EQS OF MOTION W. TERNING & VERHAAREN'S EQS. 2.4, 4.1, 4.2

Hello,

I calculated the equations of motion with the Lagrangians in Eqs. 2.4, 4.1, 4.2 of Terning-Verhaaren's, where I set $\theta = 0$ but I included the dark sector with kinetic mixing. I used $g = \frac{4\pi}{e^2}$ and $g_D = \frac{4\pi}{e_D^2}$.

The equations are given on p. 2 for A_μ, p. 4 for B_μ, p. 6 for A_μ^D and p. 7 for B_μ^D. They can be further simplified.

One purpose is to check the normalization. I am a little puzzled with Eq. 4.2, which does not contain g, g_D.

In order to compare with Terning-Verhaaren's equations of motion, Eqs. 2.5-2.7 without dark photons, I obtained the equations of motion for A_μ and B_μ on p. 3. I need to fiddle a little more with the ϵ tensors to see if the last terms are equal. If I have any luck, I will let you know.

Cheers,
Marc

$$g \equiv \frac{4\pi}{e^2}, \quad \theta \to 0 \qquad g_D \equiv \frac{4\pi}{e_D^2} \qquad F^X_{\mu\nu} \equiv \partial_\mu X_\nu - \partial_\nu X_\mu$$

$$\mathcal{L} = -\frac{n^\alpha n^\mu}{8\pi n^2} g^{\beta\nu}\left[g\left(F^A_{\alpha\beta}F^A_{\mu\nu} + F^B_{\alpha\beta}F^B_{\mu\nu}\right)\right] + \frac{n^\alpha n_\mu}{16\pi n^2}\epsilon^{\mu\nu\gamma\delta} g\left(F^B_{\alpha\nu}F^A_{\gamma\delta} - F^A_{\alpha\nu}F^B_{\gamma\delta}\right) - J_\mu A^\mu - g K_\mu B^\mu$$

$$-\frac{n^\alpha n^\mu}{8\pi n^2} g^{\beta\nu}\left[g_D\left(F^{A_D}_{\alpha\beta}F^{A_D}_{\mu\nu} + F^{B_D}_{\alpha\beta}F^{B_D}_{\mu\nu}\right)\right] + \frac{n^\alpha n_\mu}{16\pi n^2}\epsilon^{\mu\nu\gamma\delta} g_D\left(F^{B_D}_{\alpha\nu}F^{A_D}_{\gamma\delta} - F^{A_D}_{\alpha\nu}F^{B_D}_{\gamma\delta}\right) - e_D J^D_\mu A^\mu_D - g_D K^D_\mu B^\mu_D$$

$$\underbrace{+\frac{1}{2} m^2_{A_D} A_D^2 + \frac{1}{2} m^2_{B_D} B_D^2}_{\text{FROM HOOK-HUANG NOT IN TERNING}} \underbrace{+ \epsilon e_D e \frac{n^\alpha n^\mu}{n^2} g^{\beta\nu}}_{\text{TERNING EQ. (4.2) SHOULD IT READS } 2\epsilon e e_D \frac{n^\alpha n^\mu}{8\pi n^2} g^{\beta\nu} g X \ldots ?} \left(F^{A_D}_{\alpha\beta} F^A_{\mu\nu} F^{B_D}_{\alpha\beta} F^B_{\mu\nu}\right)$$

LINES 1,2 HAVE 6 TERMS
LINE 3 HAS 4 TERMS

EULER-EQ W/R A WE NEED ONLY L1T1, L1T3, L1T4, L3T3, L1T5

$$\frac{\partial \mathcal{L}_{L1T5}}{\partial A_\sigma} = \boxed{-J^\sigma}$$

$$\partial_\rho\left(\frac{\partial \mathcal{L}_{L1T1}}{\partial(\partial_\rho A_\sigma)}\right) = \partial_\rho \frac{\partial}{\partial(\partial_\rho A_\sigma)}\left(-\frac{n^\alpha n^\mu}{8\pi n^2} g^{\beta\nu} g\right) \overbrace{(\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial_\mu A_\nu - \partial_\nu A_\mu)}^{\partial_\alpha A_\beta \partial_\mu A_\nu - \partial_\alpha A_\beta \partial_\nu A_\mu - \partial_\beta A_\alpha \partial_\mu A_\nu + \partial_\beta A_\alpha \partial_\nu A_\mu}$$

