

## 31 JULY 2020 - PROGRESS REPORT

We checked the equations of motion with the Lagrangians in Eqs. (2.4), (4.1), (4.2) of Terning-Verhaaren's Ref. [1], with  $\theta = 0$  and including the dark sector with kinetic mixing. Mass terms are added. Notation  $g = \frac{4\pi}{e^2}$  and  $g_D = \frac{4\pi}{e_D^2}$ .

$$\begin{aligned}
 g &= \frac{4\pi}{e^2}, \theta \rightarrow 0 & g_D &= \frac{4\pi}{e_D^2} & F_{\mu\nu}^X &\equiv \partial_\mu X_\nu - \partial_\nu X_\mu \\
 \mathcal{L} &= -\frac{m^A m^\mu}{8\pi m^2} g^{\rho\nu} \left[ g \left( F_{\alpha\rho}^A F_{\mu\nu}^A + F_{\alpha\rho}^B F_{\mu\nu}^B \right) \right] + \frac{m^X m^\mu}{16\pi m^2} \epsilon^{\mu\nu\rho\delta} g \left( F_{\mu\nu}^B F_{\rho\delta}^A - F_{\mu\nu}^A F_{\rho\delta}^B \right) - J_\mu^\mu - g K_\mu^A B_\mu^A \\
 &- \frac{m^A m^\mu}{8\pi m^2} g^{\rho\nu} \left[ g_D \left( F_{\alpha\rho}^{A_D} F_{\mu\nu}^{A_D} + F_{\alpha\rho}^{B_D} F_{\mu\nu}^{B_D} \right) \right] + \frac{m^D m^\mu}{16\pi m^2} \epsilon^{\mu\nu\rho\delta} g_D \left( F_{\mu\nu}^{B_D} F_{\rho\delta}^{A_D} - F_{\mu\nu}^{A_D} F_{\rho\delta}^{B_D} \right) - e_D J_\mu^{A_D} - g_D K_\mu^D B_\mu^D \\
 &- \underbrace{\frac{1}{2} m_{A_D}^2 A_D^2 - \frac{1}{2} m_{B_D}^2 B_D^2}_{\text{From Hook-Huang}} + \underbrace{e e_D e \frac{m^A m^\mu}{m^2} g^{\rho\nu} \left( F_{\alpha\rho}^{A_D} F_{\mu\nu}^A - F_{\alpha\rho}^{B_D} F_{\mu\nu}^B \right)}_{\substack{\text{TERNING Eq.(4.2) should it} \\ \text{READS } 2e e_D \frac{m^A m^\mu}{8\pi m^2} g^{\rho\nu} g^{X\mu} ?}}
 \end{aligned}$$

Lines 1,2 have 6 terms  
 Line 3 has 4 terms

Equations of motion -for the visible sector only- are in Eqs. (2.5)-(2.7) of Ref. [1] and reduce to

$$\boxed{\frac{g}{4\pi} \frac{m_A}{m^2} \left( m^\mu \partial_\nu F_A^{\alpha\nu} - m^\nu \partial_\nu F_A^{\alpha\mu} - \epsilon^{\mu\nu\alpha} {}_\beta {}^\gamma \partial_\nu F_B^{\gamma\beta} \right) = J^\mu}$$

$$\boxed{\frac{m_A}{4\pi m^2} \left( m^\mu \partial_\nu F_B^{\alpha\nu} - m^\nu \partial_\nu F_B^{\alpha\mu} + \epsilon^{\mu\nu\alpha} {}_\beta {}^\gamma \partial_\nu F_A^{\gamma\beta} \right) = K^\mu}$$

We wanted to check that these are the same as the equation of motion from the Lagrangian -with visible sector only- with respect to the visible photon  $A_\mu$  (free index  $\sigma$ ):

$$\begin{aligned} & \frac{g}{4\pi n^2} M_\mu (m^\sigma)_{\beta} F_A^{\mu\sigma} + m^\sigma)_{\beta} F_A^{\sigma\mu}) + \frac{g n_\mu}{16\pi n^2} [n^\alpha (\epsilon^{\mu\nu\delta\sigma} - \epsilon^{\mu\nu\sigma\delta}) \partial_\nu F_{\alpha\delta}^\sigma + (m^\delta \epsilon^{\mu\sigma\delta\sigma} - m^\sigma \epsilon^{\mu\delta\delta\sigma}) \partial_\nu F_{\alpha\delta}^\sigma] \\ & + \frac{e e_D e}{n^2} M_\alpha (m^\sigma)_{\beta} F_A^{\mu\delta} - m^\sigma)_{\beta} F_A^{\mu\delta}) = J^\mu \end{aligned}$$

