Search and Discovery Statistics in HEP Lecture 3

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This presentation would have not been possible without the tremendous help of
the following people throughout many years

Louis Lyons, Alex Read, Bob Cousins Glen Cowan ,Kyle Cranmer
Ofer Vitells & Jonathan Shlomi

What can you expect from the Lectures

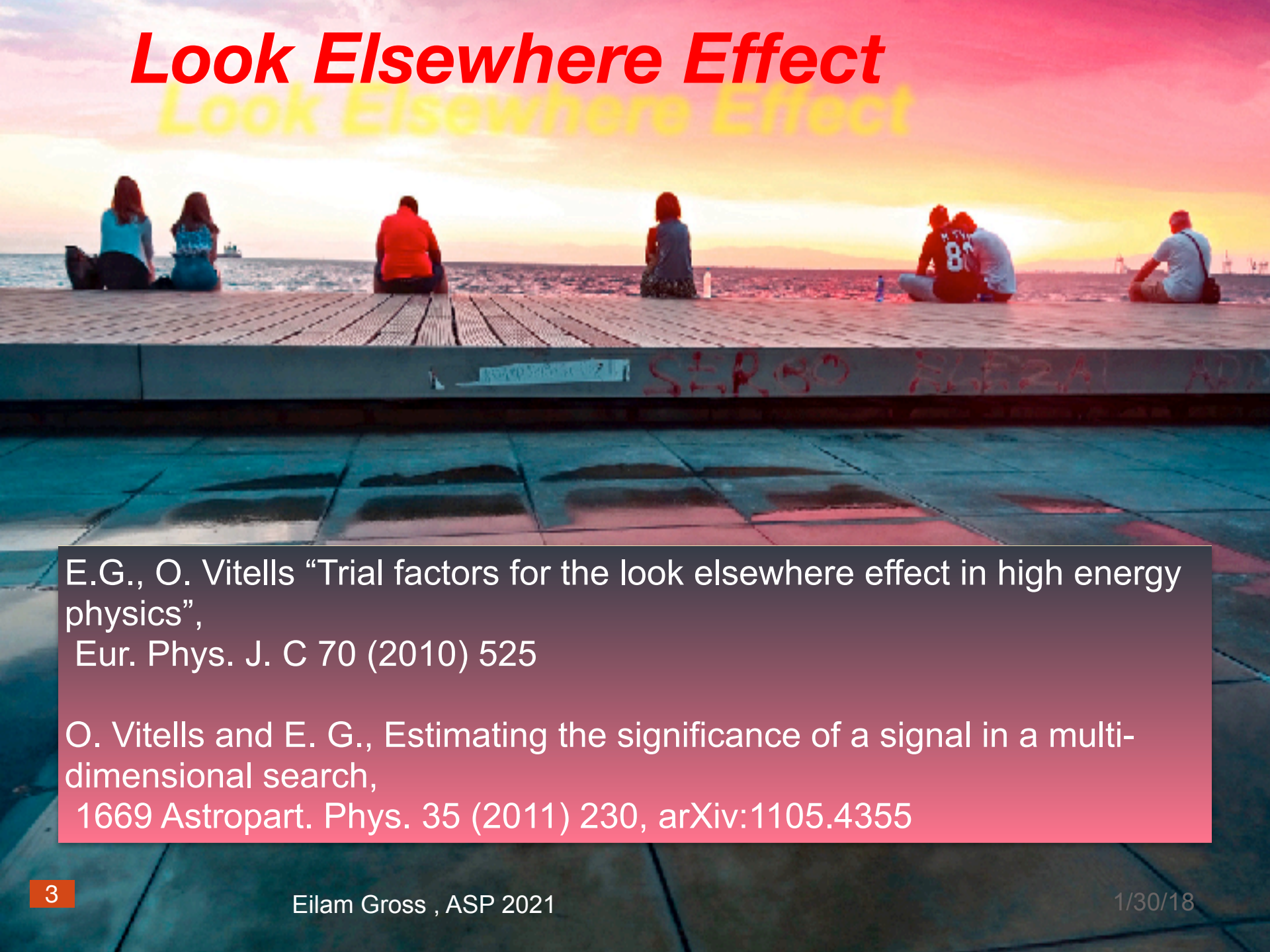
Lecture 1: Basic Concepts
Histograms, PDF, Testing Hypotheses,
LR as a Test Statistics, p-value, POWER, CLs
Measurements

Lecture 2: Wald Theorem, Asymptotic Formalism, Asimov Data Set, Feldman-Cousins, PL & CLs, Asimov Significance

Lecture 3: **Look Elsewhere Effect**
1D LEE the non-intuitive thumb rule (upcrossings, trial #~Z)
2D LEE (Euler Characteristic)

Lecture 4: Basic Introduction to Deep Learning

Look Elsewhere Effect

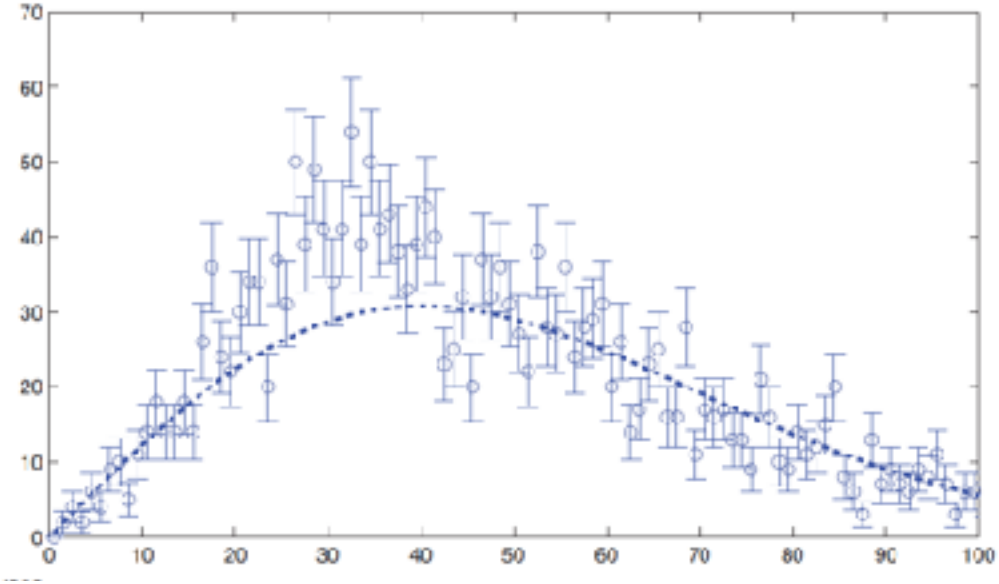


E.G., O. Vitells “Trial factors for the look elsewhere effect in high energy physics”,
Eur. Phys. J. C 70 (2010) 525

O. Vitells and E. G., Estimating the significance of a signal in a multi-dimensional search,
1669 Astropart. Phys. 35 (2011) 230, arXiv:1105.4355

Look Elsewhere Effect

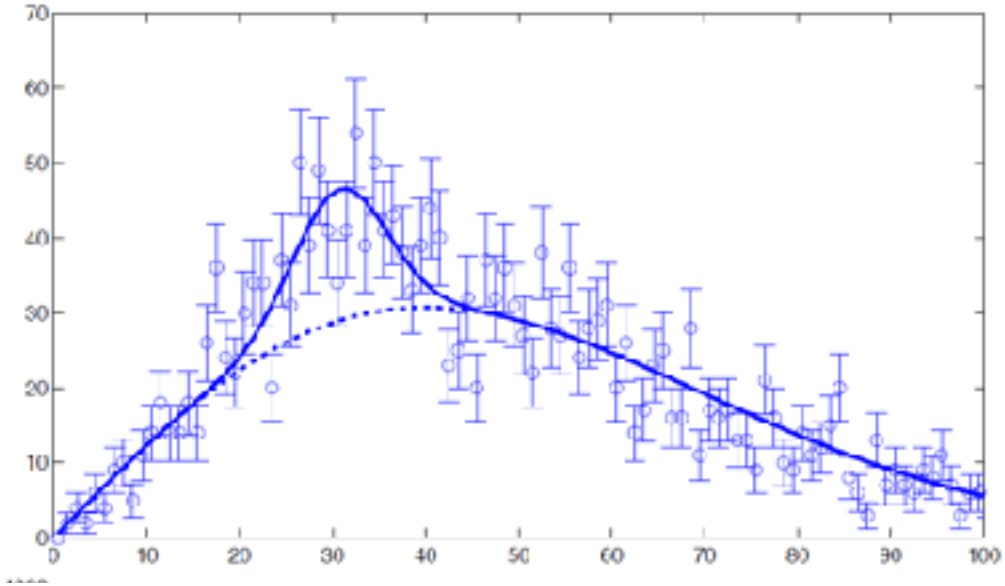
- Is there a signal here?



Look Elsewhere Effect

- Looks like a signal at m=30
- What is its significance?

Test the BG hypothesis
At m=30



$$q_0(\theta) = \begin{cases} -2\log\dfrac{L(\mu=0)}{L(\hat{\mu},\theta)} \\ 0 \end{cases}$$

$$q_{fix,obs} = -2\ln\frac{L(b)}{L(\hat{\mu}s(m=30)+b)}$$

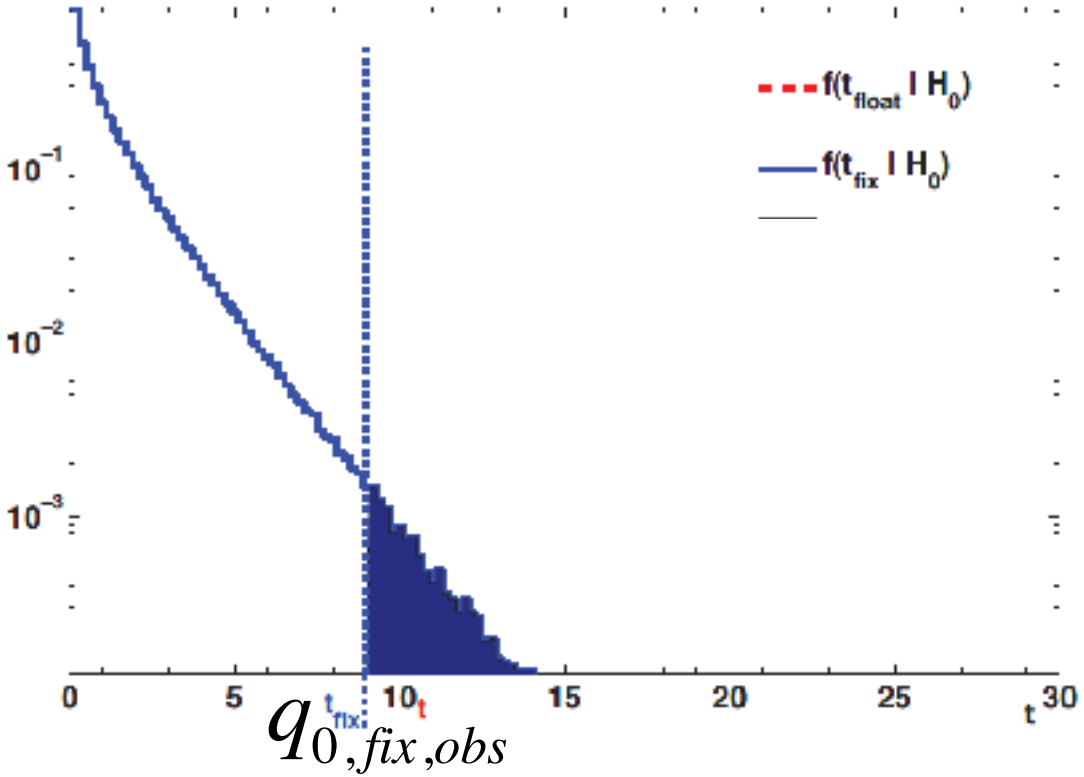
$$Z = \sqrt{q_{0,fix,obs}}$$

Look Elsewhere Effect

$$q_{0,fix} = -2ln\frac{L(\mu=0)}{L(\hat{\mu}s(30)+b)}$$

$$f(q_{0,fix} \mid H_0) \sim \chi^2$$

$$p_{fix} = \int_{q_{fix,obs}}^{\infty} f(q_0 \mid H_0) dq_0$$

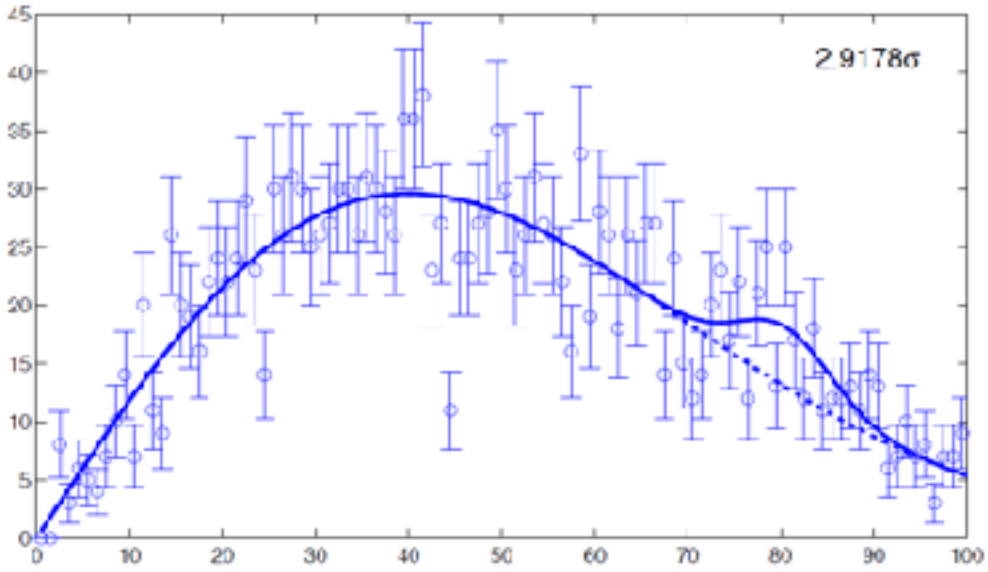


p_{fix} answers the question :

What is the probability to have a fluctuation as or bigger than the observed one?

