Quantum Link Models for the Quantum Simulation of Gauge Theories

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Outline

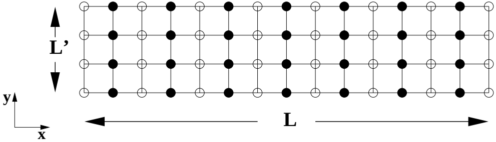
Outline

$\mathbb{C}P(N-1)$ Models from $SU(N)$ quantum spins

$$T_x^a, \quad a \in \{1, 2, ..., N^2 - 1\}, \quad [T_x^a, T_y^b] = i\delta_{xy} f_{abc} T_x^c$$

Spin ladder Hamiltonian

$$H = -J \sum_{x \in A} [T_x^a T_{x+\hat{1}}^{a*} + T_x^a T_{x+\hat{2}}^a] - J \sum_{x \in B} [T_x^{a*} T_{x+\hat{1}}^a + T_x^{a*} T_{x+\hat{2}}^{a*}]$$



Conserved $SU(N)$ spin

$$T^a = \sum_{x \in A} T_x^a - \sum_{x \in B} T_x^{a*}, \quad [T^a, H] = 0$$

B. B. Beard, M. Pepe, S. Riederer, UJW, PRL 94 (2005) 010603.

Goldstone boson fields in $\mathbb{C}P(N-1) = SU(N)/U(N-1)$

$$P(x)^\dagger = P(x), \quad \mathrm{Tr} P(x) = 1, \quad P(x)^2 = P(x)$$

Low-energy effective action

$$\begin{aligned} S[P] &= \int_0^\beta dt \int_0^L dx \int_0^{L'} dy \ \mathrm{Tr} \left\{ \rho_s' \partial_y P \partial_y P \right. \\ &+ \left. \rho_s \left[\partial_x P \partial_x P + \frac{1}{c^2} \partial_t P \partial_t P \right] - \frac{1}{a} P \partial_x P \partial_t P \right\} \end{aligned}$$

Topological charge

$$Q[P] = \frac{1}{\pi i} \int_0^\beta dt \int_0^L dx \ \mathrm{Tr}[P \partial_x P \partial_t P] \in \Pi_2[SU(N)/U(N-1)] = Z$$

Dimensional reduction to the $(1+1)$-d $\mathbb{C}P(N-1)$ model

$$S[P] = \int_0^\beta dt \int_0^L dx \; \mathrm{Tr} \left\{ \frac{1}{g^2} \left[\partial_x P \partial_x P + \frac{1}{c^2} \partial_t P \partial_t P \right] - n P \partial_x P \partial_t P \right\}$$

Asymptotically free coupling and emergent θ-vacuum angle

$$\frac{1}{g^2} = \frac{\rho_s L'}{c}, \quad \theta = n\pi$$

Very large correlation length

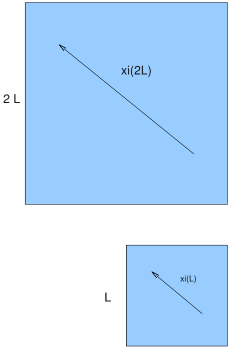
$$\xi \propto \exp\left(\frac{4\pi}{N g^2}\right) = \exp\left(\frac{4\pi L' \rho_s}{c N}\right) \gg L'$$

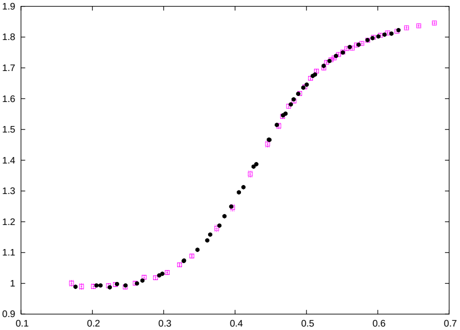
Measured correlation lengths in the $\mathbb{C}P(2)$ model