$$= \partial_\rho \, -\frac{n^\alpha n^\mu}{8\pi n^2} g^{\beta\nu} g \left(\delta^\rho_\alpha \delta^\sigma_\beta \partial_\mu A_\nu + \delta^\rho_\mu \delta^\sigma_\nu \partial_\alpha A_\beta - \delta^\rho_\alpha \delta^\sigma_\beta \partial_\nu A_\mu - \delta^\rho_\nu \delta^\sigma_\mu \partial_\alpha A_\beta \right.$$
$$\left. - \delta^\rho_\beta \delta^\sigma_\alpha \partial_\mu A_\nu - \delta^\rho_\mu \delta^\sigma_\nu \partial_\beta A_\alpha + \delta^\rho_\beta \delta^\sigma_\alpha \partial_\nu A_\mu + \delta^\rho_\nu \delta^\sigma_\mu \partial_\beta A_\alpha\right)$$

$$= -\frac{g\,\partial_\rho}{8\pi n^2}\left(n^\rho n_\mu \partial^\mu A^\sigma + n^\rho n_\mu \partial^\mu A^\sigma - n^\rho n_\mu \partial^\sigma A^\mu - n_\mu n^\sigma \partial^\mu A^\rho \right.$$
$$\left. - n^\sigma n_\mu \partial^\mu A^\rho - n_\mu n^\rho \partial^\sigma A^\mu + n^\sigma n_\mu \partial^\rho A^\mu + n^\sigma n_\mu \partial^\rho A^\mu\right)$$

$$= -\frac{g}{8\pi n^2}\partial_\rho\left(2 n^\rho n_\mu F_A^{\mu\sigma} + 2 n^\sigma n_\mu F_A^{\rho\mu}\right) = \boxed{-\frac{g}{4\pi n^2} n_\mu \left(n^\rho \partial_\rho F_A^{\mu\sigma} + n^\sigma \partial_\rho F_A^{\rho\mu}\right)}$$

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$$\partial_\rho \frac{\partial \mathcal{L}_{LIT3/4}}{\partial(\partial_\rho A_\sigma)} = \partial_\rho \frac{\partial}{\partial(\partial_\rho A_\sigma)} \frac{n^\alpha n_\mu g}{16\pi M^2} \epsilon^{\mu\nu\gamma\delta}\left[(\partial_\gamma A_\delta - \partial_\delta A_\gamma) F^B_{\alpha\nu} - (\partial_\alpha A_\nu - \partial_\nu A_\alpha) F^B_{\gamma\delta}\right]$$

$$= \frac{g}{16\pi M^2}\partial_\rho\, n^\alpha n_\mu \epsilon^{\mu\nu\gamma\delta}\left(\delta^\rho_\gamma \delta^\sigma_\delta F^B_{\alpha\nu} - \delta^\rho_\delta \delta^\sigma_\gamma F^B_{\alpha\nu} - \delta^\rho_\alpha \delta^\sigma_\nu F^B_{\gamma\delta} + \delta^\rho_\nu \delta^\sigma_\alpha F^B_{\gamma\delta}\right)$$

$$= \frac{g n_\mu}{16\pi M^2}\partial_\rho \left(n^\alpha \epsilon^{\mu\nu\rho\sigma} F^B_{\alpha\nu} - n^\alpha \epsilon^{\mu\nu\sigma\rho} F^B_{\alpha\nu} - n^\rho \epsilon^{\mu\sigma\gamma\delta} F^B_{\gamma\delta} + n^\sigma \epsilon^{\mu\rho\gamma\delta} F^B_{\gamma\delta}\right)$$

$$= \boxed{\frac{g n_\mu}{16\pi M^2}\left[n^\alpha\left(\epsilon^{\mu\nu\rho\sigma} - \epsilon^{\mu\nu\sigma\rho}\right)\partial_\rho F^B_{\alpha\nu} - \left(n^\rho \epsilon^{\mu\sigma\gamma\delta} - n^\sigma \epsilon^{\mu\rho\gamma\delta}\right)\partial_\rho F^B_{\gamma\delta}\right]}$$