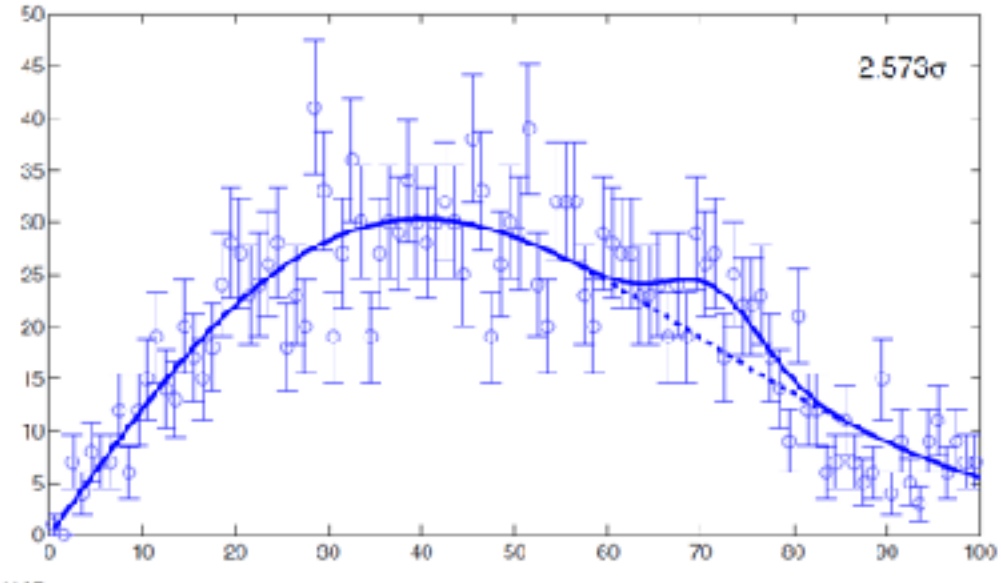
Look Elsewhere Effect

- Would you ignore this signal, had you seen it?



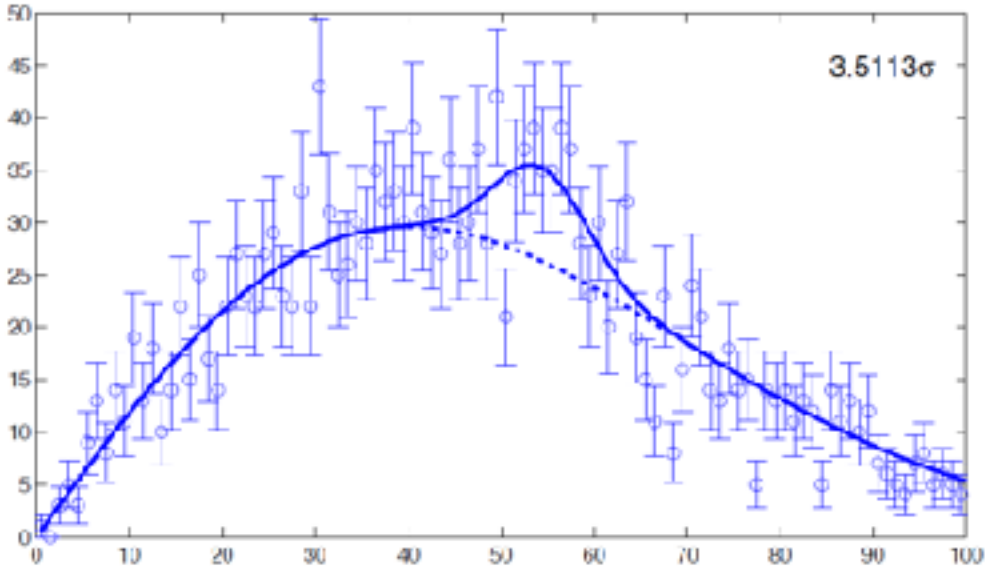
Look Elsewhere Effect

- Or this?



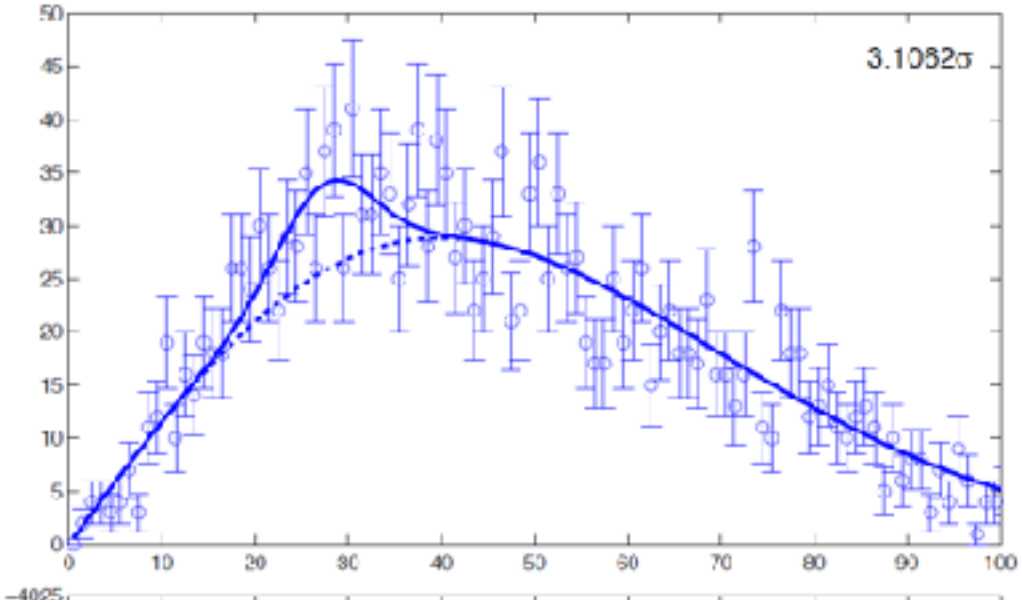
Look Elsewhere Effect

- Or this?



Look Elsewhere Effect

- Or this?
- Obviously NOT!
- ALL THESE "SIGNALS" ARE BG FLUCTUATIONS



The right question :

What is the probability to have a fluctuation as or bigger than the observed one

ANYWHERE in the mass search range?

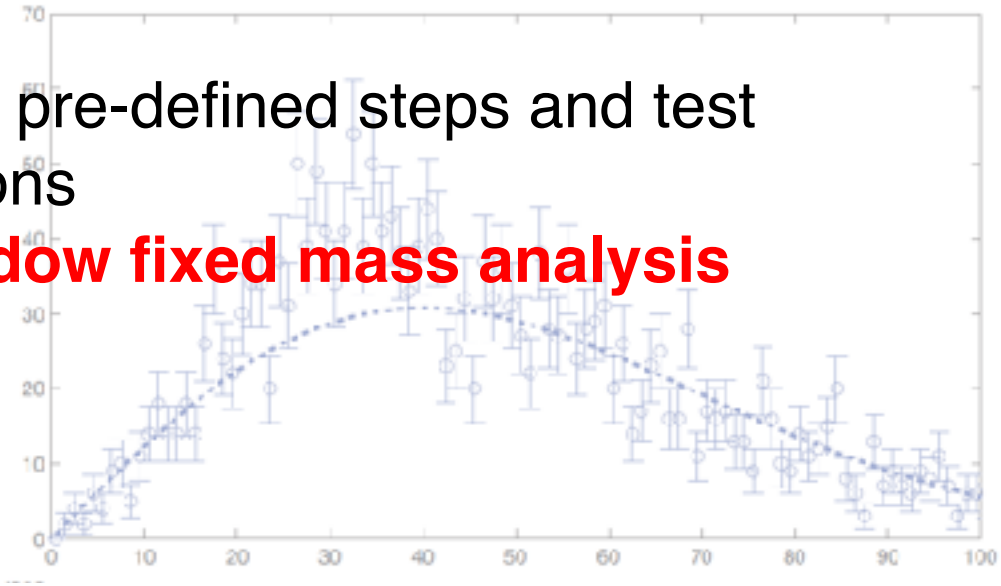
Look Elsewhere Effect

- Having no idea where the signal might be there are two equivalent options

- **OPTION I:**

scan the mass range in pre-defined steps and test any disturbing fluctuations

Perform a sliding window fixed mass analysis



$$q_{0,float} = \max_{m}\left(q_0(m)\right)$$

- **OPTION II:**

Perform a floating mass analysis

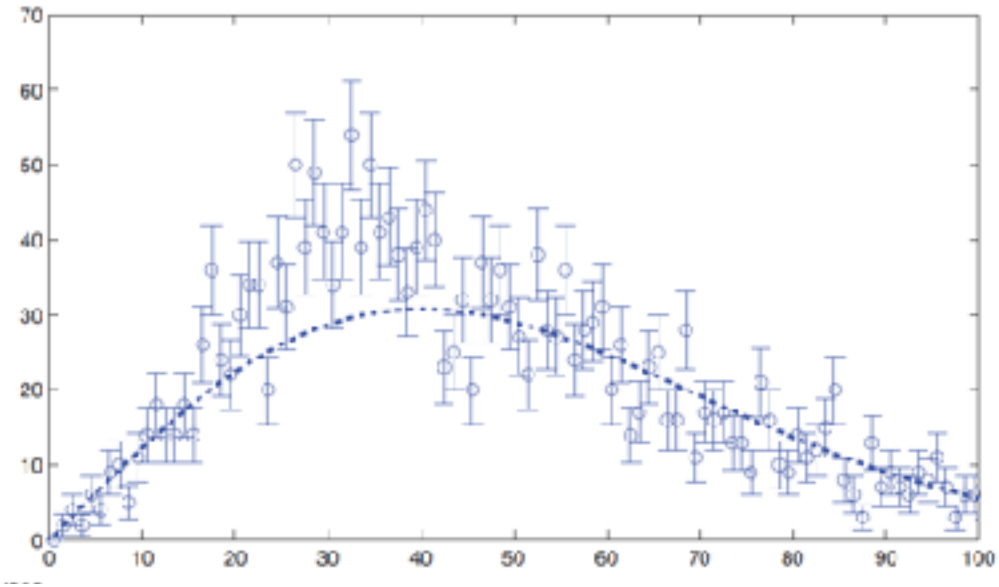
$$q_{0,float} = q_0(\hat{m}) = -2\ln\frac{L(b)}{L(\hat{\mu}s(\hat{m})+b)}$$

$$p_{float} = \int_{q_{float,obs}}^{\infty} f(q_{0,float}\,|\,H_0)\,dq_{0,float}$$