$$\xi(2) = 4.0(1), \quad \xi(4) = 17.6(2), \quad \xi(6) = 61(2)$$

Step-scaling approach

M. Lüscher, P. Weisz, and U. Wolff, Nucl. Phys. B359 (1991) 221



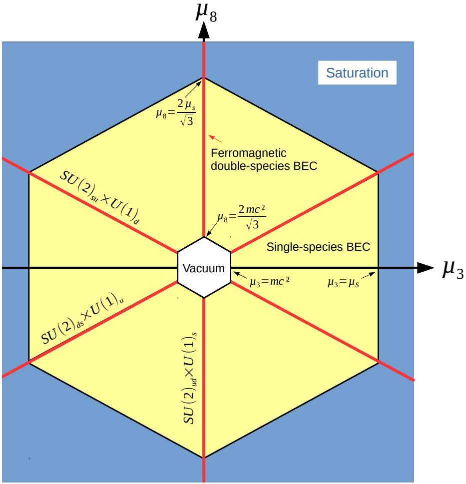


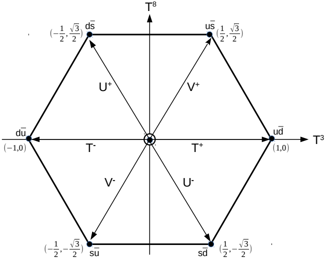
Universal step-scaling function

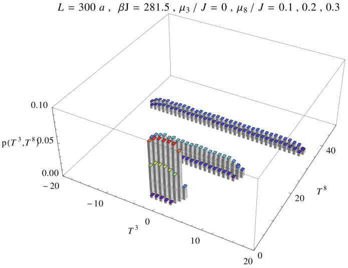
$$F(z) = \xi(2L)/\xi(L) \ , \quad z = \xi(L)/L$$

B. B. Beard, M. Pepe, S. Riederer, UJW, PRL 94 (2005) 010603.

Ferromagnetic Double-Species BEC in the $\mathbb{C}P(2)$ Model



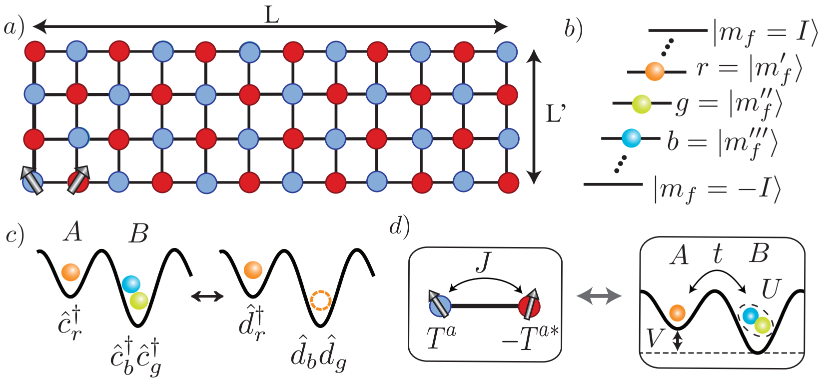




W. Evans, U. Gerber, M. Hornung, UJW, Annals Phys. 398 (2018) 92.

Ladder of $SU(N)$ quantum spins embodied with alkaline-earth atoms

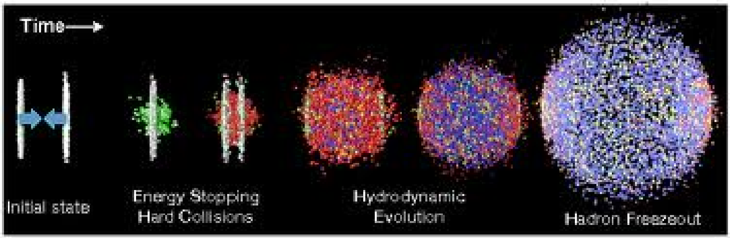
$$H = -J \sum_{\langle xy \rangle} T_x^a T_y^{a*}, \quad [T_x^a, T_y^b] = i\delta_{xy} f_{abc} T_x^c$$

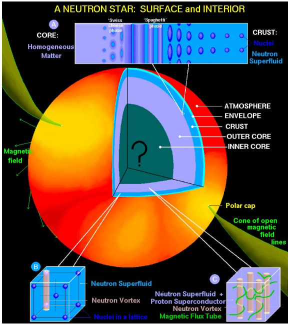


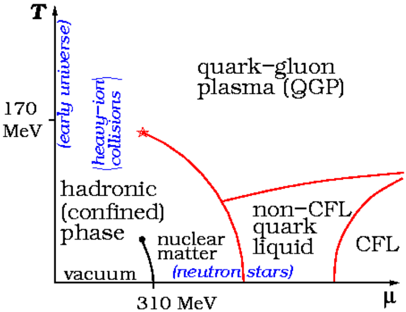
C. Laflamme, W. Evans, M. Dalmonte, U. Gerber, H. Mejia-Diaz, W. Bietenholz, UJW, and P. Zoller, Annals Phys. 360 (2016) 117.