$$\partial_\rho \frac{\partial \mathcal{L}_{L3T3}}{\partial(\partial_\rho A_\sigma)} = \partial_\rho \frac{\partial}{\partial(\partial_\rho A_\sigma)} \epsilon e_D e \frac{n^\alpha n^\mu}{M^2} g^{\beta\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) F^{A_D}_{\alpha\beta}$$

$$= \frac{\epsilon e_D e}{M^2}\partial_\rho\, n^\alpha n^\mu g^{\beta\nu}\left(\delta^\rho_\mu \delta^\sigma_\nu - \delta^\rho_\nu \delta^\sigma_\mu\right) F^{A_D}_{\alpha\beta} = \frac{\epsilon e_D e}{M^2}\partial_\rho\left(n_\alpha n^\rho F_{A_D}^{\alpha\sigma} - n_\alpha n^\sigma F_{A_D}^{\alpha\rho}\right)$$

$$= \boxed{\frac{\epsilon e_D e\, n_\alpha}{M^2}\left(n^\rho \partial_\rho F_{A_D}^{\alpha\sigma} - n^\sigma \partial_\rho F_{A_D}^{\alpha\rho}\right)}$$

ALL THE TERMS: EL EQ FOR A FREE INDEX σ

$$\boxed{\begin{aligned}&\frac{g}{4\pi M^2} n_\mu \left(n^\rho \partial_\rho F_A^{\mu\sigma} + n^\sigma \partial_\rho F_A^{\rho\mu}\right) + \frac{g n_\mu}{16\pi M^2}\left[n^\alpha\left(\epsilon^{\mu\nu\sigma\rho} - \epsilon^{\mu\nu\rho\sigma}\right)\partial_\rho F^B_{\alpha\nu} + \left(n^\rho \epsilon^{\mu\sigma\gamma\delta} - n^\sigma \epsilon^{\mu\rho\gamma\delta}\right)\partial_\rho F^B_{\gamma\delta}\right] \\ &+ \frac{\epsilon e_D e}{M^2} n_\alpha \left(n^\sigma \partial_\rho F_{A_D}^{\alpha\rho} - n^\rho \partial_\rho F_{A_D}^{\alpha\sigma}\right) = J^\sigma\end{aligned}}$$

3

TERNING'S EQS. 2.5–2.7 $\quad \mathrm{Im}\,\tau = \frac{4\pi}{e^2} = g$ with $\theta = 0$ so that $\tau = ig$

EQ 2.5 $\quad \frac{g}{4\pi}\partial_\nu\left(F^{\mu\nu} + i\,{}^*F^{\mu\nu}\right) = J^\mu + igK^\mu$

$$\frac{g}{4\pi}\partial_\nu F^{\mu\nu} = J^\mu \qquad (1)$$

$$\frac{g}{4\pi}\partial_\nu {}^*F^{\mu\nu} = gK^\mu \qquad (2)$$

EQ 2.6 $\quad F_{\mu\nu} = \frac{n^\alpha}{n^2}\left(n_\mu F^A_{\alpha\nu} - n_\nu F^A_{\alpha\mu} - \epsilon_{\mu\nu\alpha}{}^\beta n^\gamma F^B_{\gamma\beta}\right)$

FROM (1) $\quad \boxed{\frac{g}{4\pi}\frac{n_\alpha}{n^2}\left(n^\mu \partial_\nu F_A^{\alpha\nu} - n^\nu \partial_\nu F_A^{\alpha\mu} - \epsilon^{\mu\nu\alpha}{}_\beta n_\gamma \partial_\nu F_B^{\gamma\beta}\right) = J^\mu}$

FROM (2) + EQ 2.7 $\quad \boxed{\frac{n_\alpha}{4\pi n^2}\left(n^\mu \partial_\nu F_B^{\alpha\nu} - n^\nu \partial_\nu F_B^{\alpha\mu} + \epsilon^{\mu\nu\alpha}{}_\beta n_\gamma F_A^{\gamma\beta}\right) = K^\mu}$