Sliding Window

- Scan and perform a fixed mass analysis at each point

$$q_0 = -2ln\frac{L(\mu=0)}{L(\hat{\mu}s(m)+b)}$$



- The scan resolution must be less than the signal mass resolution

Sliding Window

$$q_0 = -2ln\frac{L(\mu = 0)}{L(\hat{\mu}s(m)+b)}$$

Sliding Window

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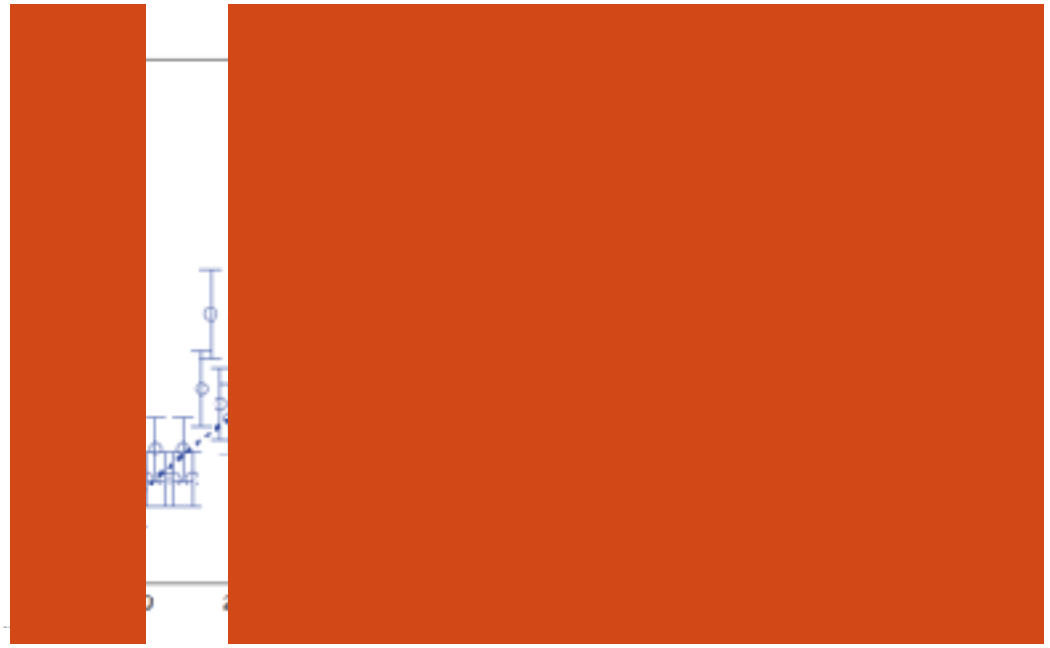
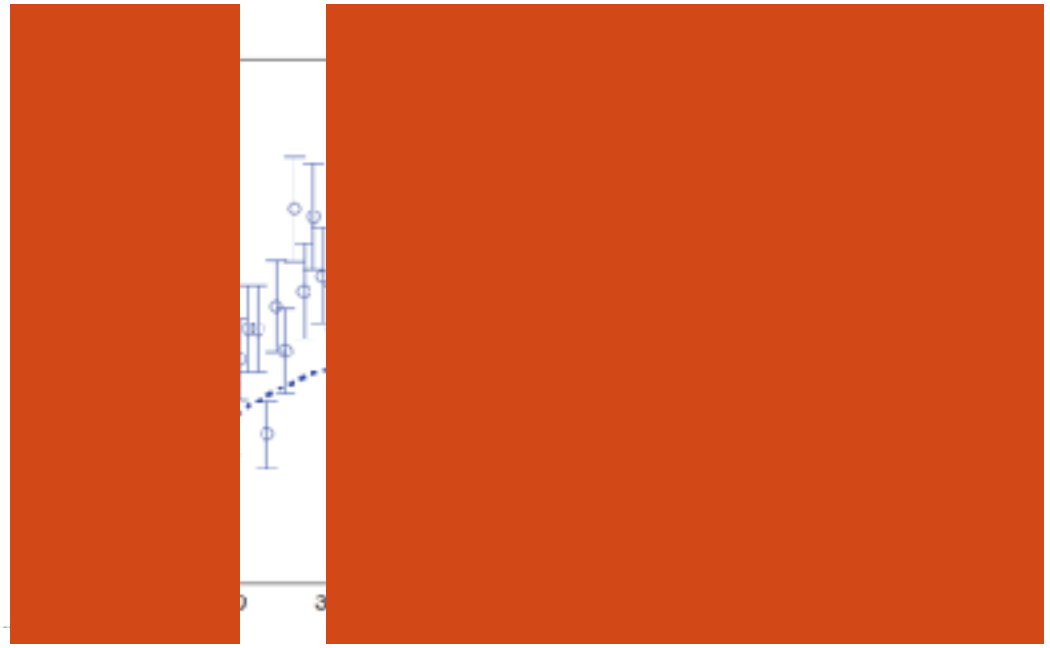


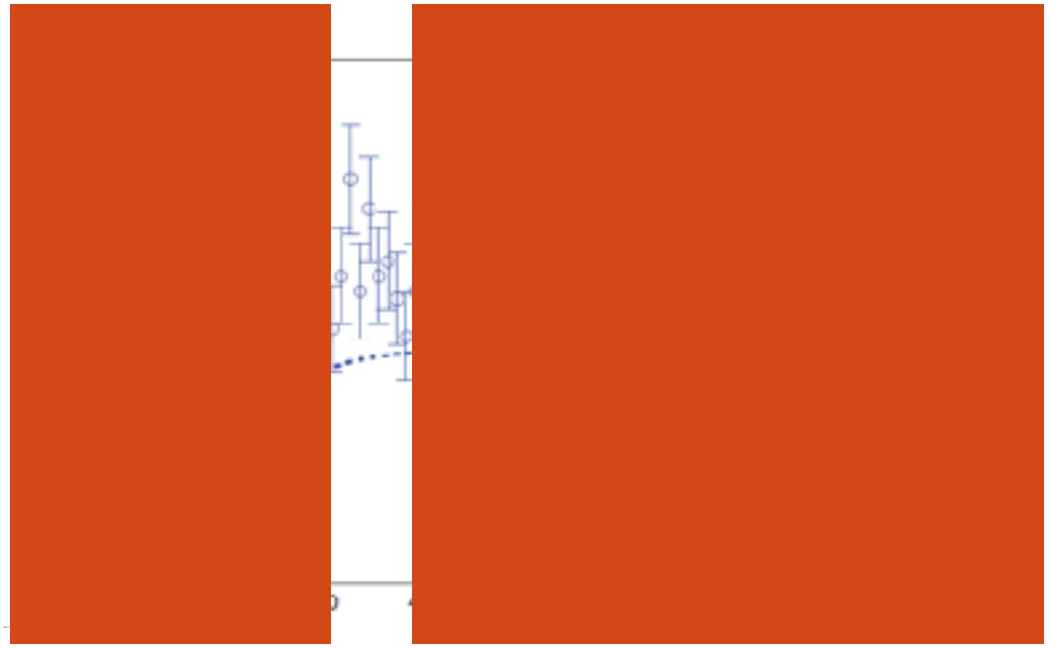
Sliding Window

$$q_0 = -2ln\frac{L(\mu = 0)}{L(\hat{\mu}s(m)+b)}$$



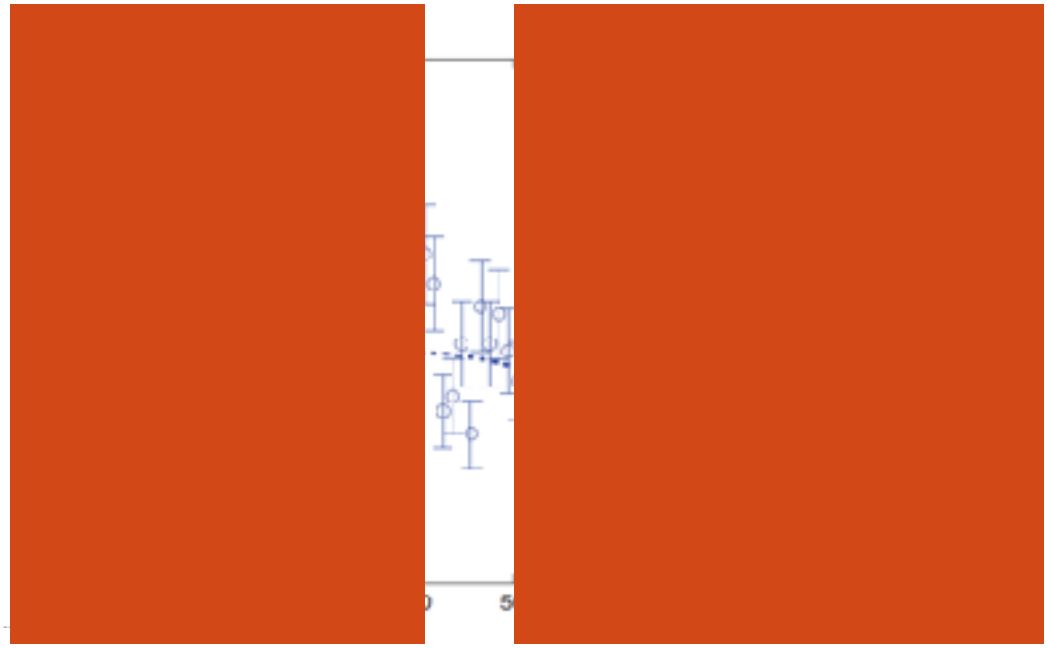
Sliding Window

$$q_0 = -2ln\frac{L(\mu = 0)}{L(\hat{\mu}s(m)+b)}$$



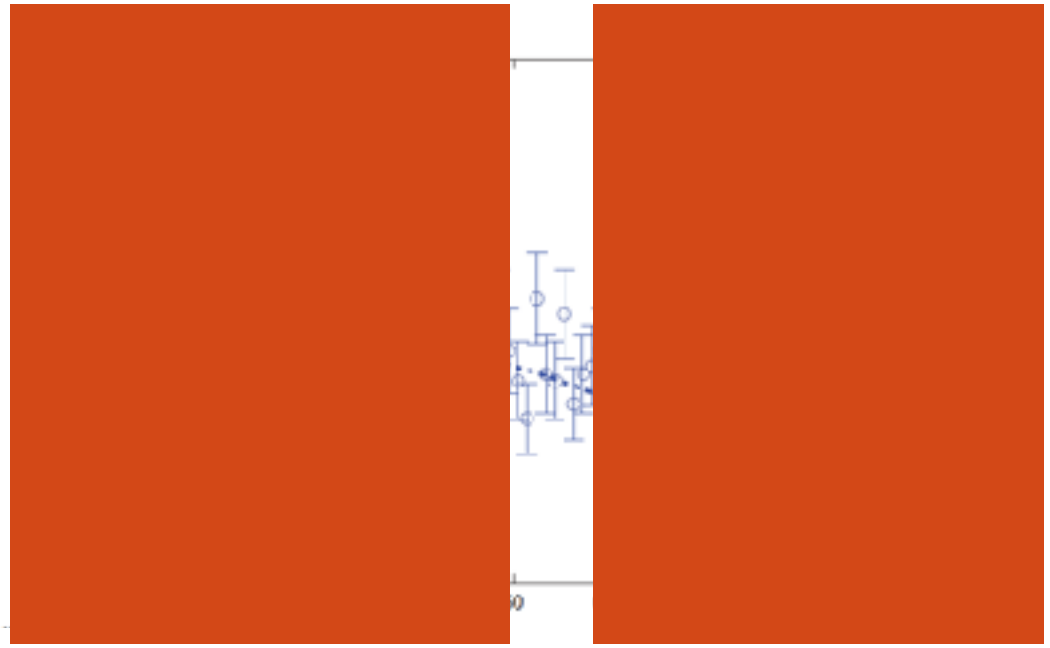
Sliding Window

$$q_0 = -2ln\frac{L(\mu = 0)}{L(\hat{\mu}s(m)+b)}$$



Sliding Window

$$q_0 = -2ln\frac{L(\mu = 0)}{L(\hat{\mu}s(m)+b)}$$



Sliding Window

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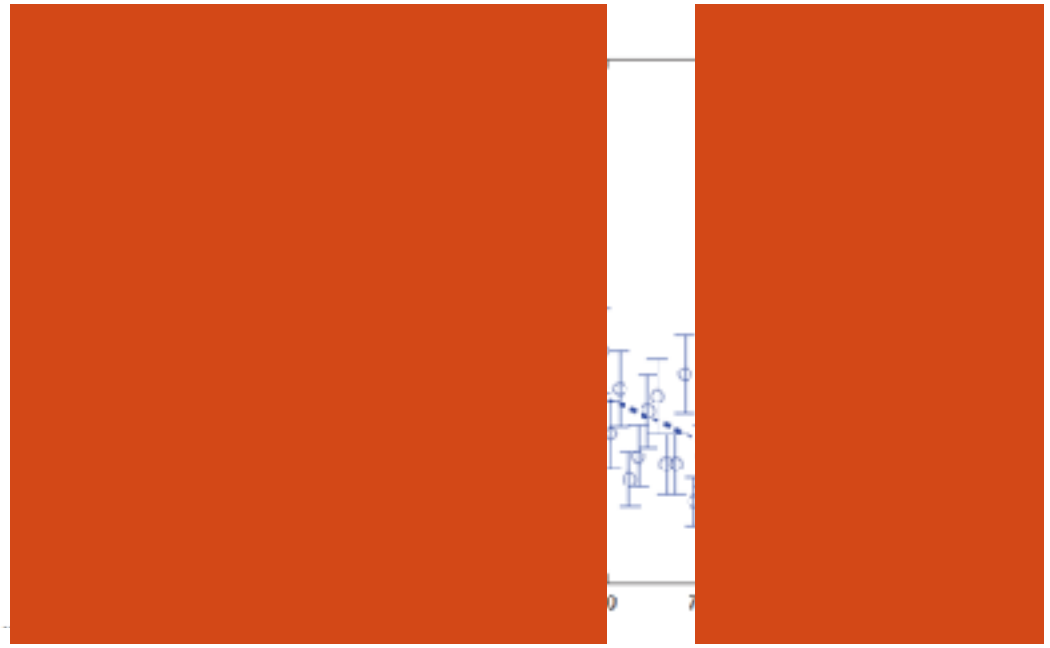