Outline

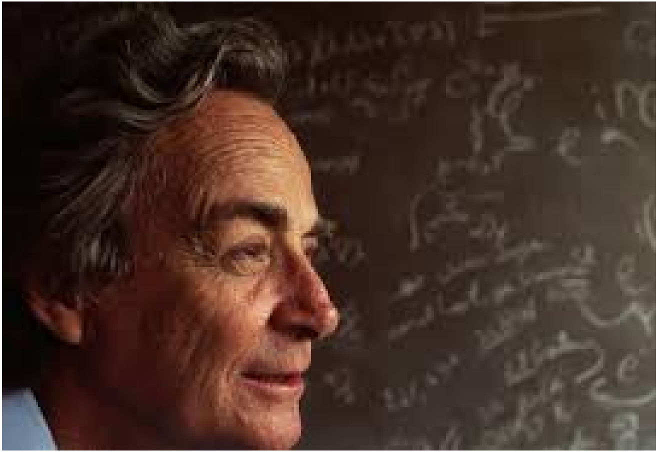
Can heavy-ion collision physics or nuclear astrophysics benefit from quantum simulations in the long run?







Richard Feynman's vision of 1982



"It does seem to be true that all the various field theories have the same kind of behavior, and can be simulated in every way, apparently, with little latticeworks of spins and other things."

Different descriptions of dynamical Abelian gauge fields:

Maxwell's classical electromagnetic gauge fields

$$\vec{\nabla} \cdot \vec{E}(\vec{x}, t) = \rho(\vec{x}, t), \quad \vec{\nabla} \cdot \vec{B}(\vec{x}, t) = 0, \quad \vec{B}(\vec{x}, t) = \vec{\nabla} \times \vec{A}(\vec{x}, t)$$

Quantum Electrodynamics (QED) for perturbative treatment

$$E_i = -i \frac{\partial}{\partial A_i}, \quad [E_i(\vec{x}), A_j(\vec{x}\,')] = i\delta_{ij}\delta(\vec{x} - \vec{x}\,'), \quad \left[\vec{\nabla} \cdot \vec{E} - \rho\right] |\Psi[A]\rangle = 0$$

Wilson's $U(1)$ lattice gauge theory for classical simulation

$$U_{xy} = \exp\left(ie \int_x^y d\vec{l} \cdot \vec{A}\right) = \exp(i\varphi_{xy}) \in U(1), \quad E_{xy} = -i \frac{\partial}{\partial \varphi_{xy}},$$

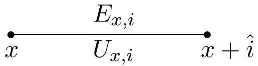
$$[E_{xy}, U_{xy}] = U_{xy}, \quad \left[\sum_i (E_{x,x+\hat{i}} - E_{x-\hat{i},x}) - \rho\right] |\Psi[U]\rangle = 0$$

$U(1)$ quantum link models for quantum simulation

$$U_{xy} = S^+_{xy}, \quad U^\dagger_{xy} = S^-_{xy}, \quad E_{xy} = S^3_{xy},$$

$$[E_{xy}, U_{xy}] = U_{xy}, \quad [E_{xy}, U^\dagger_{xy}] = -U^\dagger_{xy}, \quad [U_{xy}, U^\dagger_{xy}] = 2E_{xy}$$

Hamiltonian formulation of Wilson's $U(1)$ lattice gauge theory



$$U = \exp(i\varphi), \ U^\dagger = \exp(-i\varphi) \in U(1)$$

Electric field operator E

$$E = -i\partial_\varphi, \ [E,U] = U, \ [E,U^\dagger] = -U^\dagger, \ [U,U^\dagger] = 0$$

Generator of $U(1)$ gauge transformations

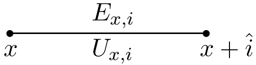
$$G_x = \sum_i (E_{x-\hat{i},i} - E_{x,i}), \quad [H, G_x] = 0$$