The last line is irrelevant because we check the equations of motion in the visible sector. In the second bracket of line 1: the first two terms are equal, the last term in  $n^\sigma$  is equal to zero when expressed in terms of the potentials  $A$ , and the term  $n^\sigma$  remains. These two terms are each equal to 1/2 of the following term, so that the equations of motion w/r  $A$  are satisfied. Details are in Marc30Jl2020.pdf attached at p.4.

Note that one can reduce the last term of p.3 Eq 1,  $- \frac{g n_\mu n^\nu}{4\pi n^2} \epsilon^{\mu\nu\delta\sigma} \partial_\nu F_{\alpha\delta}^\sigma$ , by noting

$$\epsilon^{\mu\nu\delta\sigma} (\partial_\nu \partial_\delta B_\sigma - \underbrace{\partial_\nu \partial_\sigma B_\delta}_{\text{symm.} \neq 0}) = \epsilon^{\mu\nu\delta\sigma} \partial_\nu \partial_\delta B_\sigma, \text{ TO}$$

$$-\frac{g n_\mu n^\nu}{4\pi n^2} \epsilon^{\mu\nu\delta\sigma} \partial_\nu \partial_\delta B_\sigma$$

## 1. NEXT STEPS

- Clean up the full equations of motion, with the dark sector, for  $A, B, A_D, B_D$ .
- Write out the interaction terms  $-JA - gKB - e_D J_D A_D - g_D K_D B_D$  in terms of the diagonal basis given in the following equations of Ref. [1]:

$$\begin{pmatrix} A_\mu \\ A_{D\mu} \end{pmatrix} = \begin{pmatrix} \cos \phi + \epsilon e e_D \sin \phi & -\sin \phi + \epsilon e e_D \cos \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} \bar{A}_\mu \\ \bar{A}_{D\mu} \end{pmatrix}, \quad (4.4)$$

$$\begin{pmatrix} B_\mu \\ B_{D\mu} \end{pmatrix} = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi - \epsilon e e_D \cos \phi & \cos \phi + \epsilon e e_D \sin \phi \end{pmatrix} \begin{pmatrix} \bar{B}_\mu \\ \bar{B}_{D\mu} \end{pmatrix}. \quad (4.5)$$

$$\begin{pmatrix} e\bar{J}_\mu \\ e_D\bar{J}_{D\mu} \end{pmatrix} = \begin{pmatrix} \cos\phi + \epsilon ee_D \sin\phi & \sin\phi \\ -\sin\phi + \epsilon ee_D \cos\phi & \cos\phi \end{pmatrix} \begin{pmatrix} eJ_\mu \\ e_D J_{D\mu} \end{pmatrix}, \quad (4.6)$$

$$\begin{pmatrix} \bar{K}_\mu/e \\ \bar{K}_{D\mu}/e_D \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi - \epsilon ee_D \cos\phi \\ -\sin\phi & \cos\phi + \epsilon ee_D \sin\phi \end{pmatrix} \begin{pmatrix} K_\mu/e \\ K_{D\mu}/e_D \end{pmatrix}. \quad (4.7)$$

- Obtain the corresponding Feynman rules involving  $A$ ,  $A_D$ ,  $B$ ,  $B_D$ , and the fermion fields  $\Psi$ ,  $\Psi_D$ .
- Verify the gauge fields propagator, which should look like Eq. (4.56) from Ref. [2] (with  $B$  denoted by  $\tilde{A}$  below)

$$\begin{aligned} D_{AA}^{\mu\nu}(k) &= D_{\tilde{A}\tilde{A}}^{\mu\nu}(k) = i \left[ -\eta^{\mu\nu} + \frac{k^\mu n^\nu - n^\mu k^\nu}{n \cdot k} \right] \frac{1}{k^2 + i\varepsilon}, \\ D_{A\tilde{A}}^{\mu\nu}(k) &= D_{\tilde{A}A}^{\mu\nu}(k) = i\varepsilon^{\mu\nu\rho\sigma} \frac{n_\rho k_\sigma}{n \cdot k} \frac{1}{k^2 + i\varepsilon}. \end{aligned} \quad (4.56)$$