Sliding Window

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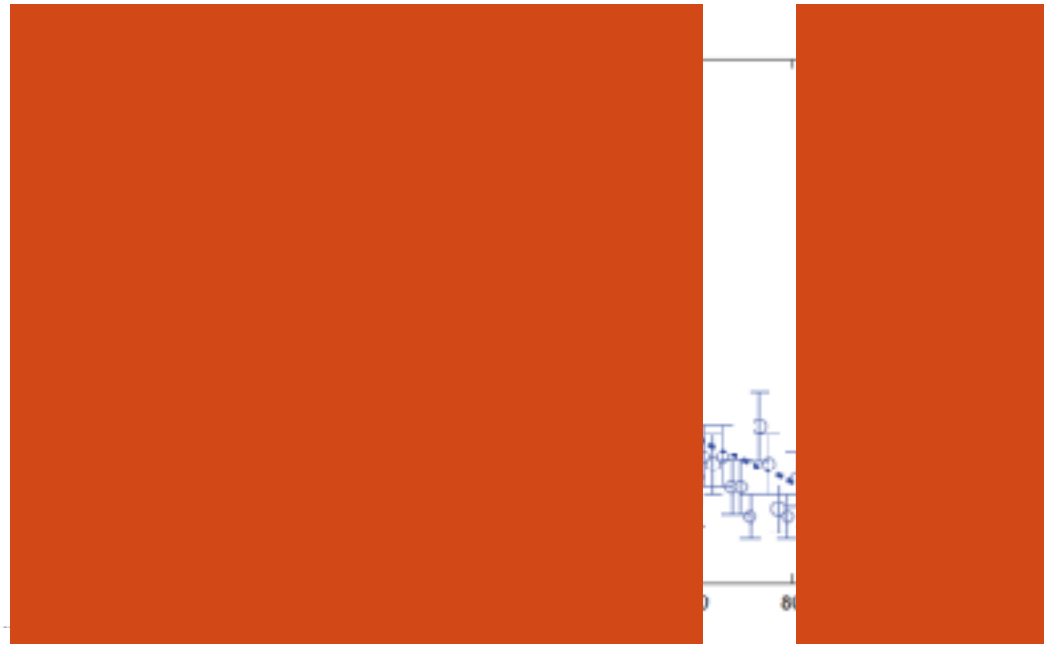
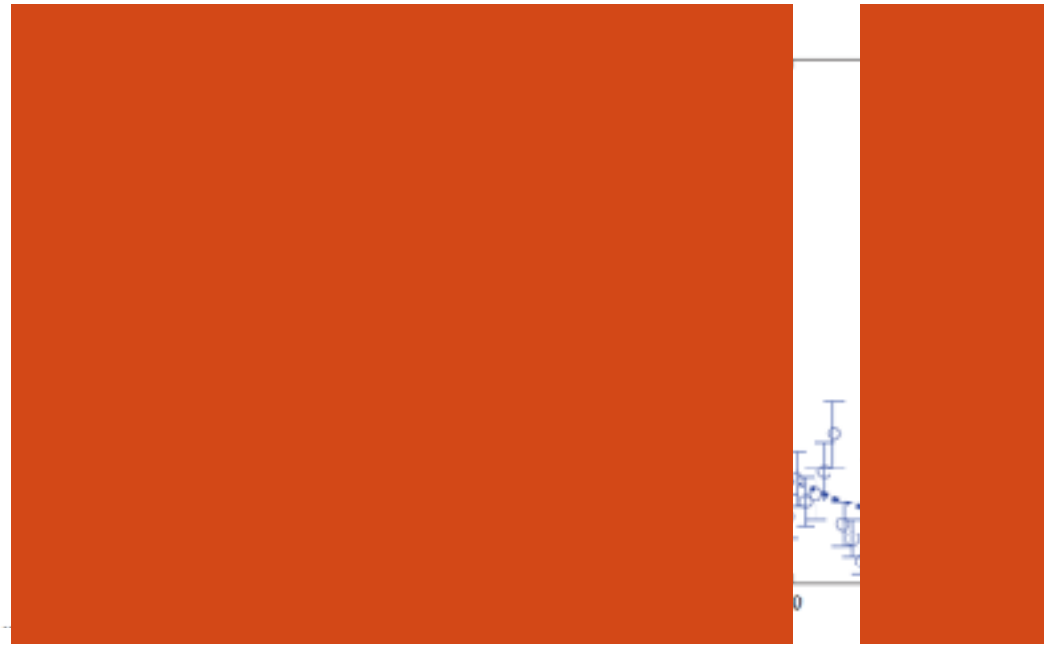


Sliding Window

$$q_0 = -2ln\frac{L(\mu = 0)}{L(\hat{\mu}s(m)+b)}$$



Sliding Window

- Assuming the signal can be only at one place
- pick the one with the MAXIMUM SIGNIFICANCE

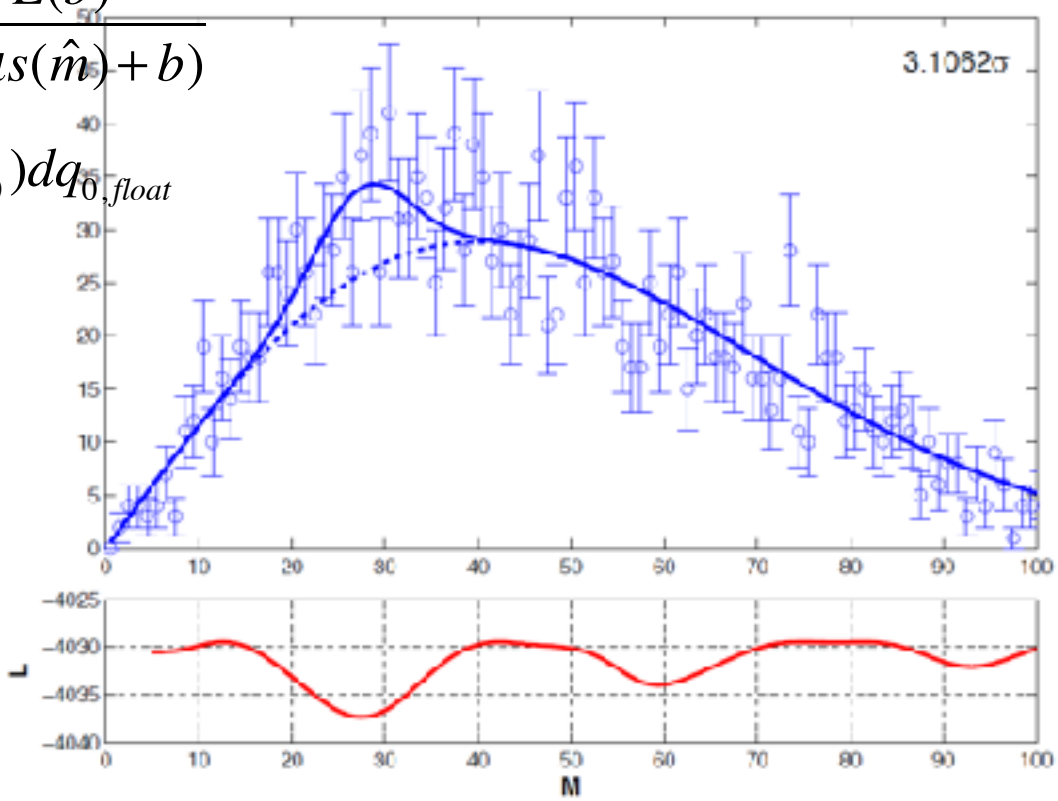
$$q_{0,float} = \max_{m}\left(q_0(m)\right)$$

Look Elsewhere Effect: Floating Mass

OPTION II

$$q_{0,float} = q_0(\hat{m}) = -2\ln\frac{L(b)}{L(\hat{\mu}s(\hat{m})+b)}$$

$$p_{float} = \int_{q_{float,obs}}^{\infty} f(q_{0,float} \mid H_0)\, dq_{0,float}$$

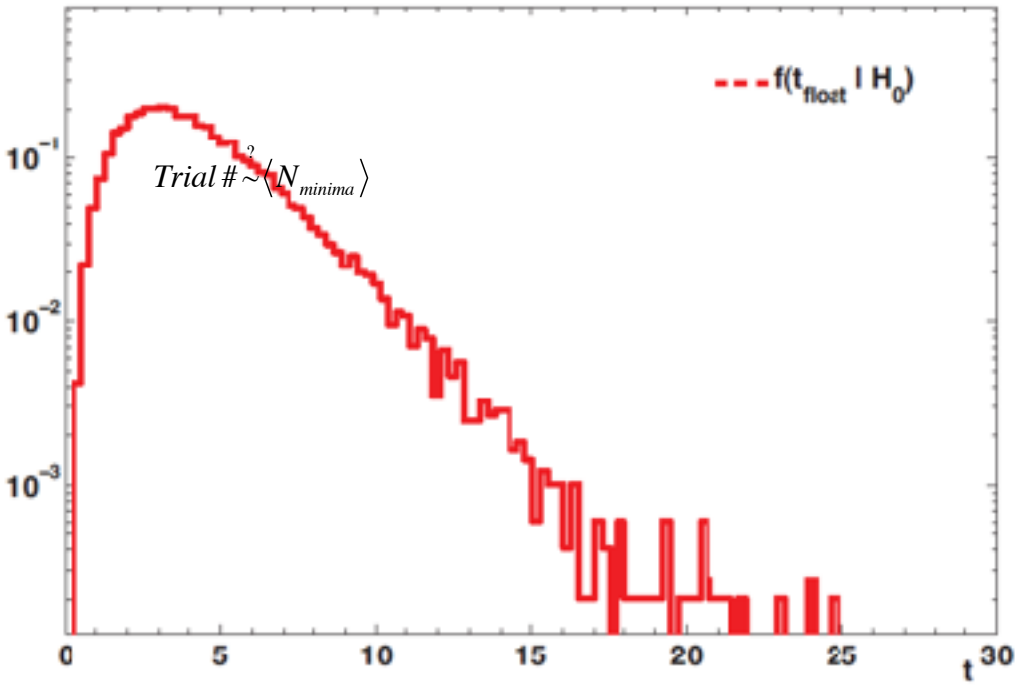


Look Elsewhere Effect

- The distribution $f(q_{float}|H_0)$ does not follow a chi-squared with 2dof because the mass parameter is not defined under the null hypothesis

$$for\ any\ m_{fix}\quad q_0(\hat{m}) \geq q_0(m_{fix})$$

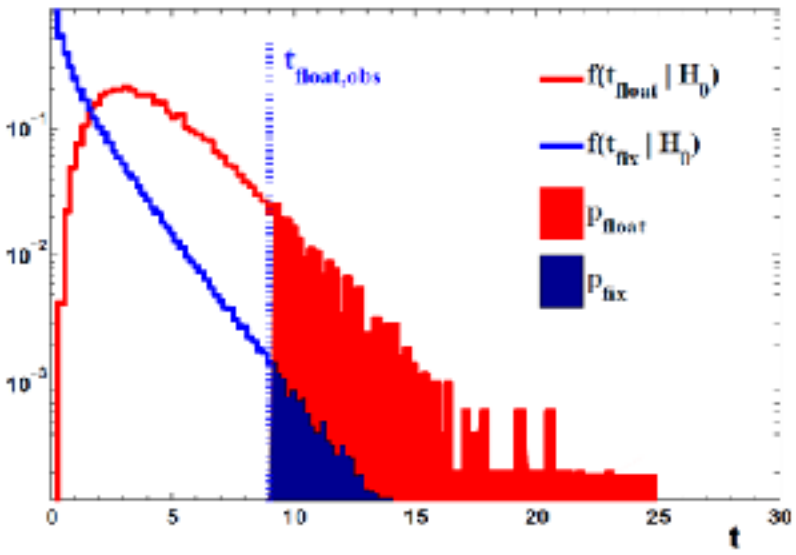
$$The\ \chi^2_1\ distribution\ is\ pushed\ to\ the\ right$$