$U(1)$ gauge invariant Hamiltonian

$$H = \frac{g^2}{2} \sum_{x,i} E_{x,i}^2 - \frac{1}{2g^2} \sum_{x,i\neq j} (U_{x,i} U_{x+\hat{i},j} U^\dagger_{x+\hat{j},i} U^\dagger_{x,j} + \text{h.c.})$$

operates in an infinite-dimensional Hilbert space per link

Quantum link formulation of $U(1)$ lattice gauge theory



$$U = S^+, \; U^\dagger = S^-$$

Electric field operator E

$$E = S^3, \; [E, U] = U, \; [E, U^\dagger] = -U^\dagger, \; [U, U^\dagger] = 2E$$

Generator of $U(1)$ gauge transformations

$$G_x = \sum_i (E_{x-\hat{i},i} - E_{x,i}), \quad [H, G_x] = 0$$

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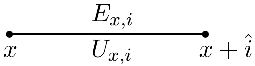
operates in a finite-dimensional Hilbert space per link

$U(1)$ quantum links from spins $\frac{1}{2}$

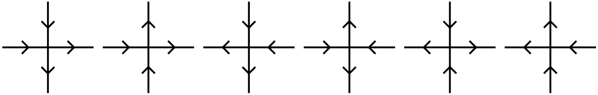


$U = S^1 + iS^2 = S^+, \; U^\dagger = S^1 - iS^2 = S^-$

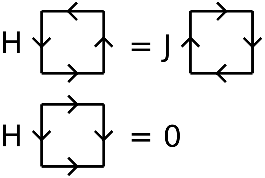
Electric flux operator E

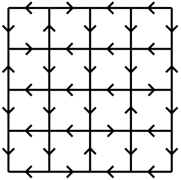
$$E = S^3, \; [E, U] = U, \; [E, U^\dagger] = -U^\dagger, \; [U, U^\dagger] = 2E$$

Gauss law



Ring-exchange plaquette Hamiltonian



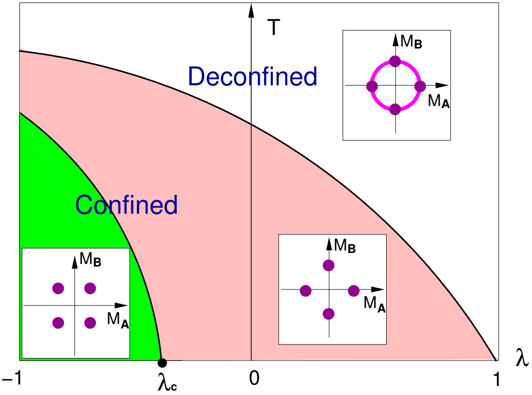


D. Horn, Phys. Lett. B100 (1981) 149
P. Orland, D. Rohrlich, Nucl. Phys. B338 (1990) 647
S. Chandrasekharan, UJW, Nucl. Phys. B492 (1997) 455

Hamiltonian with Rokhsar-Kivelson term

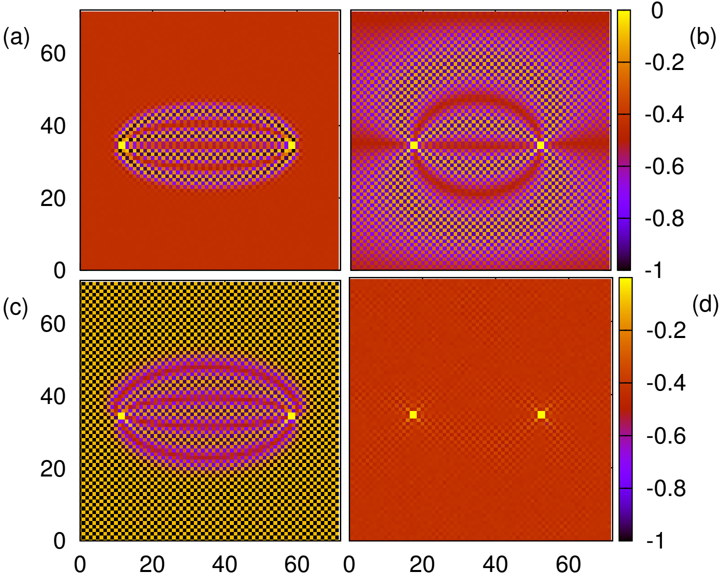
$$H = -J \left[\sum_{\square} (U_\square + U_\square^\dagger) - \lambda \sum_{\square} (U_\square + U_\square^\dagger)^2 \right]$$

