1 - \cos^2 \theta_{ij} \right)$$

- can classify events using $y_{23}, y_{34}, y_{45}, y_{56} \ldots$
- resolution parameter related to minimum transverse momentum between jets

Satisfies fundamental requirements:

1. **Collinear safe**: collinear particles recombine early on
2. **Infrared safe**: soft particles do not influence the clustering sequence

$\Rightarrow$ collinear + infrared safety important: it means that cross-sections can be computed at higher order in pQCD (no divergences)!
The CA and the anti-\( k_\tau \) algorithm

The Cambridge/Aachen: sequential algorithm like \( k_\tau \), but uses only angular properties to define the distance parameters

\[
\begin{align*}
  d_{ij} &= \frac{\Delta R_{ij}^2}{R^2} \\
  d_{iB} &= 1 \\
  \Delta R_{ij}^2 &= (\phi_i - \phi_j)^2 + (y_i - y_j)^2
\end{align*}
\]

*Dotshitzer et al ’97; Wobisch & Wengler ’99*

The anti-\( k_t \) algorithm: designed not to recombine soft particles together

\[
\begin{align*}
  d_{ij} &= \min\{1/k_{ti}^2, 1/k_{tj}^2\} \Delta R_{ij}^2 / R^2 \\
  d_{iB} &= 1/k_{ti}^2
\end{align*}
\]

*Cacciari, Salam, Soyez ’08*
Cone algorithms

1. A particle $i$ at rapidity and azimuthal angle $(y_i, \Phi_i) \subset$ cone $C$ iff
   \[ \sqrt{(y_i - y_C)^2 + (\phi_i - \phi_C)^2} \leq R_{\text{cone}} \]

2. Define
   \[ \bar{y}_C \equiv \frac{\sum_{i \in C} y_i \cdot p_{T,i}}{\sum_{i \in C} p_{T,i}} \quad \bar{\phi}_C \equiv \frac{\sum_{i \in C} \phi_i \cdot p_{T,i}}{\sum_{i \in C} p_{T,i}} \]

3. If weighted and geometrical averages coincide $(y_C, \phi_C) = (\bar{y}_C, \bar{\phi}_C)$ a stable cone $(\Rightarrow$ jet) is found, otherwise set $(y_C, \phi_C) = (\bar{y}_C, \bar{\phi}_C)$ & iterate

4. Stable cones can overlap. Run a split-merge on overlapping jets: merge jets if they share more than an energy fraction $f$, else split them and assign the shared particles to the cone whose axis they are closer to.
   Remark: too small $f (<0.5)$ creates very large jets, not recommended
Cone algorithms

- The question is where does one start looking for stable cone?
- The direction of these trial cones are called seeds.
- Ideally, place seeds everywhere, so as not to miss any stable cone.
- Practically, this is unfeasible. Speed of recombination grows fast with the number of seeds. So place only some seeds, e.g. at the $(y, \Phi)$-location of particles.

*Seeds make cone algorithms infrared unsafe*
Jets: infrared unsafety of cones

3 hard $\Rightarrow$ 2 stable cones
3 hard + 1 soft $\Rightarrow$ 3 stable cones

*Soft emission changes the hard jets $\Rightarrow$ algorithm is IR unsafe*

**Midpoint algorithm:** take as seed position of emissions and midpoint between two emissions (postpones the infrared safety problem)
Seedless cones

Solution:
use a seedless algorithm, i.e. consider all possible combinations of particles as candidate cones, so find all stable cones \(\Rightarrow\) jets \[Blazey '00\]

The problem:
clustering time growth as \(N^2 \cdot 10^{17}\) years to cluster the event \(\Rightarrow\) prohibitive beyond fixed order (N=4,5)

Better solution:
SISCone recasts the problem as a computational geometry problem, the identification of all distinct circular enclosures for points in 2D and finds a solution to that \(\Rightarrow\) \(N^2 \ln N\) time IR safe algorithm

\[Salam, Soyez '07\]
Jet substructure: $Z/W + H \rightarrow bb$

$\Rightarrow$ **Light Higgs hard**: promising channel associated production with mainly produced in association with $Z/W$, with decay $H \rightarrow bb$ is dominant, but overwhelmed by QCD backgrounds
Recall why searching for $pp \to WH(bb)$ is hard:

$\sigma(pp \to WH(bb)) \sim \text{few pb}$  \hspace{1cm} $\sigma(pp \to Wbb) \sim \text{few pb}$