trial#

- Assume a maximal local fluctuation at mass
- The observed q_0 is given by

$$q_{0,obs} = -2ln\frac{L(\mu=0)}{L(\hat{\mu}s(m)+b)}$$



$$p_{fix} = \int_{q_{0,obs}}^{\infty} f(q_{0,fix} | H_0) dq_{0,fix}$$

$$p_{float} = \int_{q_{0,obs}}^{\infty} f(q_{0,float} | H_0) dq_{0,float}$$

$$trial\,\# = \frac{p_{float}}{p_{fix}}$$

Can we calculate analytically the floating mass p-value

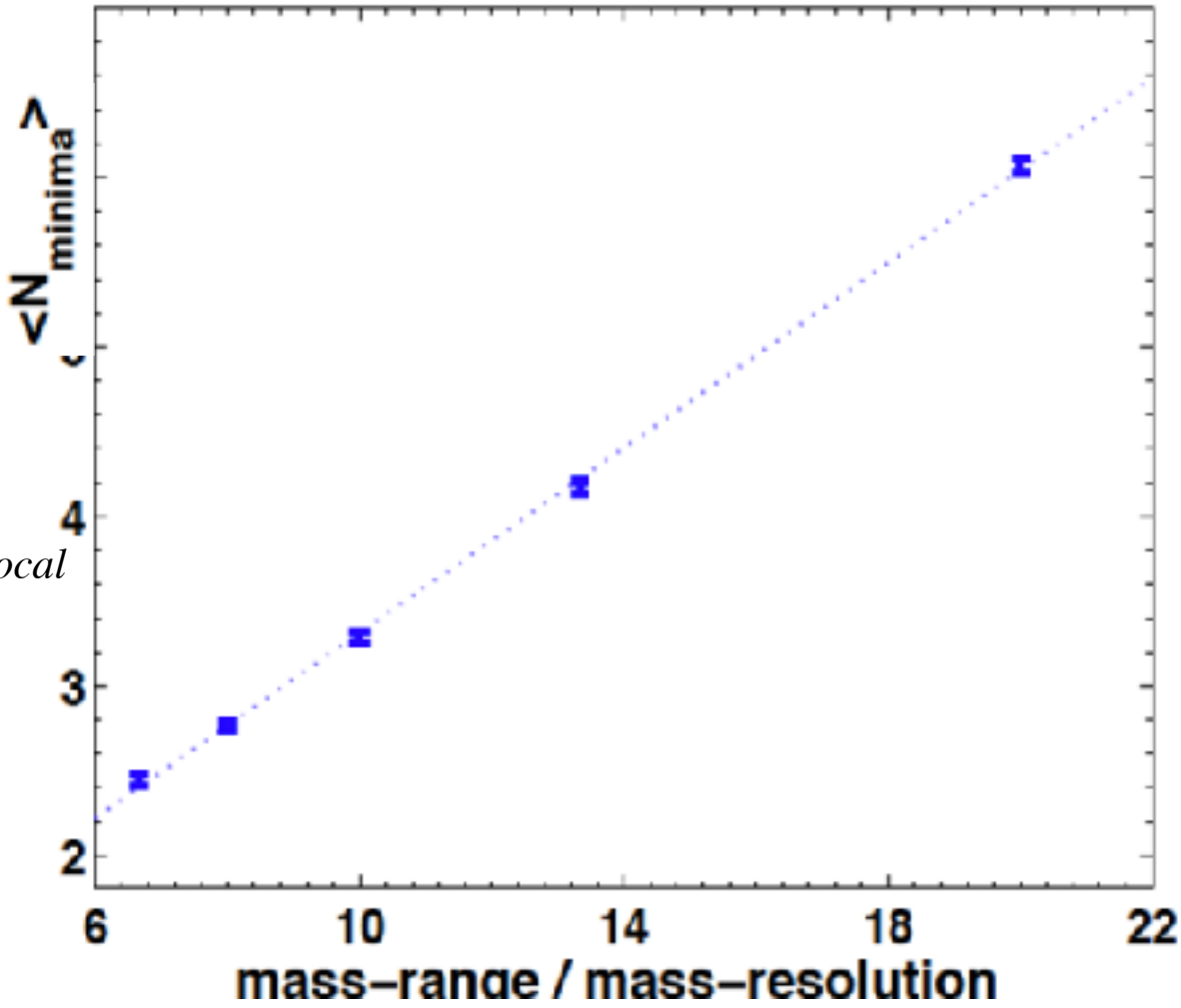
Wrong Thumb Rule Wrong Wrong Wrong

$$\langle N_{minima} \rangle \sim \frac{Mass\ Range}{Mass\ Resolution}$$

$$Trial\ \# \sim \langle N_{minima} \rangle$$

$$Trial\ \# \overset{?}{=} \langle N_{minima} \rangle\, p_{local}$$

The answer is NO



The right question :

What is the probability to have a fluctuation as or bigger than the observed one

ANYWHERE in the mass search range?

Let θ be a nuisance parameter undefined under the null hypothesis.

Define $q(\hat{\theta}) = max_{\theta}(q(\theta))$

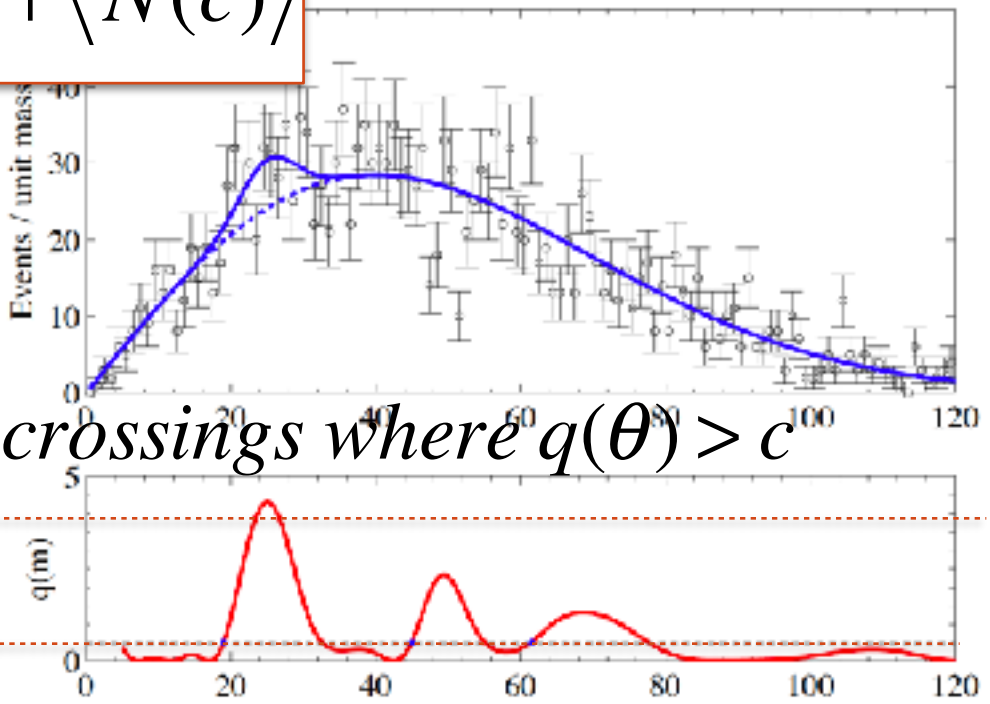
Davies (1987) finds, for $c >> 1$

$$P\left(q(\hat{\theta}) > c\right) \sim P(\chi^2_1 > c) + \langle N(c) \rangle$$

$\langle N(c) \rangle$ = *Number of upcrossings* $q(\theta) > c$

Davies Formula

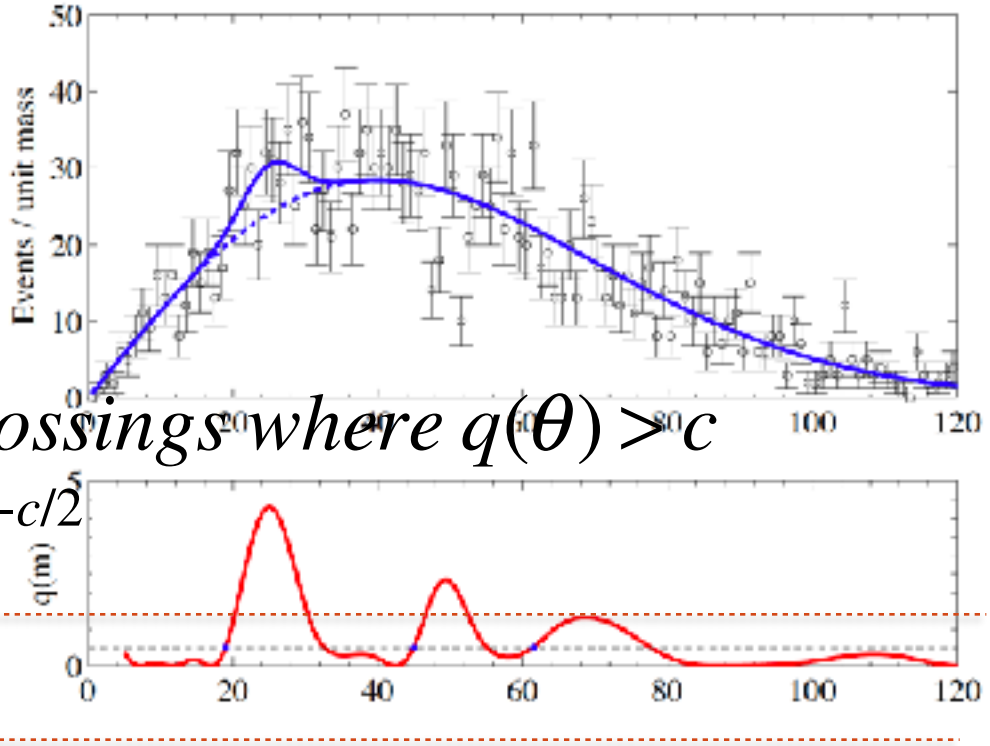
$$P\left(q(\hat{\theta}) > c\right) \sim P(\chi_1^2 > c) + \langle N(c) \rangle$$



$$\langle N(c) \rangle = \textit{Number of upcrossings where } q(\theta) > c$$

$$\textit{for } c >> 1 \rightarrow \langle N(c) \rangle << 1$$

Davies Formula



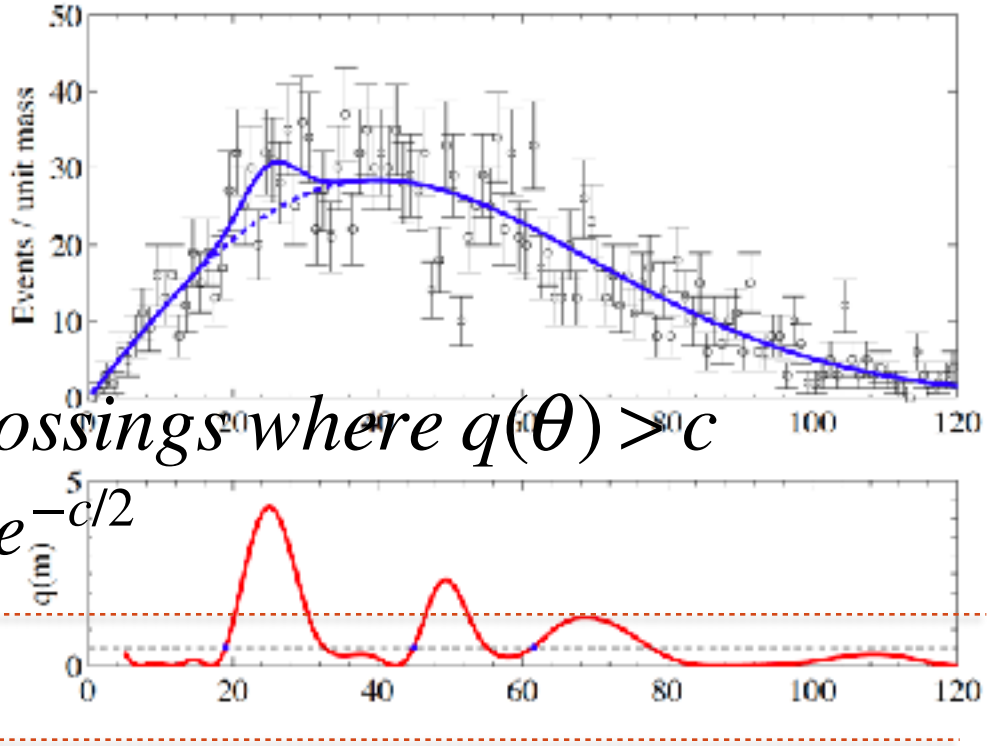
$$\langle N(c)\rangle = \text{Number of upcrossings where } q(\theta) > c$$

$$< N(c) > \sim P(\chi^2_2) > c \sim e^{-c/2}$$

$$P\left(q(\hat{\theta}) > c\right) \sim P(\chi^2_1 > c) + \langle N(c)\rangle$$

$$P(q(\hat{\theta}) > c) \sim P(\chi^2_1 > c) + \mathcal{N} P(\chi^2_2 > c)$$

Davies Formula



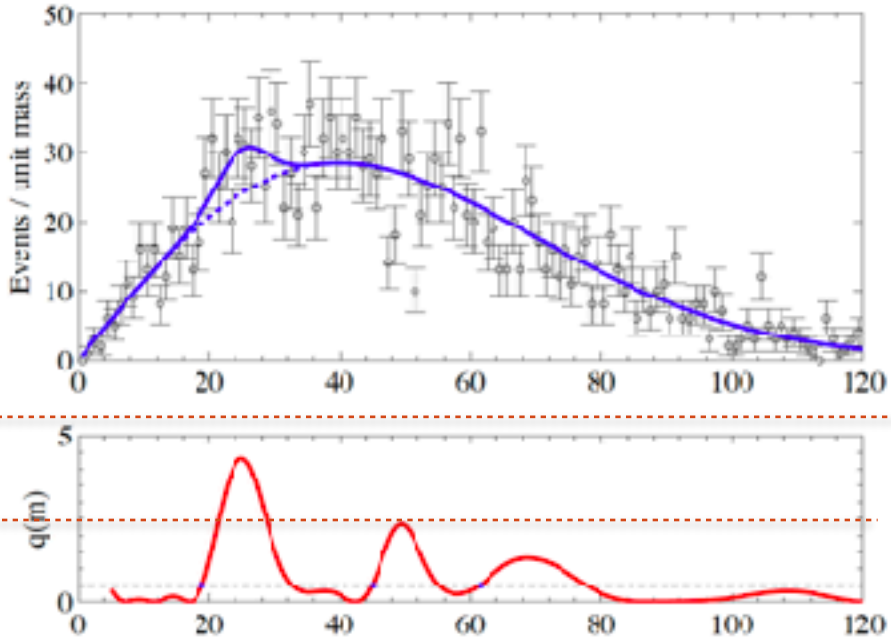
$$\langle N(c) \rangle = \text{Number of upcrossings where } q(\theta) > c$$

$$< N(c) > \sim [P(\chi^2_2) > c] \sim e^{-c/2}$$

$$< N(c) > = \frac{< N(c) >}{< N(c_0) >} < N(c_0) > = e^{-(c-c_0)/2} < N(c_0) >$$