Phase diagram



D. Banerjee, F.-J. Jiang, P. Widmer, UJW, JSTAT (2013) P12010.

Energy density of charge-anti-charge pair $Q = \pm 2$



D. Banerjee, F.-J. Jiang, P. Widmer, UJW, JSTAT (2013) P12010.

Outline

$U(N)$ quantum link operators

$$U^{ij} = S_1^{ij} + iS_2^{ij},\ U^{ij\dagger} = S_1^{ij} - iS_2^{ij},\ i,j \in \{1,2,\ldots,N\},\ [U^{ij},(U^\dagger)^{kl}] \neq 0$$

$SU(N)_L \times SU(N)_R$ gauge transformations of a quantum link

$$[L^a, L^b] = if_{abc}L^c,\ [R^a, R^b] = if_{abc}R^c,\ a,b,c \in \{1,2,\ldots,N^2-1\}$$

$$[L^a, R^b] = [L^a, E] = [R^a, E] = 0$$

Infinitesimal gauge transformations of a quantum link

$$[L^a, U] = -\lambda^a U,\ [R^a, U] = U\lambda^a,\ [E, U] = U$$

Algebraic structures of different quantum link models

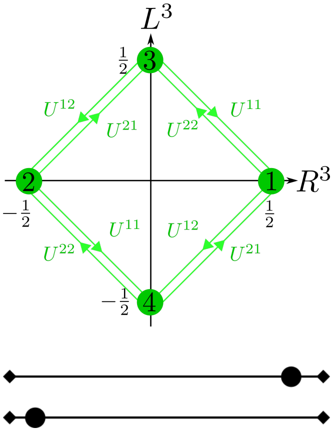
$U(N)$: $U^{ij},\ L^a,\ R^a, E,\ 2N^2+2(N^2-1)+1 = 4N^2-1$ $SU(2N)$ generators

$SO(N)$: $O^{ij},\ L^a,\ R^a,\ N^2+2\frac{N(N-1)}{2} = N(2N-1)$ $SO(2N)$ generators

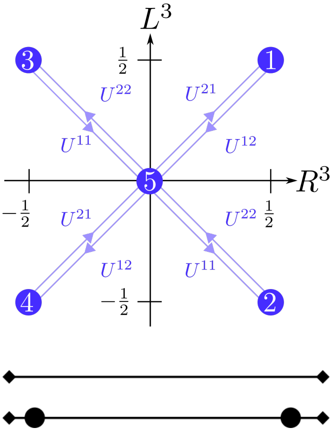
$Sp(N)$: $U^{ij},\ L^a,\ R^a,\ 4N^2+2N(2N+1) = 2N(4N+1)$ $Sp(2N)$ generators

R. Brower, S. Chandrasekharan, UJW, Phys. Rev. D60 (1999) 094502

$Sp(1) = SU(2)$ quantum link model with an $Sp(2) = SO(5)$ embedding algebra

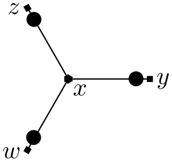


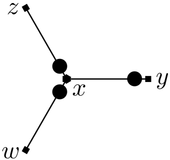
Spinor representation $\{4\}$



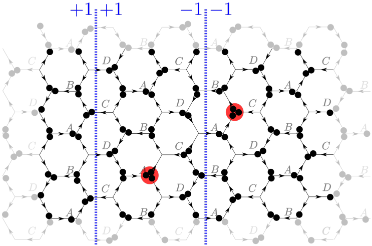
Vector representation $\{5\}$

Gauss law for the $Sp(1) = SU(2)$ quantum link model on the honeycomb lattice

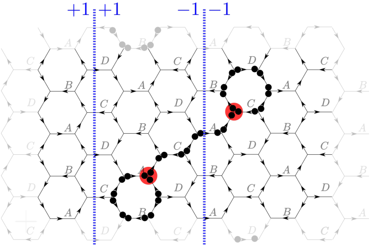




Flux strings in the $Sp(1) = SU(2)$ quantum link model

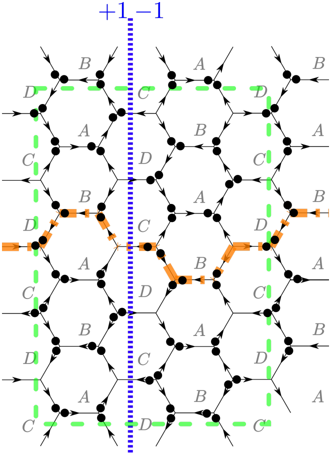


Spinor representation {4}



Vector representation {5}

Half a unit of $\mathbb{Z}(2)$ center electric flux in the $Sp(1) = SU(2)$ quantum link model on the honeycomb lattice