$\sigma(pp \to tt) \sim 800 \text{pb}$  \hspace{1cm} $\sigma(pp \to Wjj) \sim \text{few } 10^4 \text{pb}$  \hspace{1cm} $\sigma(pp \to bb) \sim 400 \text{pb}$

$\Rightarrow$ signal extraction very difficult

Conclusion [ATLAS TDR]:
The extraction of a signal from $H \to bb$ decays in the WH channel will be very difficult at the LHC even under the most optimistic assumptions [...]
Z/W+ H (→bb) rescued

But ingenious suggestions open up to window of opportunity

Central idea: require high-$p_T$ W and Higgs boson in the event
- leads to back-to-back events where two b-quarks are contained within the same jet
- high $p_T$ reduces the signal but reduces the background much more
- improve acceptance and kinematic resolution
Z/W+ H (→bb) rescued?

Then use a jet-algorithm geared to exploit the specific pattern of H → bb versus g → gg, q → gg

- QCD partons prefer soft emissions (hard → hard + soft)
- Higgs decay prefers symmetric splitting
- try to beat down contamination from underlying event
- try to capture most of the perturbative QCD radiation

1. **cluster** the event with e.g. CA algo and large-ish R
2. undo last recomb: large mass drop + symmetric + b tags
3. **filter** away the UE: take only the 3 hardest sub-jets
**Z/W+ H (→bb) rescued?**

Mass of the three hardest sub-jets:

- with common & channel specific cuts: 
  \[ p_{tV}, p_{tH} > 200\text{GeV} \]
- real/fake b-tag rate: 0.7/0.01
- NB: very neat peak for 
  \[ WZ (Z \rightarrow bb) \]
  Important for calibration

\[ Butterworth, Davison, Rubin, Salam '08 \]

This and other works opened a new field of jet-substructure…
(would be a whole new lecture)
Recap on jets

Two major jet classes: sequential ($k_t$, CA, ...) and cones (UA1, midpoint, ...)

Jet algo is fully specified by: clustering + recombination + split merge or removal procedure + all parameters

Standard/old cones based on seeds are IR unsafe

SISCone is a infrared safe cone algorithm (no seeds)

anti-kt the default sequential algorithm used in LHC analyses

using IR unsafe algorithms you can not use perturbative QCD calculations

Very active novel field of jet substructure [example of ZH(bb)]
Next, we want to review the application of perturbative QCD in high-energy LHC collisions

Before discussing calculations, it is important to understand the kinematics in proton-proton collisions

The total longitudinal momentum of the colliding system is unknown (one can measure missing transverse momentum, but not missing longitudinal one)
LHC kinematics

A more common parametrisation relies on rapidity and transverse mass

$$(E, p_x, p_y, p_z) = (m_T \cosh y, |p_T| \cos \phi, |p_T| \sin \phi, m_Y \sinh y)$$

With

$$y = \frac{1}{2} \ln \frac{E + p_z}{E - p_z} \quad \quad p_T = \sqrt{p_x^2 + p_y^2} \quad \quad m_T = \sqrt{p_T^2 + m^2}$$

**Exercise:** check that the two parametrisations are equivalent

**Exercise:** check that the rapidity transform linearly under a longitudinal boost

**Exercise:** given two particles, can you easily construct a boost-invariant quantity?
LHC kinematics

For particles with negligible mass the rapidity coincides with the pseudo-rapidity

\[ y = \eta \equiv \frac{1}{2} \log \frac{1 + \cos \theta}{1 - \cos \theta} = -\log \tan \frac{\theta}{2} \]

The pseudo-rapidity can then be easily translated to the detector geometric acceptance as used in experimental measurements

<table>
<thead>
<tr>
<th>$\Theta$</th>
<th>$10^{-4}$</th>
<th>$10^{-3}$</th>
<th>$10^{-2}$</th>
<th>0.1</th>
<th>0.5</th>
<th>1</th>
<th>$\pi/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta$</td>
<td>9.9</td>
<td>7.6</td>
<td>5.3</td>
<td>3</td>
<td>1.36</td>
<td>0.6</td>
<td>0</td>
</tr>
</tbody>
</table>

(The other hemisphere has same but negative numbers)
Rapidity coverage of LHC detectors

The achieved maximum rapidity coverage is important in LHC detectors

- For ATLAS and CMS: muons can be detected only in the central regions, while for jets and hadrons, hadronic calorimetry extends up to 4.5-5 (essential for processes like vector boson fusion Higgs production)
- LHCb covers better the forward region, but only forward one
- Studies are ongoing to determine the required/possible rapidity coverage of future detectors
Rapidity is also interesting from a theoretical point of view, as the single particle phase space is uniform in rapidity:

\[
\frac{d^3p}{2E(2\pi)^3} = \frac{1}{2(2\pi)^3} d^2p_T dy
\]