- We need to address the issue of normalization and renormalization in Section 3 of Ref. [1]. That this issue is non-trivial is discussed in Shnir's book:

The procedure of renormalization of electric and magnetic charges was the subject of intensive discussions for a long time (see, for example, the reviews [41,45]). According to the conclusion reached by Schwinger [459], and supported in [191,419], both electric and magnetic charges are renormalized in a similar way, that is

$$e_r^2 = Z_e e_0^2, \quad g_r^2 = Z_g g_0^2, \quad Z_e = Z_g < 1. \quad (4.59)$$

By contrast, Coleman [43] and other authors [158,242,496] concluded that

$$Z_e = Z_g^{-1}, \quad (4.60)$$

that is, when the energy increases, the effect of vacuum fluctuations lead not only to the standard electric charge screening, but simultaneously to the effect of anti-screening of the magnetic charge.

## REFERENCES

- [1] J. Terning, C. B. Verhaaren, JHEP 12 (2018) 123
- [2] Ya. Shnir, Magnetic Monopoles (Springer, 2005)

**30 JULY 2020: EQS OF MOTION W. TERNING & VERHAAREN'S  
EQS. 2.4, 4.1, 4.2**

Hello,

I calculated the equations of motion with the Lagrangians in Eqs. 2.4, 4.1, 4.2 of Terning-Verhaaren's, where I set  $\theta = 0$  but I included the dark sector with kinetic mixing. I used  $g = \frac{4\pi}{e^2}$  and  $g_D = \frac{4\pi}{e_D^2}$ .

The equations are given on p. 2 for  $A_\mu$ , p. 4 for  $B_\mu$ , p. 6 for  $A_\mu^D$  and p. 7 for  $B_\mu^D$ . They can be further simplified.

One purpose is to check the normalization. I am a little puzzled with Eq. 4.2, which does not contain  $g$ ,  $g_D$ .

In order to compare with Terning-Verhaaren's equations of motion, Eqs. 2.5-2.7 without dark photons, I obtained the equations of motion for  $A_\mu$  and  $B_\mu$  on p. 3. Except for a global sign, the equation of motion for  $A_\mu$  p. 3 coincides with equation -obtained from the Lagrangian in Eq. 2.4- at the bottom of p.2. Note (see p.8) that the last term of the  $A_\mu$  equation on p. 3 reduces to

$$(1) \quad -\frac{g}{4\pi n^2} n_\alpha n^\gamma \epsilon^{\mu\nu\alpha\beta} \partial_\nu \partial_\gamma B_\beta,$$

because the contraction of  $\epsilon^{\mu\nu\alpha\beta}$  with  $\partial_\nu \partial_\beta B_\gamma$  eliminates that last term. This should be equal to p.2, second bracket of line 1. On p. 8,

- (a) the first term  $\propto (\epsilon^{\mu\nu\sigma\rho} - \epsilon^{\mu\nu\rho\sigma})$  is equal to  $-1/2 \times$ Eq. (1);
- (b) the second term  $\propto n^\rho \epsilon^{\mu\sigma\gamma\delta} \partial_\rho F_{\gamma\delta}$  also seems to be to  $-1/2 \times$ Eq. (1), and
- (c) finally the last term,  $\propto n^\sigma \epsilon^{\mu\rho\gamma\delta} \partial_\rho F_{\gamma\delta} = 0$  because of the symmetry in the derivatives of  $A$ .