Spinor representation $\{4\}$

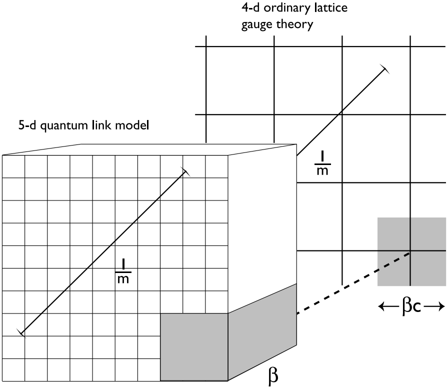
D. Banerjee, F.-J. Jiang, T. Z. Olesen, P. Orland, UJW,
Phys. Rev. B97 (2018) 205108.

Low-energy effective action for a quantum link model in a $(4+1)$-d massless non-Abelian Coulomb phase

$$S[G_\mu] = \int_0^\beta dx_5 \int d^4x \; \frac{1}{2e^2} \left(\mathrm{Tr}\; G_{\mu\nu} G_{\mu\nu} + \frac{1}{c^2} \mathrm{Tr}\; G_{\mu 5} G_{\mu 5} \right),$$

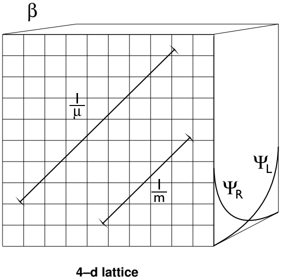
undergoes dimensional reduction from $4+1$ to 4 dimensions

$$S[G_\mu] \to \int d^4x \; \frac{1}{2g^2} \mathrm{Tr}\; G_{\mu\nu} G_{\mu\nu}, \;\; \frac{1}{g^2} = \frac{\beta}{e^2}, \;\; \frac{1}{m} \sim \exp\left(\frac{24\pi^2 \beta}{11 N e^2} \right)$$



Quarks as Domain Wall Fermions

$$
\begin{aligned}
H &= J \sum_{x,\mu\neq\nu} \mathrm{Tr}[U_{x,\mu} U_{x+\hat{\mu},\nu} U^\dagger_{x+\hat{\nu},\mu} U^\dagger_{x,\nu}] + J' \sum_{x,\mu} [\det U_{x,\mu} + \det U^\dagger_{x,\mu}] \\
&+ \frac{1}{2} \sum_{x,\mu} [\Psi^\dagger_x \gamma_0 \gamma_\mu U_{x,\mu} \Psi_{x+\hat{\mu}} - \Psi^\dagger_{x+\hat{\mu}} \gamma_0 \gamma_\mu U^\dagger_{x,\mu} \Psi_x] + M \sum_x \Psi^\dagger_x \gamma_0 \Psi_x \\
&+ \frac{r}{2} \sum_{x,\mu} [2\Psi^\dagger_x \gamma_0 \Psi_x - \Psi^\dagger_x \gamma_0 U_{x,\mu} \Psi_{x+\hat{\mu}} - \Psi^\dagger_{x+\hat{\mu}} \gamma_0 U^\dagger_{x,\mu} \Psi_x].
\end{aligned}
$$



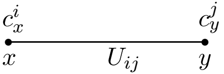
$$
\mu = 2M \exp(-M\beta), \quad \frac{1}{m} \propto \exp\left(\frac{24\pi^2 \beta}{(11N - 2N_f)e^2}\right), \quad M > \frac{24\pi^2}{(11N - 2N_f)e^2}
$$

Outline

Fermionic rishons at the two ends of a link

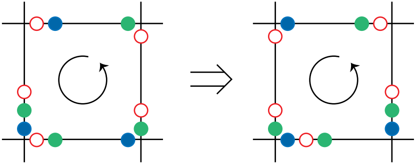
$$\{c_x^i, c_y^{j\dagger}\} = \delta_{xy}\delta_{ij}, \ \{c_x^i, c_y^j\} = \{c_x^{i\dagger}, c_y^{j\dagger}\} = 0$$