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EL EQ FOR B L1T2, L1T3 - L1T4, L1T6, L3T4

$$\frac{\partial \mathcal{L}_{L1T6}}{\partial B_\sigma} = \boxed{-g K^\sigma}$$

L1T2 SIMILAR TO L1T1 WITH $A \to B$, THUS

$$\boxed{\partial_\rho \frac{\partial \mathcal{L}_{L1T2}}{\partial(\partial_\rho B_\sigma)} = -\frac{g}{4\pi m^2} n_\mu \left(n^\rho \partial_\rho F_B^{\mu\sigma} + n^\sigma \partial_\rho F_B^{\rho\mu} \right)}$$

$$\partial_\rho \frac{\partial \mathcal{L}_{L1T34}}{\partial(\partial_\rho B_\sigma)} = \frac{g}{16\pi m^2} \partial_\rho \frac{\partial}{\partial(\partial_\rho B_\sigma)} n^\alpha n_\mu \epsilon^{\mu\nu\gamma\delta} \left[(\partial_\alpha B_\nu - \partial_\nu B_\alpha) F^A_{\gamma\delta} - (\partial_\gamma B_\delta - \partial_\delta B_\gamma) F^A_{\alpha\nu} \right]$$

$$= \frac{g}{16\pi m^2} \partial_\rho n^\alpha n_\mu \epsilon^{\mu\nu\gamma\delta} \left[\delta^\rho_\alpha \delta^\sigma_\nu F^A_{\gamma\delta} - \delta^\rho_\nu \delta^\sigma_\alpha F^A_{\gamma\delta} - \delta^\rho_\gamma \delta^\sigma_\delta F^A_{\alpha\nu} + \delta^\rho_\delta \delta^\sigma_\gamma F^A_{\alpha\nu} \right]$$

$$= \frac{g}{16\pi m^2} \partial_\rho \left(n^\rho n_\mu \epsilon^{\mu\sigma\gamma\delta} F^A_{\gamma\delta} - n^\sigma n_\mu \epsilon^{\mu\rho\gamma\delta} F^A_{\gamma\delta} - n^\alpha n_\mu \epsilon^{\mu\nu\rho\sigma} F^A_{\alpha\nu} + n^\alpha n_\mu \epsilon^{\mu\nu\sigma\rho} F^A_{\alpha\nu} \right)$$

$$= \boxed{\frac{g n_\mu}{16\pi m^2} \left[n^\alpha \left(\epsilon^{\mu\nu\sigma\rho} - \epsilon^{\mu\nu\rho\sigma} \right) \partial_\rho F^A_{\alpha\nu} + \left(n^\sigma \epsilon^{\mu\rho\gamma\delta} - n^\rho \epsilon^{\mu\sigma\gamma\delta} \right) \partial_\rho F^A_{\gamma\delta} \right]}$$

$$\partial_\rho \frac{\partial \mathcal{L}_{L3T4}}{\partial(\partial_\rho B_\sigma)} = \partial_\rho \frac{\partial}{\partial \partial_\rho b_\sigma} - \frac{\epsilon e_D e}{m^2} n^\alpha n^\mu g^{\beta\nu} (\partial_\mu B_\nu - \partial_\nu B_\mu) F^{B_D}_{\alpha\beta}$$

SAME AS L3T3 $\times -1$ AND $A \to B$

$$= \boxed{\frac{\epsilon e_D e}{m^2} n_\alpha \left(n^\sigma \partial_\rho F_{B_D}^{\alpha\rho} - n^\rho \partial_\rho F_{B_D}^{\alpha\sigma} \right)}$$

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SUMMARY: E-L EQ FOR B WITH FREE INDEX σ

$$\boxed{\begin{aligned}&\frac{g}{4\pi m^2} n_\mu \left(n^\rho \partial_\rho F_B^{\mu\sigma} + n^\sigma \partial_\rho F_B^{\rho\mu}\right) + \frac{g n_\mu}{16\pi m^2}\left[n^\alpha\left(\epsilon^{\mu\nu\rho\sigma} - \epsilon^{\mu\nu\sigma\rho}\right)\partial_\rho F^A_{\alpha\nu} + \left(n^\sigma \epsilon^{\mu\rho\gamma\delta} - n^\rho \epsilon^{\mu\sigma\gamma\delta}\right)\partial_\rho F^A_{\gamma\delta}\right] \\ &+ \frac{\epsilon e_D e}{m^2} n_\alpha \left(n^\rho \partial_\rho F_{B_D}^{\alpha\sigma} - n^\sigma \partial_\rho F_{B_D}^{\alpha\rho}\right) = g K^\sigma\end{aligned}}$$

TO BE COMPARED WITH LAST EQ. ON P. 3 (EXTRA F_{B_D} HERE...)