**Exercise:** derive the above expression (change variables and include the Jacobian of the transformation).

The above relation has already deep implications: for instance incoherent radiation (e.g. soft underlying event) is to a large extend uniform in rapidity.
Dijet production

Before discussing higher-order corrections, let’s discuss go through the leading order calculation of one of the main LHC process: di-jet production

Sample diagrams (all must be included)

Many partonic subprocesses contribute

<table>
<thead>
<tr>
<th>Process</th>
<th>$\frac{d\sigma}{d\Phi_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$qq' \rightarrow qq'$</td>
<td>$\frac{1}{2} \frac{4}{9} \left( \frac{s^2+u^2}{t^2} + \frac{s^2+\tilde{u}^2}{\tilde{u}^2} \right) - \frac{8}{27} \frac{s^2}{t^2}$</td>
</tr>
<tr>
<td>$qq \rightarrow qq$</td>
<td>$\frac{1}{2} \frac{4}{9} \left( \frac{s^2+u^2}{t^2} + \frac{s^2+\tilde{u}^2}{\tilde{u}^2} \right) - \frac{8}{27} \frac{s^2}{t^2}$</td>
</tr>
<tr>
<td>$q\bar{q} \rightarrow q'\bar{q}'$</td>
<td>$\frac{1}{2} \frac{4}{9} \left( \frac{s^2+\tilde{u}^2}{\tilde{u}^2} + \frac{\tilde{t}^2+\tilde{u}^2}{\tilde{u}^2} \right) - \frac{8}{27} \frac{\tilde{u}^2}{t^2}$</td>
</tr>
<tr>
<td>$q\bar{q} \rightarrow gg$</td>
<td>$\frac{1}{2} \frac{32}{27} \left( \frac{\tilde{t}^2+\tilde{u}^2}{\tilde{u}\tilde{t}} - \frac{8}{3} \frac{\tilde{t}^2+u^2}{\tilde{s}^2} \right)$</td>
</tr>
<tr>
<td>$g\bar{g} \rightarrow q\bar{q}$</td>
<td>$\frac{1}{6} \frac{i^2+u^2}{u\tilde{t}} - \frac{3}{8} \frac{i^2+u^2}{s^2}$</td>
</tr>
<tr>
<td>$gg \rightarrow q\bar{q}$</td>
<td>$\frac{1}{2} \frac{6}{9} \left( \frac{i^2+u^2}{u\tilde{t}} - \frac{8}{3} \frac{i^2+\tilde{u}^2}{\tilde{s}^2} \right)$</td>
</tr>
<tr>
<td>$gg \rightarrow gg$</td>
<td>$\frac{1}{2} \frac{9}{2} \left( f - \frac{i\tilde{u}}{\tilde{s}^2} - \frac{\tilde{s}u}{\tilde{t}^2} - \frac{\tilde{s}u}{\tilde{u}^2} \right)$</td>
</tr>
</tbody>
</table>

Mandelstam variables: $\hat{s} = (p_1 + p_2)^2$, $\hat{t} = (p_1 + p_3)^2$, $\hat{u} = (p_1 + p_4)^2$
Dijet production

The hadron cross-section

\[ d\sigma^{\text{dijet}} = \sum_{ijkl} \frac{1}{1 + \delta_{kl}} \int dx_1 dx_2 f_i(x_1, \mu^2) f_j(x_2, \mu^2) \frac{d\hat{\sigma}_{ij \rightarrow kl}}{d\Phi_2} \Theta_{\text{cuts}} \]

Symmetry factor

\[ p_3 = (p_T \cosh y_3, p_T \cos \phi, p_T \sin \phi, p_T \sinh y_3) \]
\[ p_4 = (p_T \cosh y_4, -p_T \cos \phi, -p_T \sin \phi, p_T \sinh y_4) \]

PDFs for initial state partons

Matrix elements

Phase space

Measurement function

We have seen that in the LAB frame

Exercise: show that the rapidities are related to the Bjorken-x variables by

\[ x_1 = \frac{p_T}{\sqrt{s}} (e^{y_3} + e^{y_4}) \]
\[ x_2 = \frac{p_T}{\sqrt{s}} (e^{-y_3} + e^{-y_4}) \]
Exercise: show that the rapidities in the partonic centre-of-mass frame are given by
\[ \hat{y}_3 = \frac{1}{2} (y_3 - y_4) = -\hat{y}_4 \]