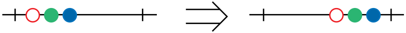
Rishon representation of link algebra



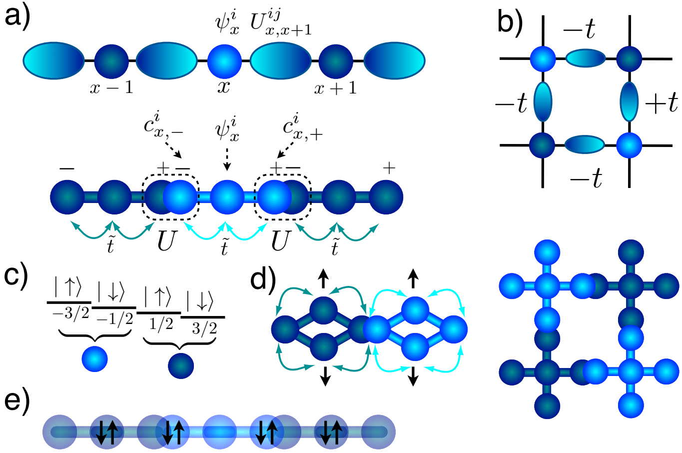
$$U_{xy}^{ij} = c_x^i c_y^{j\dagger}, \ L_{xy}^a = c_x^{i\dagger}\lambda_{ij}^a c_x^j, \ R_{xy}^a = c_y^{i\dagger}\lambda_{ij}^a c_y^j, \ E_{xy} = \frac{1}{2}(c_y^{i\dagger}c_y^i - c_x^{i\dagger}c_x^i)$$

Can a "rishon abacus" implemented with ultra-cold atoms be used as a quantum simulator?


Tr Up


det $U_{x,\mu}$

Optical lattice with ultra-cold alkaline-earth atoms (^{87}Sr or ^{173}Yb) with color encoded in nuclear spin



D. Banerjee, M. Bögli, M. Dalmonte, E. Rico, P. Stebler, UJW, P. Zoller, Phys. Rev. Lett. 110 (2013) 125303

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Some analog quantum simulator proposals

H. P. Büchler, M. Hermele, S. D. Huber, M. P. A. Fisher, P. Zoller, Phys. Rev. Lett. 95 (2005) 040402.
E. Zohar, B. Reznik, Phys. Rev. Lett. 107 (2011) 275301.
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D. Banerjee, M. Bögli, M. Dalmonte, E. Rico, P. Stebler, UJW, P. Zoller, Phys. Rev. Lett. 110 (2013) 125303.
V. Kasper, F. Hebenstreit, F. Jendrzejewski, M. K. Oberthaler, J. Berges, New J. Phys. 19 (2017) 023030; A. Mil et al., arXiv:1909.07641.

Some digital quantum simulator proposals

M. Müller, I. Lesanovsky, H. Weimer, H. P. Büchler, P. Zoller, Phys. Rev. Lett. 102 (2009) 170502; Nat. Phys. 6 (2010) 382.
L. Tagliacozzo, A. Celi, P. Orland, M. Lewenstein, Nature Communications 4 (2013) 2615.

Reviews on quantum simulators for lattice gauge theories

UJW, Annalen der Physik 525 (2013) 777, arXiv:1305.1602.
M. C. Banuls et al., arXiv:1911.00003.

Conclusions

• Quantum link models provide an alternative formulation of lattice gauge theory with a finite-dimensional Hilbert space per link, which allows implementations with ultra-cold atoms in optical lattices.

• Quantum simulator constructions have already been presented for Wilson's lattice gauge theory as well as for the $U(1)$ quantum link model with fermionic matter using ultra-cold Bose-Fermi mixtures. $\mathbb{C}P(N-1)$ models as well as non-Abelian $U(N)$ and $SU(N)$ quantum link models can be embodied by alkaline-earth atoms.

• This allows the quantum simulation of the real-time evolution of string breaking as well as false vacuum decay. Accessible effects also include chiral symmetry restoration at high baryon density or the expansion of a hot quark-gluon plasma.

• In quantum spin and quantum link models regularizing asymptotically free theories, including $(1+1)$-d $\mathbb{C}P(N-1)$ models and $(3+1)$-d QCD, the continuum limit is taken by dimensional reduction of discrete variables.

• The path towards quantum simulation of QCD will be a long one.

However, with a lot of interesting physics along the way.