Cheers,  
Marc

$$\begin{aligned}
 g &\equiv \frac{4\pi}{e^2}, \quad \theta \rightarrow 0 \quad g_B \equiv \frac{4\pi}{e^2} \\
 \mathcal{L} &= -\frac{m^2 m^{\mu}}{8\pi m^2} g^{\mu\nu} \left[ g(F_{AB} F^A_{\mu\nu} + F^B F_{\mu\nu}) \right] + \frac{m^2 m^{\mu}}{16\pi m^2} e^{\mu\nu\delta} g \left( F^B_{\mu\nu} F^A_{\delta\sigma} - F^A_{\mu\nu} F^B_{\delta\sigma} \right) - J_{\mu} A^{\mu} - g K_{\mu} B^{\mu} \\
 &\quad - \frac{mc^2 n^{\mu}}{8\pi m^2} g^{\mu\nu} \left[ g_B (F^{AB} F_{AB} + F^{B\mu} F_{\mu\nu}) \right] + \frac{mc^2 n^{\mu}}{16\pi m^2} e^{\mu\nu\delta} g_B \left( F^B_{\mu\nu} F^{AB}_{AB} - F^{AB}_{\mu\nu} F^B_{AB} \right) - e_B J^{\mu} A^{\mu} - g_B K^{\mu} B^{\mu} \\
 &\quad + \frac{1}{2} m_B^2 A_B^2 + \frac{1}{2} m_B^2 B_B^2 + \cancel{e_B e_m n^{\mu} g_B} \left( F^{AB} F_{AB} \cancel{F^B_{\mu\nu} F^B_{\mu\nu}} \right) \\
 &\quad \underbrace{\text{From floor-plans}}_{\text{not in Terning}} \quad \text{Terning Eq.(4.2) should read} \\
 &\quad \text{READS } \cancel{e_B e_m n^{\mu} g_B} \text{?} \quad \text{Line 1,2 have 6 terms} \\
 &\quad \frac{mc^2 n^{\mu}}{8\pi m^2} g^{\mu\nu} g_{\nu\lambda}?
 \end{aligned}$$

Eqn. Eq with A we need once CIT1, CIT3, CIT4, CIT3, CIT5

$$\frac{\partial L_{\text{vars}}}{\partial A^{\sigma}} = [-J^{\sigma}]$$

$$g \left( \frac{\partial L_{\text{vars}}}{\partial (\partial^{\mu} A^{\sigma})} \right) = \frac{\partial}{\partial g^{\mu\nu}} \left( \frac{-mc^2 n^{\mu} n^{\nu}}{8\pi m^2} g^{\mu\nu} \right) \left( \partial_{\mu} A_B - \partial_B A_{\mu} \right) \left( \partial_{\nu} A_B - \partial_B A_{\nu} \right)$$