Exercise: show that the scattering angle in the partonic frame is given by
\[
\cos \hat{\theta} = \tanh \hat{y}_3 = \tanh \left( \frac{y_3 - y_4}{2} \right)
\]

this relation shows that the difference in rapidities between the jets gives direct access to the
dynamics in the partonic frame

Exercise: show that in terms of rapidities the cross-section becomes
\[
\frac{d^3 \sigma_{\text{dijet}}}{dy_1 dy_2 dp_T^2} = \frac{1}{16\pi s} \sum_{ijkl} \frac{1}{1 + \delta_{kl}} \int \frac{dx_1}{x_1} \frac{dx_2}{x_2} f_i(x_1, \mu^2) f_j(x_2, \mu^2) \frac{d\hat{\sigma}_{ij \to kl}}{d\Phi_2}
\]

The above expression can be integrated numerically and provides a leading order estimate of
the cross section
Dijet production

Inclusive and dijet production are extensively studies at the LHC, both for SM measurements and in searches for New physics.

Direct determination of gluon PDF, constraints of other PDFS, measurement of $\alpha_s$, probe of QCD running at TeV scales …
Dijet production

Inclusive and dijet production are extensively studies at the LHC, both for SM measurements and in searches for New physics.

**Search for excited quarks**

**Search for gluions**
Dijet production

Inclusive and dijet production are extensively studied at the LHC, both for SM measurements and in searches for New physics.

Explore substructure of quarks

It is clear that the smaller the uncertainties, the more one can exclude exotic scenarios. Above we sketched a leading order calculation, in the following we’ll discuss higher-order corrections in a more generic case.
Perturbative calculations rely on the idea of an order-by-order expansion in the small coupling

\[ \sigma \sim A + B\alpha_s + C\alpha_s^2 + D\alpha_s^3 + \ldots \]

- **LO**
- **NLO**
- **NNLO**
- **NNNLO**

- Perturbative calculations are possible because the coupling is small at high energy
- In QCD (or in a generic QFT) the coupling depends on the energy (renormalization scale)
- So changing scale the result changes. By how much? What does this dependence mean?
- In the following will discuss these issues through examples
Hard cross section

Born level cross section straightforward in principle

\[ \sigma_{LO} = \int_{m} d\Phi_{m} |M^{(0)}(\{p_{i}\})|^{2} S(\{p_{i}\}) \]

m-particle phase space (e.g. Vegas)  
Matrix element  
measurement function (constraint on phase space)
Leading order with Feynman diagrams

Get *any* LO cross-section from the Lagrangian

1. draw all Feynman diagrams
2. put in the explicit Feynman rules and get the amplitude
3. do some algebra, simplifications
4. square the amplitude
5. integrate over phase space + flux factor + sum/average over outgoing/incoming states

Automated tools for (1-3): FeynArts/Qgraf, Mathematica/Form etc.

**Bottlenecks**

a) number of Feynman diagrams diverges factorially
b) algebra becomes more cumbersome with more particles

*But given enough computer power everything can be computed at LO*
Diagrams for gluon amplitudes

Number of diagrams for $gg \rightarrow n$ gluons

<table>
<thead>
<tr>
<th>$n$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>diag.</td>
<td>4</td>
<td>25</td>
<td>220</td>
<td>2485</td>
<td>34300</td>
<td>559405</td>
<td>10525900</td>
</tr>
</tbody>
</table>

- number of diagrams grows very fast
- complexity of each diagrams grows with $n$

Alternative methods?
Techniques beyond Feynman diagrams

✓ **Berends-Giele relations:** compute helicity amplitudes recursively using off-shell currents

\[ \times = \sum \times + \sum \times \]

*Berends, Giele ’88*

✓ **BCF relations:** compute helicity amplitudes via on-shell recursions (use complex momentum shifts)

\[ \sum_+ + \sum_- \]

*Britto, Cachazo, Feng ’04*

✓ **CSW relations:** compute helicity amplitudes by sewing together MHV amplitudes

\[ [- - + + ... + ] \]

*Cachazo, Svrcek, Witten ’04*
Is it necessary to go beyond LO?