Making Davies Formula Accessible

$$< N(c) > = \frac{< N(c) >}{< N(c_0) >} < N(c_0) > = e^{-(c-c_0)/2} < N(c_0) >$$



$$\langle N(c) \rangle \ll 1$$

$$\langle N(c) \rangle \sim e^{-c/2}$$

$$P(q(\hat{\theta}) > c) \sim P(\chi_1^2 > c) + \langle N(c_0) \rangle \frac{\langle N(c) \rangle}{\langle N(c_0) \rangle}$$

$$P(q(\hat{\theta}) > c) \sim P(\chi_1^2 > c) + \langle N(c_0) \rangle e^{-(c-c_0)/2}$$

Gross Vitells Formula

Trial

$$P(\chi_1^2 > c) \xrightarrow{c>>1} \sqrt{\frac{2}{c}} \frac{e^{-c/2}}{\Gamma\left(\frac{1}{2}\right)}$$

$$P(\chi_2^2 > c) \xrightarrow{c>>1} e^{-c/2}$$

$$trial\,\# = \frac{P(q(\hat{\theta}) > c)}{P(q(\theta) > c)} \approx$$

$$\approx 1 + \mathcal{N} \frac{P(\chi_2^2 > c)}{P(\chi_1^2 > c)} \Rightarrow$$

$$trial\,\# \approx 1 + \mathcal{N} \sqrt{\frac{c}{2}} \Gamma(1/2) \Rightarrow$$

$$trial\,\# \approx 1 + \sqrt{\frac{\pi}{2}} \mathcal{N} Z_{fix}$$

Example: The 750 GeV Resonance

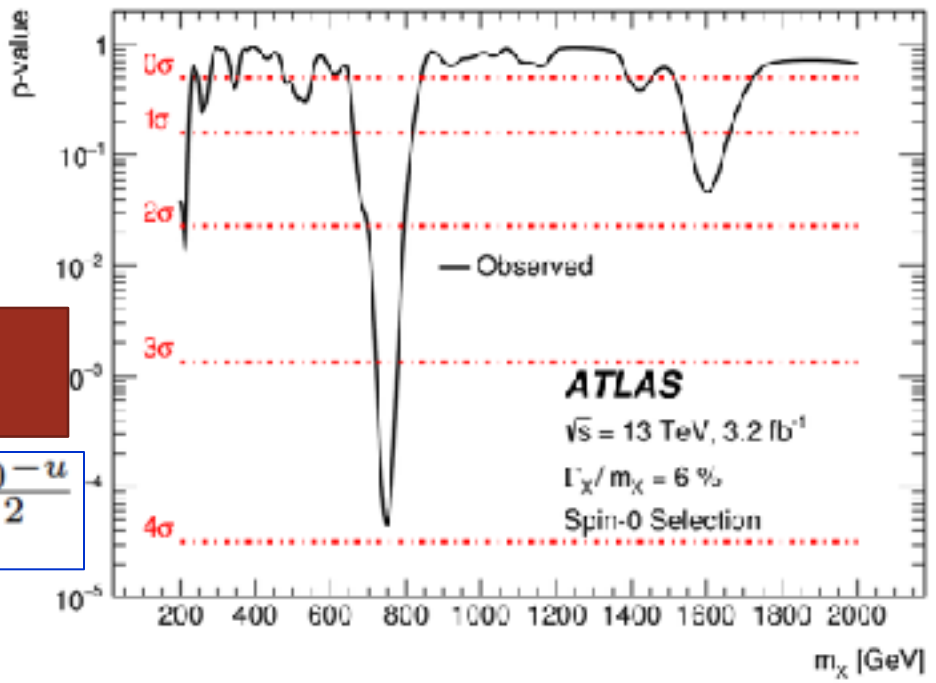
Spin 0 2015

Largest significance

$m_x \sim 750\text{GeV}, \Gamma_x \sim 45\text{GeV}(6$

Local Z = **3.9σ**

Any peak with Z>3.8σ
with m=500-2000 will draw our attention



$$P_{global}(u) \approx p_{local}(u) + E(n_{u_0})e^{\frac{u_0-u}{2}}$$

$p_{local} = 5 \cdot 10^{-5}$

$u_0 = 0$

$n_{u_0} = 7 \pm 2.6$

$u = Z^2 = 3.9^2 = 15.2$

$p_{global} = 5 \cdot 10^{-5} + (7 \pm 2.6)e^{-15.2/2} = (2.2 - 4.8)10^{-3}$

$Z_{global} \sim 2.7 \pm 0.1\sigma$

The LEE is even stronger when you consider another dimension (the width range (0-10%m) should also be taken into account)

A real life example

$$P(q_0 > u) \le E[N_u] + P(q_0(0) > u)$$

$$E[N_u] = \mathbb{N}_1 e^{-u/2}$$

$$\mathbb{N}_1 \cong \langle N_{u_0} \rangle e^{u_0/2}$$

$$P(q_0 > u) = \mathbb{N}_1 e^{-u/2} + \frac{1}{2} P(\chi_1^2 > u)$$

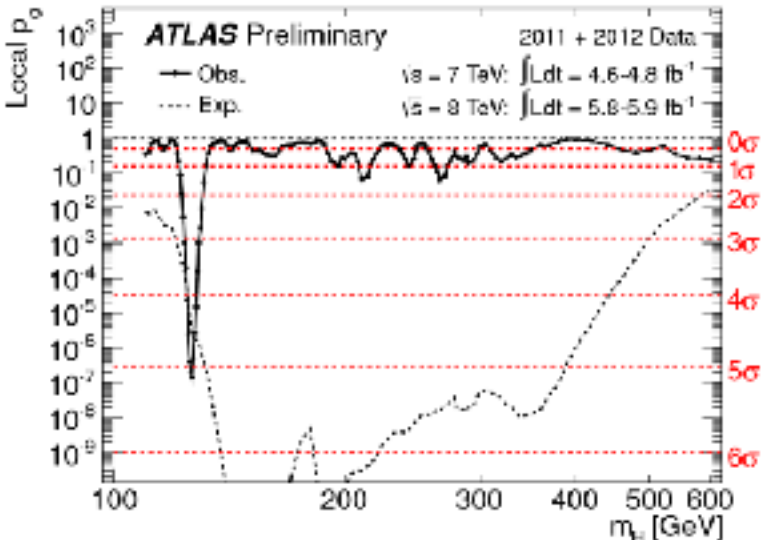
$$p_{global} = \mathbb{N}_1 e^{-u/2} + p_{local}$$

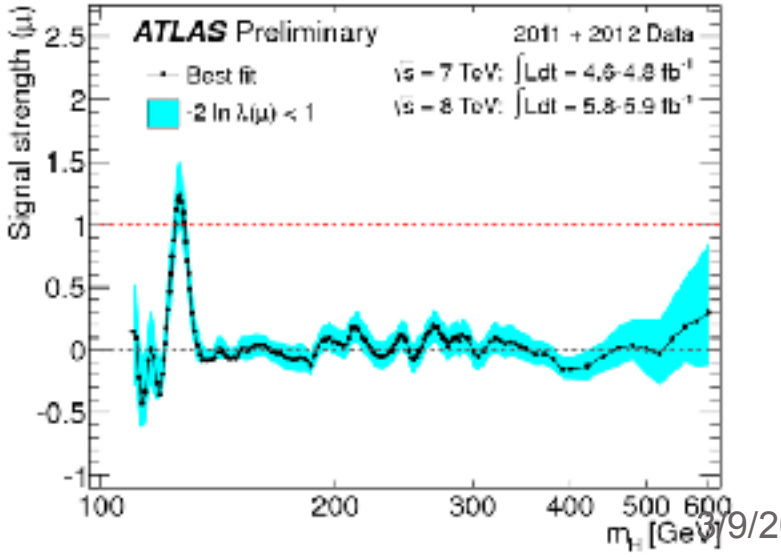
$$p_{global} = \langle N_{u_0} \rangle e^{\frac{u_0 - u}{2}} + p_{local}$$

$$N_{u_0=0} = 9 \pm 3$$

$$p_{global} = 9 \cdot e^{-25/2} + O(10^{-7}) = 3.3 \cdot 10^{-5}$$

$5\sigma \rightarrow 4\sigma$ trial#~100





The 2D LEE

Define the Problem

- Let $n = \mu s(m,\Gamma) + b$
- m, Γ are nuisance parameters undefined under the null hypothesis $\mu = 0$
- What is the pdf of

$$\hat{q}_0 \equiv q_0(\hat{m},\hat{\Gamma}) = -2\ln\frac{L(\mu=0)}{L(\hat{\mu},\hat{m},\hat{\Gamma})} = \max_{m,\Gamma} q_0(m,\Gamma)$$

under the null hypothesis

Define the Problem

- To generalize the problem , let θ be the nuisance parameter, undefined under the null hypothesis, and let us try to find out the pdf of

$$\hat{q}_0 \equiv q_0(\hat{\theta}) = -2\ln\frac{L(\mu=0)}{L(\hat{\mu},\hat{\theta})} = \max_{\theta} q_0(\theta)$$

for which we want to calculate

$$p-value = P\left(\max_{\theta}\left[q_0(\theta)\right] \geq u\right),\ u = Z^2$$

Chi Squared Random Field

- For fixed θ $\quad q_0(\theta) = -2\ln\dfrac{L(\mu=0)}{L(\hat{\mu},\theta)} \sim \chi^2_1$

- $q_0(\theta)$ is a chi squared random field over the space of θ
  (a random variable indexed by a continuous parameter(s))
- We are interested in

$$\hat{q}_0 \equiv q_0(\hat{\theta}) = -2\ln\frac{L(\mu=0)}{L(\hat{\mu},\hat{\theta})} = \max_{\theta} q_0(\theta)$$

for which we want to calculate

$$p-value = P\left(\max_{\theta}\left[q_0(\theta)\right] \geq u\right),\ u = Z^2$$

Chi Squared Random Field

- We are only interested in positive signals
(downward fluctuations of the background are not considered as an evidence against the background)

$$q_0(\theta) = \begin{cases} -2\log\dfrac{L(\mu=0)}{L(\hat{\mu},\theta)} \\ 0 \end{cases} \qquad q_0(\theta) \sim \frac{1}{2}\chi_1^2$$

[H. Chernoff, Ann. Math. Stat. 25, 573578 (1954)]

Chi Squared Random Field

- We are only interested in positive signals (downward fluctuations of the background are not considered as an evidence against the background)

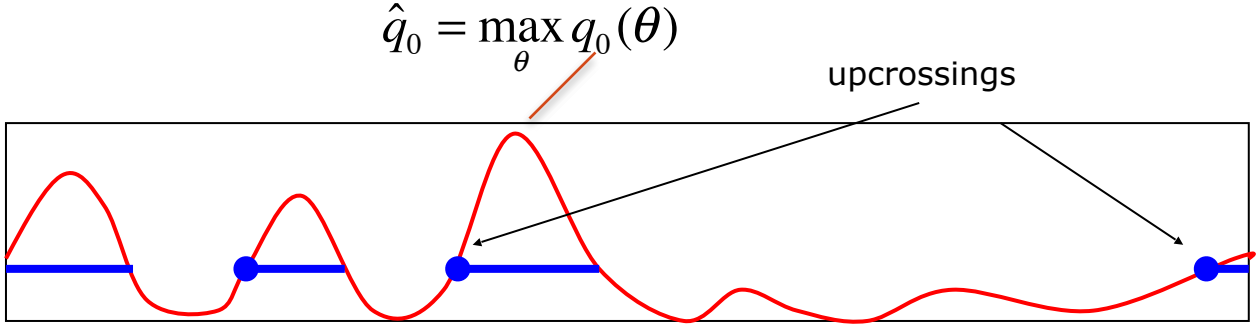
$$q_0(\theta) = \begin{cases} -2\log\dfrac{L(\mu=0)}{L(\hat{\mu},\theta)} & \\ 0 & \end{cases} \qquad q_0(\theta) \sim \frac{1}{2}\chi^2_1$$

[H. Chernoff, Ann. Math. Stat. 25, 573578 (1954)]

- $q_0(\theta) = \left(\dfrac{\hat{\mu}(\theta)}{\sigma}\right)^2$ $\hat{\mu}(\theta)$ is a Gaussuan Random Field over θ