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EULER-LAGRANGE EQ. FOR A_D L2T1 L2T3-L2T4 L2T5 L3T1 L3T3

$$\frac{\partial \mathcal{L}_{L2T5-L3T1}}{\partial A^D_\sigma} = \boxed{-e_D J_D^\sigma + M_{A_D}^2 A_D^\sigma}$$

L2T1 SAME AS L1T1 WITH $A \to A_D, g \to g_D$: $\boxed{\partial_\rho \frac{\partial \mathcal{L}_{L2T1}}{\partial(\partial_\rho A^D_\sigma)} = -\frac{g_D}{4\pi m^2} n_\mu \left(n^\rho \partial_\rho F_{A_D}^{\mu\sigma} + \mu^\sigma \partial_\rho F_{A_D}^{\rho\mu}\right)}$

L2T3-4 SAME AS L1T3-4 WITH $A \to A_D$, $B \to B_D$, $g \to g_D$: (FROM p.2)

$$\boxed{\partial_\rho \frac{\partial \mathcal{L}_{L2T3-4}}{\partial(\partial_\rho A^D_\sigma)} = \frac{g_D n_\mu}{16\pi m^2}\left[n^\alpha\left(\epsilon^{\mu\nu\rho\sigma} - \epsilon^{\mu\nu\sigma\rho}\right)\partial_\rho F^{B_D}_{\alpha\nu} - \left(n^\rho \epsilon^{\mu\sigma\gamma\delta} - n^\sigma \epsilon^{\mu\rho\gamma\delta}\right)\partial_\rho F^{B_D}_{\gamma\delta}\right]}$$

$$\partial_\rho \frac{\partial \mathcal{L}_{L3T3}}{\partial(\partial_\rho A^D_\sigma)} = \partial_\rho \frac{\partial}{\partial \partial_\rho A^D_\sigma} + \frac{\epsilon e_D e}{m^2} n^\alpha n^\mu g^{\beta\nu} \left(\partial_\alpha A^D_\beta - \partial_\beta A^D_\alpha\right) F^A_{\mu\nu}$$

$$= \frac{\epsilon e_D e}{m^2} \partial_\rho n^\alpha n^\mu g^{\beta\nu} \left(\delta^\rho_\alpha \delta^\sigma_\beta - \delta^\rho_\beta \delta^\sigma_\alpha\right) F^A_{\mu\nu} = \frac{\epsilon e_D e}{m^2} \partial_\rho \left(n^\rho n^\mu F^{A\ \sigma}_{\mu} - n^\sigma n^\mu F^{A\ \rho}_{\mu}\right)$$

$$= \boxed{\frac{\epsilon e_D e}{m^2} n_\alpha \left(n^\rho \partial_\rho F_A^{\alpha\sigma} - n^\sigma \partial_\rho F_A^{\alpha\rho}\right)}$$ (i.e. SAME AS A WITH $A_D \to A$)

E-L FOR A_D:

$$\boxed{\begin{aligned} &\frac{g_D}{4\pi m^2} n_\mu \left(n^\rho \partial_\rho F_{A_D}^{\mu\sigma} + n^\sigma \partial_\rho F_{A_D}^{\rho\mu}\right) + \frac{g_D n_\mu}{16\pi m^2}\left[n^\alpha\left(\epsilon^{\mu\nu\sigma\rho} - \epsilon^{\mu\nu\rho\sigma}\right)\partial_\rho F^{B_D}_{\alpha\nu} + \left(n^\rho \epsilon^{\mu\sigma\gamma\delta} - n^\sigma \epsilon^{\mu\rho\gamma\delta}\right)\partial_\rho F^{B_D}_{\gamma\delta}\right] \\ &+ \frac{\epsilon e_D e}{m^2} n_\alpha \left(n^\sigma \partial_\rho F_A^{\alpha\rho} - n^\rho \partial_\rho F_A^{\alpha\sigma}\right) = e_D J_D^\sigma - M_{A_D}^2 A_D^\sigma \end{aligned}}$$