$$\begin{aligned}
 &= g_B - \frac{mc^2 n^{\mu} n^{\nu}}{8\pi m^2} g \left( \delta_{AB}^{\mu\nu} \partial_{\mu} A_B + \delta_{AB}^{\nu\mu} \partial_{\nu} A_B - \delta_{\mu\nu}^{\mu\nu} \partial_{\mu} A_B - \delta_{\mu\nu}^{\nu\mu} \partial_{\nu} A_B \right. \\
 &\quad \left. - \delta_{AB}^{\mu\mu} \partial_{\mu} A_B - \delta_{AB}^{\nu\nu} \partial_{\nu} A_B + \delta_{\mu\nu}^{\mu\nu} \partial_{\mu} A_B + \delta_{\mu\nu}^{\nu\mu} \partial_{\nu} A_B \right) \\
 &= -\frac{\partial g_B}{\partial A^{\sigma}} \left( \cancel{mc^2 n^{\mu} n^{\nu}} A^{\sigma} - m^2 m^{\mu} n^{\nu} \cancel{A^{\sigma}} - m^2 m^{\nu} n^{\mu} \cancel{A^{\sigma}} \right. \\
 &\quad \left. - m^2 m^{\mu} n^{\nu} \cancel{A^{\sigma}} - m^2 m^{\nu} n^{\mu} \cancel{A^{\sigma}} + m^2 m^{\mu} n^{\nu} \cancel{A^{\sigma}} \right) \\
 &= -\frac{g}{8\pi m^2} \left( 2n^{\mu} n^{\nu} F^{\sigma\mu}_{\sigma} + 2n^{\sigma} n^{\mu} F^{\nu\mu}_{\sigma} \right) = -\frac{g}{4\pi m^2} \left[ m^2 \cancel{n^{\mu} F^{\sigma\mu}} + m^2 \cancel{n^{\sigma} F^{\nu\mu}} \right]
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial f_{1134}}{\partial (\partial_S A_\sigma)} &= \frac{\partial_S \frac{\partial}{\partial (\partial_S A_\sigma)}}{\frac{\partial (\partial_S A_\sigma)}{\partial m^2}} \frac{m^2 g e m^{05} \left[ (\partial_\mu A_\delta - \partial_\mu A_\delta) F_{\alpha\beta}^\beta - (\partial_\nu A_\delta - \partial_\nu A_\delta) F_{\delta\beta}^\beta \right]}{16\pi m^2} \\
 &= \frac{\partial}{16\pi m^2} \partial_S m^2 g e m^{05} \left( \delta_{\mu\alpha}^{\nu\delta} F_{\alpha\beta}^\beta - \delta_{\mu\beta}^{\nu\delta} F_{\alpha\delta}^\beta - \delta_{\mu\delta}^{\nu\beta} F_{\alpha\beta}^\beta + \delta_{\mu\beta}^{\nu\delta} F_{\delta\beta}^\beta \right) \\
 &= \frac{\partial m^2}{16\pi m^2} \partial_S \left( m^2 e m^{05} F_{\alpha\beta}^\beta - m^2 e m^{05} F_{\alpha\delta}^\beta - m^2 e m^{05} F_{\delta\beta}^\beta + m^2 e m^{05} F_{\delta\delta}^\beta \right) \\
 &= \boxed{\frac{g m^2}{16\pi m^2} \left[ m^2 (e m^{05} - e m^{05}) \partial_S F_{\alpha\beta}^\beta - (m^2 e m^{05} - m^2 e m^{05}) \partial_S F_{\delta\beta}^\beta \right]}
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial f_{1234}}{\partial (\partial_S A_\sigma)} &= \frac{\partial_S \frac{\partial}{\partial (\partial_S A_\sigma)}}{\frac{\partial (\partial_S A_\sigma)}{\partial m^2}} \frac{e e g e m^2 m^2 g^2 (\partial_\mu A_\nu - \partial_\nu A_\mu) F_{\alpha\beta}^\alpha F_{\delta\beta}^\delta}{m^2} \\
 &= \frac{e e g e}{m^2} \partial_S m^2 g^2 (\delta_{\mu\alpha}^{\nu\delta} - \delta_{\mu\delta}^{\nu\alpha}) F_{\alpha\beta}^\alpha F_{\delta\beta}^\delta = \frac{e e g e}{m^2} \partial_S \left( m^2 g^2 (m^2 g^2 F_{\alpha\beta}^\alpha F_{\delta\beta}^\delta - m^2 g^2 F_{\alpha\delta}^\alpha F_{\delta\beta}^\beta) \right) \\
 &= \boxed{\frac{e e g e m^2}{m^2} (m^2 g^2 F_{\alpha\beta}^\alpha F_{\delta\beta}^\delta - m^2 g^2 F_{\alpha\delta}^\alpha F_{\delta\beta}^\beta)}
 \end{aligned}$$

THE TERMS: E L E A FOR A FREE INDEX  $\sigma'$

$$\begin{aligned}
 &\frac{g}{4\pi m^2} m (m^2 \partial_S F_A^{\mu\nu} + m^2 \partial_S F_A^{\mu\nu}) F_A^{\delta\mu} + \frac{g m e}{16\pi m^2} \left[ m^2 (e m^{05} g^2 - e m^{05} g^2) \partial_S F_{\alpha\beta}^\beta + (m^2 e m^{05} g^2 - m^2 e m^{05} g^2) \partial_S F_{\delta\beta}^\beta \right] \\
 &+ \frac{e e g e}{m^2} m^2 (m^2 g^2 F_A^{\alpha\beta} - m^2 g^2 F_A^{\alpha\beta}) = J^{\alpha'}
 \end{aligned}$$

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Terning's Eq. 2.5-2.7

$$\text{Eq. 2.5} \quad \frac{g}{4\pi} \partial_\nu (F^{\mu\nu} + i^* F^{\mu\nu}) = J^\mu + ig K^\mu \quad \boxed{\frac{g}{4\pi} \partial_\nu F^{\mu\nu} = J^\mu} \quad (1)$$

$$\frac{g}{4\pi} \partial_\nu (F^{\mu\nu} + i^* F^{\mu\nu}) = J^\mu + ig K^\mu \quad \boxed{\frac{g}{4\pi} \partial_\nu F^{\mu\nu} = g K^\mu} \quad (2)$$

$$\text{Eq. 2.6} \quad F_{\mu\nu} = \frac{m^A}{m^2} (m_A F^A_{\mu\nu} - m_B F^B_{\mu\nu} - \epsilon_{\mu\nu\lambda} \beta_{AB} F^B_\lambda)$$

$$\text{From (1)} \quad \boxed{\frac{g}{4\pi} \frac{m_A}{m^2} (m^A \partial_\nu F^{A\nu} - m^B \partial_\nu F^{B\nu} - \epsilon^{\mu\nu\lambda} \beta_{AB} F^B_\lambda) = J^\mu}$$

$$\text{From (2)} \quad \boxed{\frac{m_A}{4\pi m^2} (m^A \partial_\nu F^{A\nu} - m^B \partial_\nu F^{B\nu} + \epsilon^{\mu\nu\lambda} \beta_{AB} F^B_\lambda) = K^\mu}$$