1-D Random Fields

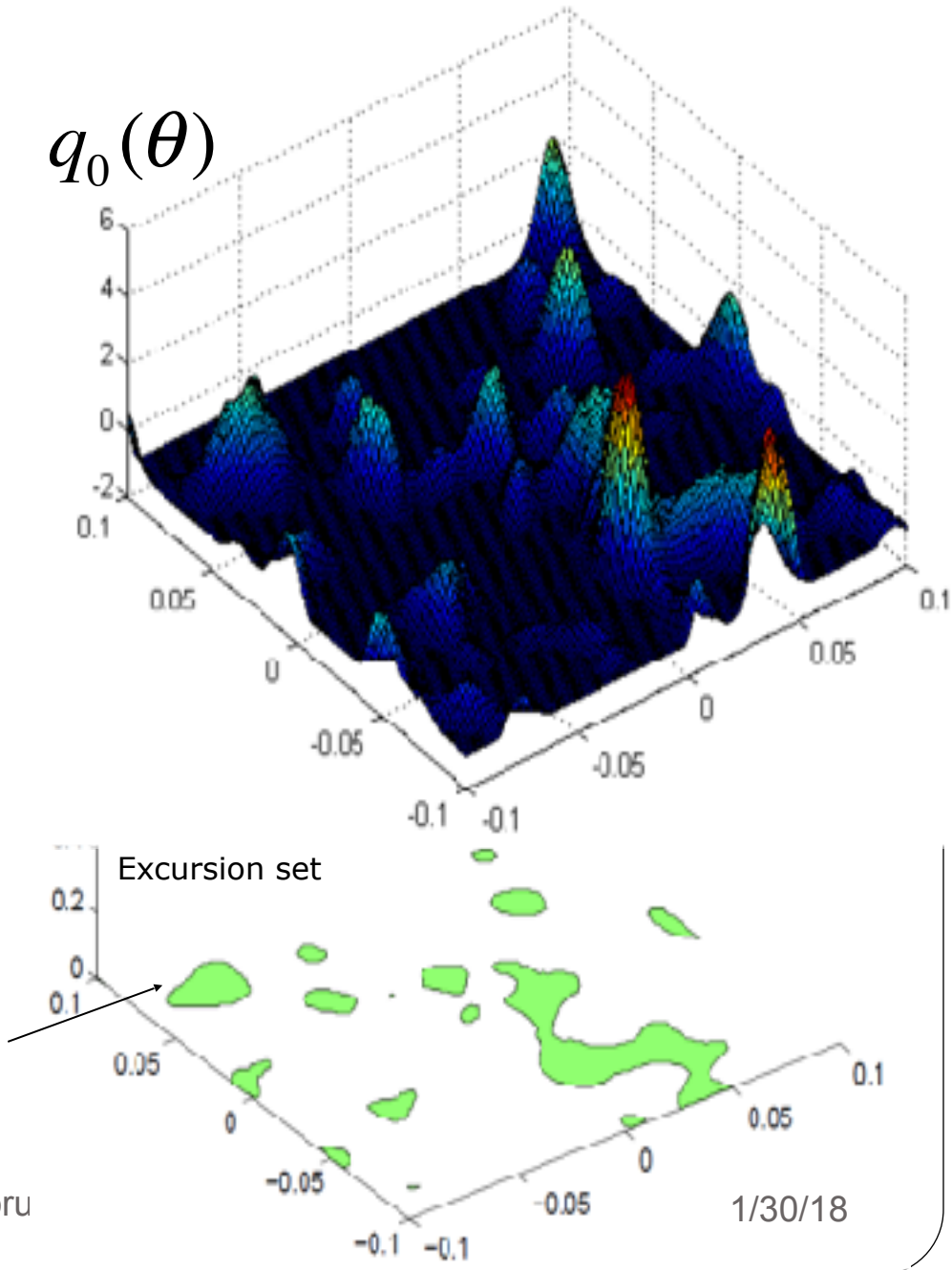
- In 1-D points where the field becomes larger than u are called upcrossings.



- The probability that the global maximum is above the level u is called *exceedance probability*.
(p-value of $q_0(\hat{\theta})$) $p = P\left(\max_\theta \left[q_0(\theta) \right] \geq u \right),\ u = Z^2$

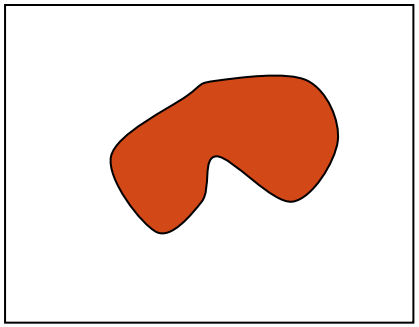
Random fields (>1 D)

- The set of points where the field has values larger then some number *u* is called the *excursion set* A_u above the level *u.*

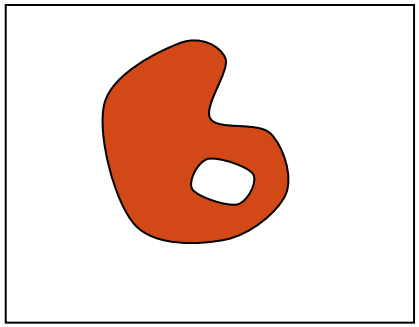


Euler characteristic

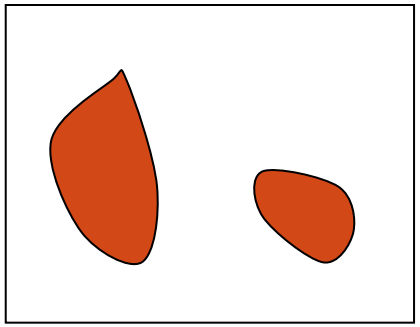
- Number of disconnected components minus number of `holes'



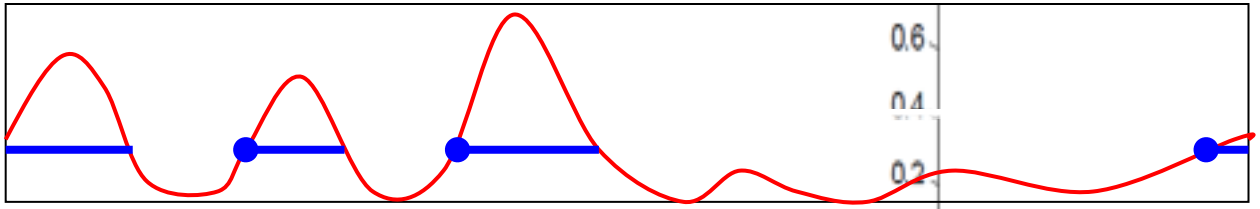
φ=1



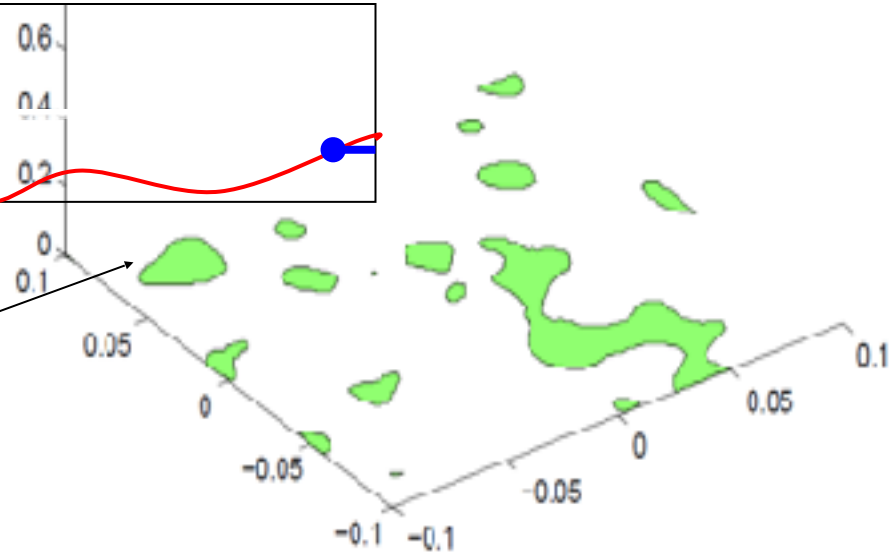
φ=0



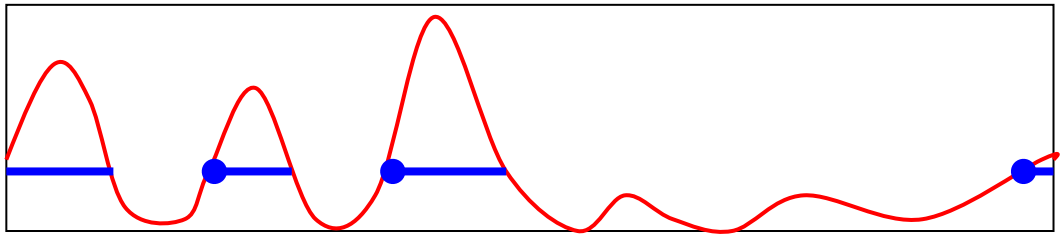
φ=2



Excursion set



1D Euler characteristic



$$\mathrm{E}[\varphi(A_u)] = \sum_{d=0}^{n} \mathrm{N}_d \rho_d(u)$$

In 1 dimension:

$$\varphi(A_u) = N_u + 1_{[q_0(0)>u]}$$

$$E\left[\varphi(A_u)\right] = E\left[N_u\right] + P(q_0(0) > u)$$

$$= N_0 P(\chi_1^2 > u) + N_1 e^{-u/2}$$

$$N_0 = \varphi(manifold) = 1$$

$$E\left[\varphi(A_u)\right] = P(\chi_1^2 > u) + N_1 e^{-u/2}$$

This is Davies Formula

The general case

$$N_0 = \varphi(manifold)$$

$$\rho_0(u) = P(\chi_s^2 > u)$$

In general for high-level excursions

$$E\left[\varphi(A_u)\right] \xrightarrow[u\gg 1]{} P\left(\max_{\theta}\left[q_0(\theta)\right] \geq u\right)$$

[J. Taylor, A. Takemura, and R. J. Adler, Ann. Probab. 33, 4 (2005)]

2-d example: search for neutrino sources (IceCube)

For a chi² field in 2 dimensions:

$$E\left[\vartheta(A_u)\right] = \frac{1}{2}P(\chi_2^2 > u) + \left(\mathcal{N}_1 + \mathcal{N}_2\sqrt{u}\right)e^{-u/2}$$

Estimate E[φ] at two levels, e.g. 0 and 1, and solve for $\mathcal{N}_1$ and $\mathcal{N}_2$

From 20 bkg. Simulations:

$\langle\varphi_0\rangle = 33.5 \pm 2$

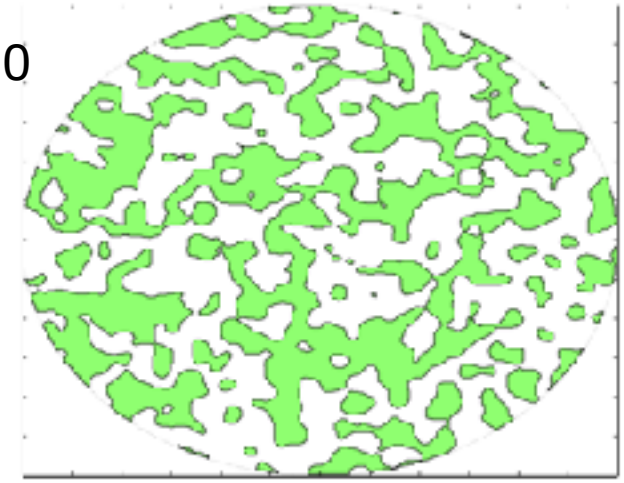
$\langle\varphi_1\rangle = 94.6 \pm 1.3$

⇩

$\mathcal{N}_1 = 33 \pm 2$

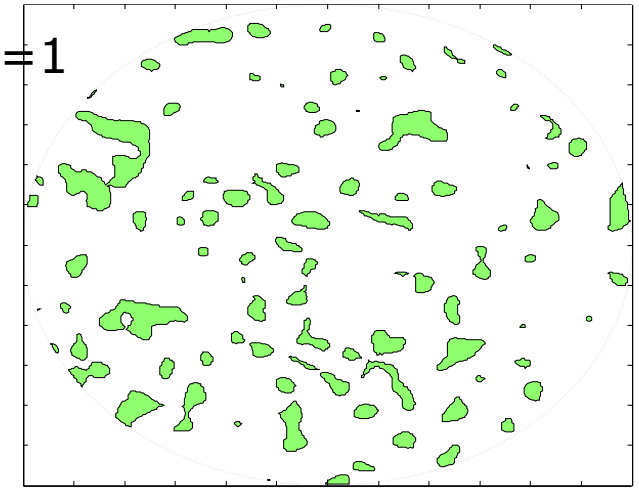
$\mathcal{N}_2 = 123 \pm 3$

u=0



φ=35

u=1

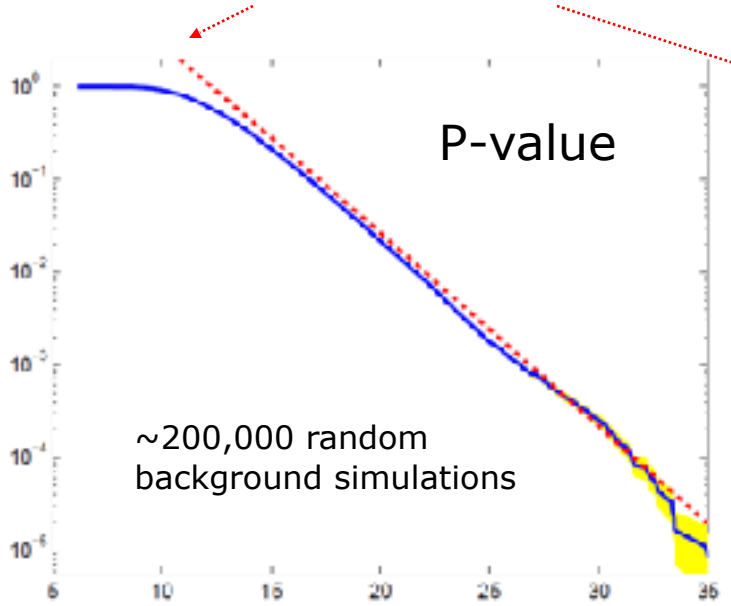


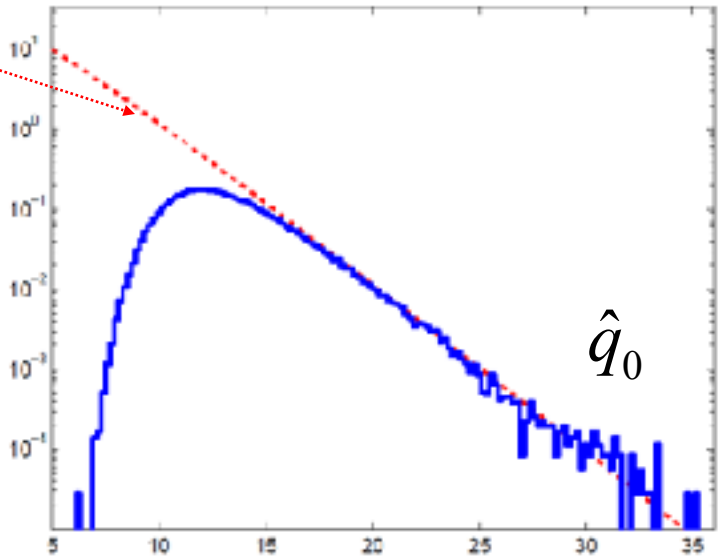
φ=95

2-d example: search for neutrino sources (IceCube)