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E-L EQ. FOR B_D in L2T2, L2T3-4, L2T6, L3T2, L3T4

$$\frac{\partial \mathcal{L}_{L2T6-L3T2}}{\partial B^D_\sigma} = \boxed{-g_D K^\sigma_D + M^2_{B_D} B^\sigma_D}$$

L2T2 SAME AS L2T1 WITH $A_D \to B_D$:
$$\boxed{\partial_\rho \frac{\partial \mathcal{L}_{L2T2}}{\partial(\partial_\rho B^D_\sigma)} = -\frac{g_D}{4\pi m^2} n_\mu \left(n^\rho \partial_\rho F^{\mu\sigma}_{B_D} + n^\sigma \partial_\rho F^{\rho\mu}_{B_D}\right)}$$

L2T3-4 AS L1T34 WITH $A \to A_D$, $B \to B_D$, $g \to g_D$ (see p. 4)

$$\boxed{\partial_\rho \frac{\partial \mathcal{L}_{L2T34}}{\partial(\partial_\rho B^D_\sigma)} = \frac{g_D n_\mu}{16\pi m^2}\left[n^\alpha \left(\epsilon^{\mu\nu\sigma\rho} - \epsilon^{\mu\nu\rho\sigma}\right)\partial_\rho F^{A_D}_{\alpha\nu} - \left(n^\sigma \epsilon^{\mu\rho\gamma\delta} - n^\rho \epsilon^{\mu\sigma\gamma\delta}\right)\partial_\rho F^{A_D}_{\gamma\delta}\right]}$$

$$\partial_\rho \frac{\partial \mathcal{L}_{L3T4}}{\partial(\partial_\rho B^D_\sigma)} = \partial_\rho \frac{\partial}{\partial \partial_\rho B^D_\sigma} - \frac{\epsilon e_D e}{m^2} n^\alpha n^\mu g^{\beta\nu} F^B_{\mu\nu}\left(\partial_\alpha B^D_\beta - \partial_\beta B^D_\alpha\right)$$

$$= -\frac{\epsilon e_D e}{m^2} \partial_\rho n^\alpha n^\mu g^{\beta\nu} F^B_{\mu\nu}\left(\delta^\rho_\alpha \delta^\sigma_\beta - \delta^\rho_\beta \delta^\sigma_\alpha\right) = -\frac{\epsilon e_D e}{m^2} \partial_\rho\left(n^\rho n_\mu F^{\mu\sigma}_B - n^\sigma n_\mu F^{\mu\rho}_B\right)$$

$$= \boxed{\frac{\epsilon e_D e}{m^2} n_\mu \left(n^\sigma \partial_\rho F^{\mu\rho}_B - n^\rho \partial_\rho F^{\mu\sigma}_B\right)}$$

SAME AS L3T4 FOR B WITH $B_D \to B$ P.4

E-L FOR B_D :

$$\boxed{\begin{aligned}&\frac{g_D}{4\pi m^2} n_\mu \left(n^\rho \partial_\rho F^{\mu\sigma}_{B_D} + n^\sigma \partial_\rho F^{\rho\mu}_{B_D}\right) + \frac{g_D n_\mu}{16\pi m^2}\left[n^\alpha \left(\epsilon^{\mu\nu\rho\sigma} - \epsilon^{\mu\nu\sigma\rho}\right)\partial_\rho F^{A_D}_{\alpha\nu} + \left(n^\sigma \epsilon^{\mu\rho\gamma\delta} - n^\rho \epsilon^{\mu\sigma\gamma\delta}\right)\partial_\rho F^{A_D}_{\gamma\delta}\right] \\ &+ \frac{\epsilon e_D e}{m^2} n_\mu \left(n^\rho \partial_\rho F^{\mu\sigma}_B - n^\sigma \partial_\rho F^{\mu\rho}_B\right) = g_D K^\sigma_D - M^2_{B_D} B^\sigma_D\end{aligned}}$$