Very early observation:

at least NLO corrections are needed to describe data

Drell Yan production is one of the first processes for which NLO corrections have been computed
Leading order n-jet cross-section

- Consider the cross-section to produce n jets. The leading order result at scale $\mu$ result will be

$$\sigma_{n\text{jets}}^{\text{LO}}(\mu) = \alpha_s(\mu)^n A(p_i, \epsilon_i, \ldots)$$

- Instead, choosing a scale $\mu'$ one gets

$$\sigma_{n\text{jets}}^{\text{LO}}(\mu') = \alpha_s(\mu')^n A(p_i, \epsilon_i, \ldots) = \alpha_s(\mu)^n \left( 1 + n b_0 \alpha_s(\mu) \ln \frac{\mu^2}{\mu'^2} + \ldots \right) A(p_i, \epsilon_i, \ldots)$$

So the change of scale is an NLO effect ($\propto \alpha_s$), but this becomes more important when the number of jets increases ($\propto n$)

- Notice that at Leading Order the normalization is not under control:

$$\frac{\sigma_{n\text{jets}}^{\text{LO}}(\mu)}{\sigma_{n\text{jets}}^{\text{LO}}(\mu')} = \left( \frac{\alpha_s(\mu)}{\alpha_s(\mu')} \right)^n$$
NLO n-jet cross-section

Now consider n-jet cross-section at NLO. At scale $\mu$ the result reads

$$\sigma_{\text{njets}}^{\text{NLO}}(\mu) = \alpha_s(\mu)^n A(p_i, \epsilon_i, \ldots) + \alpha_s(\mu)^{n+1} \left( B(p_i, \epsilon_i, \ldots) - nb_0 \ln \frac{\mu^2}{Q_0^2} \right) + \ldots$$

- So the NLO result compensates the LO scale dependence. The residual dependence is NNLO.

- Scale dependence and normalization start being under control only at NLO, since compensation mechanism kicks in.

- Notice also that a good scale choice automatically resums large logarithms to all orders, while a bad one spuriously introduces large logs and ruins the PT expansion.

- Scale variation is conventionally used to estimate theory uncertainty, but the validity of this procedure should not be overrated (see later).
NLO calculations

NLO accuracy requires to dress a process with one real or one virtual parton

Sample diagrams shown. All diagrams must be included.

We won’t have time to do detailed NLO calculations, but let’s look a bit more in detail at the issue of divergences/subtraction.
Regularization procedures in QCD

Regularization: a way to make intermediate divergent quantities meaningful

- In QCD **dimensional regularization** is today the standard procedure, based on the fact that d-dimensional integrals are more convergent if one reduces the number of dimensions.

\[
\int \frac{d^4 l}{(2\pi)^4} \rightarrow \mu^{2\epsilon} \int \frac{d^d l}{(2\pi)^d}, \quad d = 4 - 2\epsilon < 4
\]

- N.B. to preserve the correct dimensions a mass scale \(\mu\) is needed

- Divergences show up as intermediate poles \(1/\epsilon\)

- This procedure works both for UV divergences and IR divergences

Alternative regularization schemes: photon mass (EW), cut-offs, Pauli-Villard ...
Compared to those methods, dimensional regularization has the big virtue that it leaves the regularized theory Lorentz invariant, gauge invariant, unitary etc.
Subtraction and slicing methods

• Consider e.g. an n-jet cross-section with some arbitrary infrared safe jet definition. At NLO, two divergent integrals, but the sum is finite

\[ \sigma_{\text{NLO}}^{J} = \int_{n+1} d\sigma_{\text{R}}^{J} + \int_{n} d\sigma_{\text{V}}^{J} \]

• Since one integrates over a different number of particles in the final state, real and virtual need to be evaluated first, and combined then

• This means that one needs to find a way of removing divergences before evaluating the phase space integrals

• Two main techniques to do this
  - phase space slicing ⇒ obsolete because of practical/numerical issues
  - subtraction method ⇒ most used in recent applications
Subtraction method

• The real cross-section can be written schematically as

\[ d\sigma^J_R = d\phi_{n+1} |M_{n+1}|^2 F^{J}_{n+1}(p_1, \ldots, p_{n+1}) \]

where \( F^J \) is the arbitrary jet-definition

• The matrix element has a non-integrable divergence

\[ |M_{n+1}|^2 = \frac{1}{x} \mathcal{M}(x) \]

where \( x \) vanishes in the soft/collinear divergent region

• IR divergences in the loop integration regularized by taking \( D=4-2\varepsilon \)

\[ 2 \text{Re}\{\mathcal{M}_V \cdot \mathcal{M}_0^*\} = \frac{1}{\varepsilon} \mathcal{V} \]
Subtraction method

- The n-jet cross-section becomes

\[ \sigma_{NLO}^J = \int_0^1 \frac{dx}{x^{1+\epsilon}} M(x) F_n^{J+1}(x) + \frac{1}{\epsilon} \mathcal{V} F_n^J \]