$$E\left[\vartheta(A_u)\right] = \frac{1}{2}P(\chi^2_2 > u) + \left(\mathcal{N}_1 + \mathcal{N}_2\sqrt{u}\right)e^{-u/2}$$

$$N_1 = 33 \pm 2$$
$$N_2 = 123 \pm 3$$





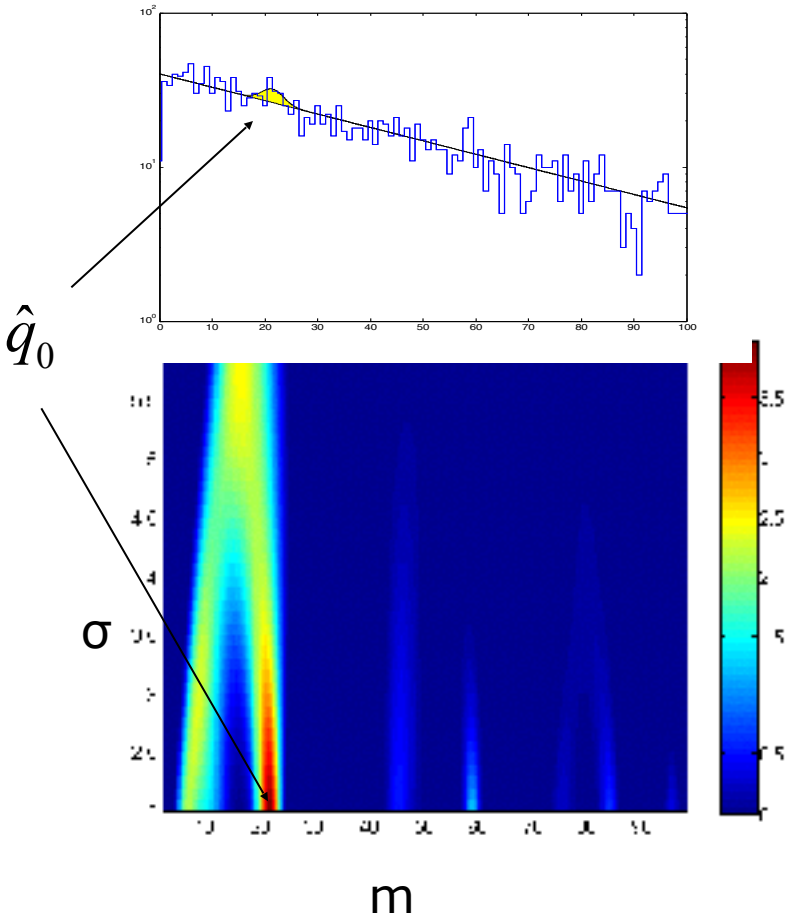
e.g.: P(max q_0>30) = (2.5 ± 0.4)x10^{-4} (estimated)
E.C. Formula : (2.28 ± 0.06)x10^{-4}

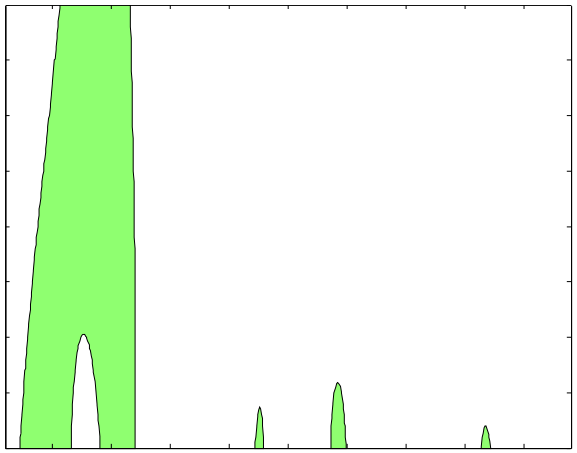
2-D exapmle #2: resonance search with unknown width

- Gaussian signal on exponential background
- Toy model : 0<m<100 , 2<σ<6
- Unbinned likelihood:

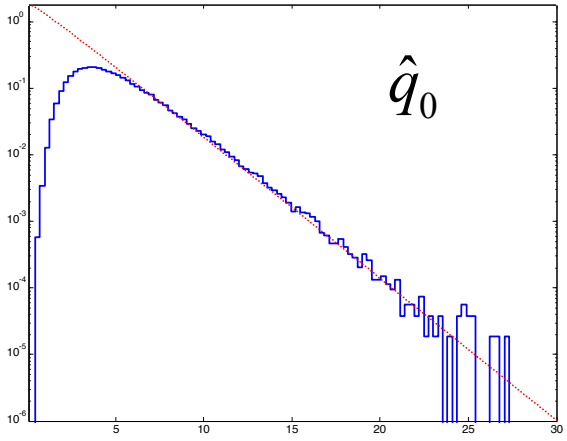
$$L = \prod_i \frac{N_s f_s(x_i) + N_b f_b(x_i)}{N_s + N_b} \times Poiss(N \mid N_s + N_b)$$

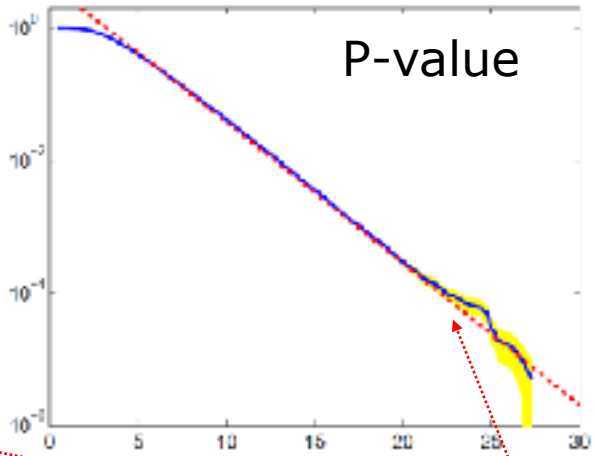
$$f_s(x;m,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}} \qquad f_b(x) = ce^{-cx}$$





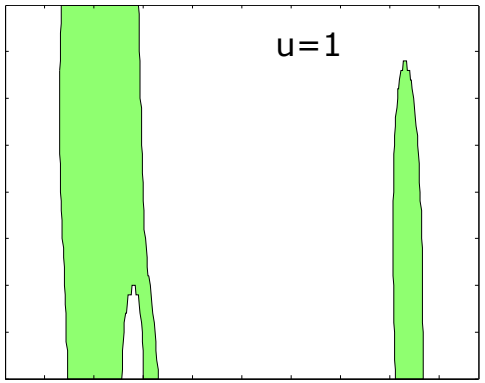
2-D exapmle #2: resonance search with unknown width



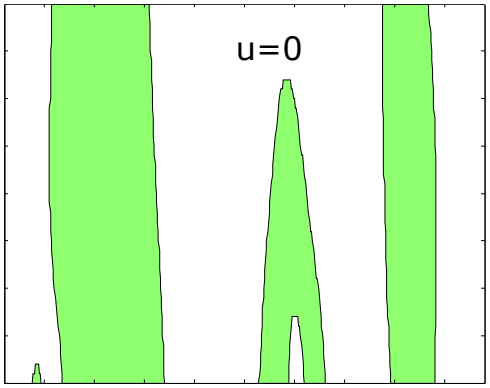


Excellent approximation above the ~2σ level

$\langle \varphi_1 \rangle = 3 \pm 0.16$



$\langle \varphi_0 \rangle = 4.5 \pm 0.2$



$$E\left[\vartheta(A_u)\right] = \frac{1}{2}P(\chi^2_2 > u) + \left(\mathcal{N}_1 + \mathcal{N}_2\sqrt{u}\right)e^{-u/2}$$

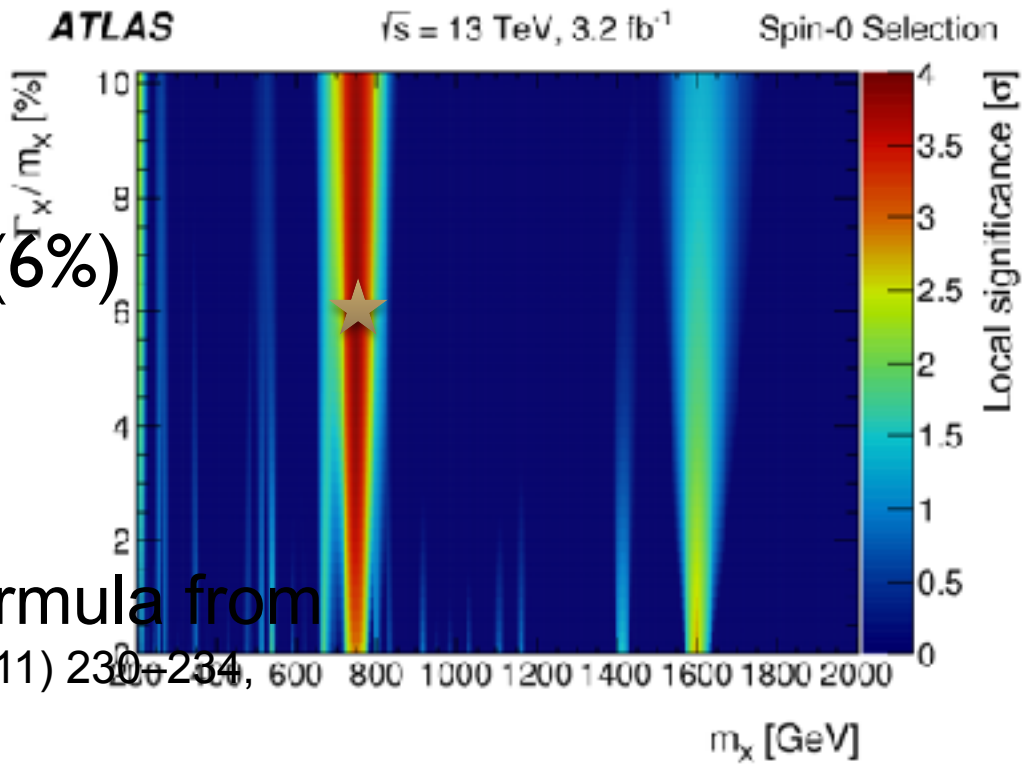
$$\mathcal{N}_1 = 4 \pm 0.2$$

$$\mathcal{N}_2 = 0.7 \pm 0.3$$

The 750 GeV saga

2015

2D Scan
Largest significance



$m_x \sim 750\text{GeV}, \Gamma_x \sim 45\text{GeV}(6\%)$

Local Z = **3.9σ**

m=200-2000 GeV
Γ_X/m_X=0-10%

Use toys or asymptotic formula from
O. Vitells et. al. Astropart. Phys. 35 (2011) 230-234
arXiv:1105.4355

$$Z_{local} = 3.9\sigma$$

$$Z_{global} = 2.1\sigma$$

2.1σ is not something to write home about

Summary

$$p_{global}(s=1,D=1) \approx E\left[\vartheta(A_u)\right] = \frac{1}{2}P(\chi_1^2 > u) + \mathcal{N}_1 e^{-u/2}$$

$$p_{global}(s=1,D=2) \approx E\left[\vartheta(A_u)\right] = \frac{1}{2}P(\chi_2^2 > u) + \left(\mathcal{N}_1 + \mathcal{N}_2\sqrt{u}\right)e^{-u/2}$$

- The procedure for estimating the p-value is simple and reliable.
- The Euler characteristic formula provides a practical way of estimating the look-elsewhere effect.
- It is easily expandable to s p.o.i and D NPs (undefined under the null hypothesis)

Thank You

Eilam Gross