EQUATION FOR B

LIT2, LIT3-LIT4, LIT6, LIT4

$$\frac{\partial \mathcal{L}_{LIT6}}{\partial \dot{\theta}_\alpha} = \begin{bmatrix} -g K^2 \end{bmatrix}$$

LIT2 similar to LIT1 with A  $\rightarrow$  B, thus

$$\boxed{\partial_S \frac{\partial \mathcal{L}_{LIT2}}{\partial (\partial_S \theta_\alpha)} = -\frac{g}{4\pi m^2} n^\sigma \mu_\alpha (\epsilon^{\mu\nu\sigma} F_{B\beta}^A + m^\sigma \partial_S F_{B\beta}^A)}$$

$$\begin{aligned} \partial_S \frac{\partial \mathcal{L}_{LIT34}}{\partial (\partial_S \theta_\alpha)} &= \frac{g}{16\pi m^2} \partial_S \frac{\partial}{\partial (\partial_S \theta_\alpha)} n^\sigma \mu_\alpha \epsilon^{\mu\nu\sigma} \left[ (\partial_\alpha \theta_\beta - \partial_\beta \theta_\alpha) F_{B\beta}^A - (\partial_\beta \theta_\beta - \partial_\beta \theta_\alpha) F_{B\alpha}^A \right] \\ &= \frac{g}{16\pi m^2} \partial_S n^\sigma \mu_\alpha \epsilon^{\mu\nu\sigma} \left[ \delta_\alpha^\beta \delta_\nu^\sigma F_{B\beta}^A - \delta_\beta^\beta \delta_\nu^\sigma F_{B\alpha}^A - \delta_\beta^\beta \delta_\nu^\sigma F_{B\beta}^A + \delta_\beta^\beta \delta_\nu^\sigma F_{B\alpha}^A \right] \\ &= \frac{g}{16\pi m^2} \partial_S (n^\sigma \mu_\alpha \epsilon^{\mu\nu\sigma} F_{B\beta}^A - n^\sigma \mu_\alpha \epsilon^{\mu\nu\sigma} F_{B\alpha}^A - n^\sigma \mu_\beta \epsilon^{\mu\nu\sigma} F_{B\beta}^A + n^\sigma \mu_\beta \epsilon^{\mu\nu\sigma} F_{B\alpha}^A) \\ &= \frac{g n_\mu}{16\pi m^2} \left[ n^\sigma (\epsilon^{\mu\nu\sigma} - \epsilon^{\mu\nu\sigma}) F_{B\alpha}^A - (n^\sigma \epsilon^{\mu\nu\sigma} - n^\sigma \epsilon^{\mu\nu\sigma}) F_{B\beta}^A \right] \end{aligned}$$

same as LIT3  $\times -1$  and A  $\rightarrow$  B

$$\boxed{\partial_S \frac{\partial \mathcal{L}_{LIT6}}{\partial (\partial_S \theta_\alpha)} = \partial_S \frac{2}{\partial \dot{\theta}_\alpha} - \frac{e e_B e}{m^2} n^\sigma \mu_\alpha g_B (\partial_\mu \theta_\beta - \partial_\beta \theta_\mu) F_{B\beta}^B}$$

$$\boxed{= \frac{e e_B e}{m^2} n_\alpha (n^\sigma \mu_\beta F_{B\beta}^B - n^\sigma \mu_\beta F_{B\alpha}^B)}$$

Summary: Eq 2 Eq for  $B$  with free index  $\sigma$

$$\frac{g}{4\pi m^2} n (m^\sigma g_B^{\mu\nu} + m^\nu g_B^{\mu\sigma}) + \frac{gn}{16\pi m^2} [n^\alpha (\epsilon^{\mu\nu\sigma\tau} - \epsilon^{\mu\sigma\nu\tau}) g_F^A + (n^\mu\epsilon^{\nu\sigma\tau} - n^\nu\epsilon^{\mu\sigma\tau}) g_F^A]$$

$$+ \frac{e e n}{m^2} (m^\sigma g_{B\mu}^{\lambda\tau} - m^\lambda g_{B\mu}^{\sigma\tau}) = g_K^{\lambda\tau}$$

To be compared with eq. on p. 3 (extra  $F_B$  here...)