- **Infrared safety** of the jet definition implies

\[ \lim_{x \to 0} F_n^{J+1}(x) = F_n^J \]

- **KLN cancelation** guarantees that

\[ \lim_{x \to 0} M(x) = \mathcal{V} \]

- One can then add and subtract the analytically computed divergent part

\[ \sigma_{NLO}^J = \int_0^1 \frac{dx}{x^{1+\epsilon}} M(x) F_n^{J+1}(x) - \int_0^1 \frac{dx}{x^{1+\epsilon}} \mathcal{V} F_n^J + \int_0^1 \frac{dx}{x^{1+\epsilon}} \mathcal{V} F_n^J + \frac{1}{\epsilon} \mathcal{V} F_n^J \]
Subtraction method

• This can be rewritten exactly as

\[ \sigma_{\text{NLO}}^J = \int_0^1 \frac{dx}{x^{1+\epsilon}} \left( \mathcal{M}(x) F_{n+1}^J - \mathcal{V} F_n^J \right) + \mathcal{O}(1) \mathcal{V} F_n^J \]

⇒ Now both terms are finite and can be evaluated numerically

• Subtracted cross-section must be calculated separately for each process (but mostly automated now). It must be valid everywhere in phase space

• Systematised in the seminal papers of Catani-Seymour (dipole subtraction, ’96) and Frixione-Kunszt-Signer (FKS method, ’96)

• Subtraction used in all recent NLO applications and public codes (Event2, Disent, MCFM, NLOjet++, Madgraph4@aMC@NLO, POWHEG ... )
Ingredients at NLO

A full N-particle NLO calculation requires:

- ✔ tree graph rates with N+1 partons
  ➔ soft/collinear divergences

- ❏ virtual correction to N-leg process
  ➔ divergence from loop integration,
    use e.g. dimensional regularization

- ✔ set of subtraction terms

bottleneck for a very long time
Virtual one-loop: two breakthrough ideas

Aim: NLO loop integral without doing the integration

1) “... we show how to use generalized unitarity to read off the (box) coefficients. The generalized cuts we use are quadrupole cuts ...”

Quadrupole cuts: 4 on-shell conditions on 4 dimensional loop momentum) freezes the integration. But rational part of the amplitude, coming from $D=4-2\varepsilon$ not 4, computed separately

Britto, Cachazo, Feng ’04
One-loop: two breakthrough ideas

Aim: NLO loop integral without doing the integration

2) The OPP method: “We show how to extract the coefficients of 4-, 3-, 2- and 1-point one-loop scalar integrals....”

\[ A_N = \sum_{[i_1|i_4]} \left( d_{i_1i_2i_3i_4} I_{i_1i_2i_3i_4}^{(D)} \right) + \sum_{[i_1|i_3]} \left( c_{i_1i_2i_3} I_{i_1i_2i_3}^{(D)} \right) + \sum_{[i_1|i_2]} \left( b_{i_1i_2} I_{i_1i_2}^{(D)} \right) \]

Ossola, Pittau, Papadopolous ’06

Coefficients can be determined by solving system of equations: no loops, no twistors, just algebra!
Virtual (one-loop) amplitude

Bottleneck for a long time... but thanks to these and other theoretical breakthrough ideas

• connection between NLO amplitudes and LO ones
• input from supersymmetry/string theory
• sophisticated algebraic methods
• connections with formal theory and pure mathematics...

the problem of computing NLO QCD corrections is now solved
Automated NLO (aka NLO revolution)

Example: single Higgs production processes (similar results available for all SM processes of similar complexity, backgrounds to Higgs studies)

<table>
<thead>
<tr>
<th>Process</th>
<th>Syntax</th>
<th>LO 13 TeV</th>
<th>Cross section (pb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single Higgs production</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>g.1 $pp \rightarrow H$ (HEFT)</td>
<td>$p \ p &gt; h$</td>
<td>$1.593 \pm 0.003 \cdot 10^1$</td>
<td>$3.261 \pm 0.010 \cdot 10^1$</td>
</tr>
<tr>
<td>g.2 $pp \rightarrow H j$ (HEFT)</td>
<td>$p \ p &gt; h \ j$</td>
<td>$8.367 \pm 0.003 \cdot 10^0$</td>
<td>$1.422 \pm 0.006 \cdot 10^1$</td>
</tr>
<tr>
<td>g.3 $pp \rightarrow H jj$ (HEFT)</td>
<td>$p \ p &gt; h \ j j$</td>
<td>$3.020 \pm 0.002 \cdot 10^0$</td>
<td>$5.124 \pm 0.020 \cdot 10^0$</td>
</tr>
<tr>
<td>g.4 $pp \rightarrow H jj$ (VBF)</td>
<td>$p \ p &gt; h \ j j \ \gamma \ j j$</td>
<td>$1.987 \pm 0.002 \cdot 10^0$</td>
<td>$1.900 \pm 0.006 \cdot 10^0$</td>
</tr>
<tr>
<td>g.5 $pp \rightarrow H jj$ (VBF)</td>
<td>$p \ p &gt; h \ j j \ \gamma \ j j$</td>
<td>$2.824 \pm 0.005 \cdot 10^{-1}$</td>
<td>$3.085 \pm 0.010 \cdot 10^{-1}$</td>
</tr>
<tr>
<td>g.6 $pp \rightarrow H W^{\pm}$</td>
<td>$p \ p &gt; h \ W^{\pm}$</td>
<td>$1.195 \pm 0.002 \cdot 10^0$</td>
<td>$1.419 \pm 0.005 \cdot 10^0$</td>
</tr>
<tr>
<td>g.12* $pp \rightarrow H W^{+} W^{-}$ (4f)</td>
<td>$p \ p &gt; h \ W^{+} W^{-}$</td>
<td>$8.325 \pm 0.139 \cdot 10^{-1}$</td>
<td>$1.063 \pm 0.003 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>g.13* $pp \rightarrow H W^{\pm} \gamma$</td>
<td>$p \ p &gt; h \ W^{\pm} \ W^{\pm}$</td>
<td>$2.518 \pm 0.006 \cdot 10^{-3}$</td>
<td>$3.309 \pm 0.011 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>g.14* $pp \rightarrow H Z W^{\pm}$</td>
<td>$p \ p &gt; h \ Z W^{\pm}$</td>
<td>$3.763 \pm 0.007 \cdot 10^{-3}$</td>
<td>$5.292 \pm 0.015 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>g.15* $pp \rightarrow H Z Z$</td>
<td>$p \ p &gt; h \ Z Z$</td>
<td>$2.093 \pm 0.003 \cdot 10^{-3}$</td>
<td>$2.538 \pm 0.007 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>g.16 $pp \rightarrow H t \ t$</td>
<td>$p \ p &gt; h \ t \ t$</td>
<td>$3.579 \pm 0.003 \cdot 10^{-1}$</td>
<td>$4.608 \pm 0.016 \cdot 10^{-1}$</td>
</tr>
<tr>
<td>g.17 $pp \rightarrow H t j$</td>
<td>$p \ p &gt; h \ t j$</td>
<td>$4.994 \pm 0.005 \cdot 10^{-2}$</td>
<td>$6.328 \pm 0.022 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>g.18 $pp \rightarrow H b \ b$ (4f)</td>
<td>$p \ p &gt; h \ b \ b$</td>
<td>$4.983 \pm 0.002 \cdot 10^{-2}$</td>
<td>$6.085 \pm 0.026 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>g.19 $pp \rightarrow H t \ t$</td>
<td>$p \ p &gt; h \ t \ t$</td>
<td>$2.674 \pm 0.041 \cdot 10^{-1}$</td>
<td>$3.244 \pm 0.025 \cdot 10^{-1}$</td>
</tr>
<tr>
<td>g.20* $pp \rightarrow H b \ b$ (4f)</td>
<td>$p \ p &gt; h \ b \ b$</td>
<td>$7.367 \pm 0.002 \cdot 10^{-2}$</td>
<td>$9.034 \pm 0.032 \cdot 10^{-2}$</td>
</tr>
</tbody>
</table>

✓ A solved problem
What lead to this remarkable progress?

The fact that

1. leading order can be computed automatically and efficiently (e.g. via recursion relations)
2. one can reduce the one-loop to product of tree-level amplitudes
3. it was well understood how to subtract singularities
4. the basis of master integrals was known

But for item 2. everything was there since the time of Passarino-Veltman (even item 2. was understood, but no efficient/practical method exited). We will later on compare this to the current status of NNLO
NLO status

Various public tools developed: Blackhat+Sherpa, GoSam+Sherpa, Helac-NLO, Madgraph5_aMC@NLO, NJet+Sherpa, OpenLoops+Sherpa, Samurai, Recola ...

• Practical limitation: high-multiplicity processes still difficult because of numerical instabilities, need long run-time on clusters to obtain stable results (edge: around 6 particles in the final state, depending on the process)

• Today focus on
  ➡ automation of NLO for BSM signals
  ➡ loop-induced processes: formally higher-order, but enhanced by gluon PDF
  ➡ automation of NLO electroweak corrections (necessary to match accuracy of NNLO)
  ➡ automation of NLO in SMEFT

Comparison to NLO is the standard now in most LHC analyses
The “unpleasant” feature that cross-sections depend on the choice of renormalization and factorization scale can be turned into something useful, i.e. a way to quantify the theoretical error.

**Example: R-ratio (again!)**

Fix both scales to the scale at which the hard process occurs (Q) and vary them up and down by a factor 2.

**NB:**
- the factor 2 is conventional
- it is a procedure that seemed to work well in practice
- in complicated processes large degree of freedom in the choice of the scale
I. LHC example of NLO: $t\bar{t}+1\text{jet}$

- Scale variation is not a perfect procedure to assess the theory uncertainty
- Ambiguities in the central scale choice (more so for more complicated processes)
- Scale variation for ratios (asymmetries etc.) underestimates the uncertainty
2. LHC example of NLO: $W+3$ jets

Scale choice: example of $W+3$ jets (problem more severe with more jets)

... large logarithms can appear in some distributions, invalidating even an NLO prediction.

Bern et al. '09
LHC data clearly already requires NNLO
Same conclusion in all measurements examined so far
With more data NLO likely to be insufficient
Why is NNLO difficult

calculation of two-loop master integrals (when many scales are involved)

methods to cancel (overlapping) divergences before integration

\[
\int d\Phi_n 2\text{Re} |M_{\text{2-loop}}| \quad \int d\Phi_n d\Phi_1 2\text{Re} |M_{\text{1-loop}}^{\text{one-loop}} M_{\text{tree}}^{\text{tree}}| \quad \int d\Phi_n d\Phi_2 |M_{\text{tree}}^{\text{2-loop}}|_{n+2}
\]

\[
\int d\Phi_n \left\{ \left( a_4 \frac{1}{\epsilon^4} + a_3 \frac{1}{\epsilon^3} + \ldots + a_0 \right) - \left( a_4 \frac{1}{\epsilon^4} + a_3 \frac{1}{\epsilon^3} + \ldots + b_0 \right) \right\}
\]

Cancellation manifest after phase space integration, but to have fully differential results must achieve cancellation before integration
Ingredients for NNLO

At NNLO the situation is very different from NLO

1. leading order of very limited importance
2. no procedure to reduce two-loop to tree-level (unitarity approaches still face many outstanding issues)
3. subtraction of singularities far from trivial
4. basis set of master integrals not known, integrals not all/always known analytically

And all this even for simple processes (no full result exists for any $2 \rightarrow 3$ scattering process)

What changed in the last years (and is undergoing more changes)

1. technology to compute integrals
2. extension of systematic subtraction to NNLO
NNLO: status
NNLO uncertainty?

NNLO *scale* uncertainty bands of 1-2%. Is the *theory* uncertainty indeed 1-2%?
N3LO: only 3 LHC process known so far

Gluon fusion Higgs production (in the large $m_t$ effective theory)

Vector boson fusion Higgs production (in the structure function approximation, i.e. double DIS process)

Drell Yan (W and Z bosons to leptons)
Higgs production at N3LO

- $O(100000)$ interference diagrams (1000 at NNLO)
- 68273802 loop and phase space integrals (47000 at NNLO)
- about 1000 master integrals (26 at NNLO)
• $\text{N}^3\text{LO}$ finally stabilizes the perturbative expansion

• also matched to resummed calculation (essentially no impact on central value at preferred scale $m_H/2$)
Error budget: one example

Gluon-fusion Higgs productions (known to N$^3$LO fully differential)

Error budget as of 2018

- More data; lattice determination $\alpha_s$; progress in $\alpha_s$ fits
- Removed by Czakon et al ’21
- can be removed
- Reduced by factor 2 through mixed EW-QCD calculations
- Missing N$^3$LO PDFs
- Missing N$^4$LO

Dulat, Lazopoulos, Mistlberger ’18
Summary of perturbative calculations

- **LO**: fully automated. Edge: 10-12 particles in the final state

- **NLO**: also automated. Edge: 4-6 particles in the final state

- **NNLO**: the new frontier. Lots of new $2 \to 2$ processes in recent years. First $2 \to 3$ calculations for the LHC

- **NNNLO**: fully inclusive Higgs production via gluon fusion (large $m_t$ effective theory), vector boson fusion (factorised approximation) and Drell Yan production