6

$$\text{Equilibrium Eq. for } A_B \quad \frac{\partial \mathcal{L}_{2+2+3-4}}{\partial A_B^\sigma} = \frac{\partial \mathcal{L}_{2+1}}{\partial A_B^\sigma} - \frac{\partial \mathcal{L}_{2+3-4}}{\partial A_B^\sigma} - \frac{\partial \mathcal{L}_{2+5}}{\partial A_B^\sigma} + \frac{\partial \mathcal{L}_{3+3}}{\partial A_B^\sigma}$$

$$\frac{\partial \mathcal{L}_{2+2+3-4}}{\partial A_B^\sigma} = -e_B \delta^{\sigma}_\sigma + m_B^2 A_B^\sigma$$

$$\text{L2+1 same as L1+1 with } A \rightarrow A_B, g \rightarrow g_B : \boxed{\frac{\partial \mathcal{L}_{2+1}}{\partial (\partial_S A_B^\sigma)} = -\frac{g_D}{4\pi m^2} \mu (m^2 \delta_{AB}^{\mu\nu} + \mu^0 \delta_{AB}^{\nu\mu})}.$$

L2+3-4 same as L1+3-4 with  $A \rightarrow A_B, B \rightarrow B_B, g \rightarrow g_B$ : (from p.2)

$$\boxed{\frac{\partial \mathcal{L}_{2+3-4}}{\partial (\partial_S A_B^\sigma)} = \frac{g_D m}{16\pi m^2} [m (e^{m\sigma} - e^{m\sigma}) \delta F_{\sigma\nu}^B - (m e^{m\sigma} - m e^{m\sigma}) \delta F_{\sigma\nu}^B]}.$$

$$\begin{aligned} \frac{\partial \mathcal{L}_{2+3}}{\partial (\partial_S A_B^\sigma)} &= \frac{\partial}{\partial \partial_S A_B^\sigma} + \text{cose } m \text{ and } g_B^\sigma (2_A A_B^\sigma - \delta_B^\sigma A_A^\sigma) F_{\mu\nu}^A \\ &= \frac{\text{cose }}{m^2} \delta_S m \text{ and } g_B^\sigma (\delta_A^\sigma \delta_B^\nu - \delta_B^\sigma \delta_A^\nu) F_{\mu\nu}^A = \frac{\text{cose }}{m^2} \delta_S (m \delta^{\mu\nu} F_\mu^\sigma - m \delta^{\nu\mu} F_\mu^\sigma) \\ &\equiv \frac{\text{cose } m}{m^2} (m \delta^{\mu\nu} - m \delta^{\nu\mu}) \quad (\text{i.e. same as 4 with } A_B \rightarrow A) \end{aligned}$$

$$\begin{aligned} \text{E-L for } A_B : \quad & \left[ \frac{g_D}{4\pi m^2} \mu (m^2 \delta_{AB}^{\mu\nu} + m^2 \delta_{AB}^{\nu\mu}) \delta F_{\mu\nu}^B + \frac{g_D m}{16\pi m^2} [m^2 (e^{m\sigma} - e^{m\sigma}) \delta F_{\mu\nu}^B + (m e^{m\sigma} - m e^{m\sigma}) \delta F_{\mu\nu}^B] \right. \\ & \left. + \frac{\text{cose } m}{m^2} (m^2 \delta_{AB}^{\mu\nu} - m^2 \delta_{AB}^{\nu\mu}) \right] = e_B \delta^{\sigma}_\sigma - m_B^2 A_B^\sigma \end{aligned}$$

$\mathcal{E}-L$  for  $B_D$  in L272, L273-4, L276, L372, L374

$$\frac{\partial L_{276-1372}}{\partial B_D^\alpha} = -g_D K_D^\sigma + m^2 B_D^\sigma$$

L272 same as L271 with  $A \rightarrow B$ :

$$g_S \frac{\partial L_{272}}{\partial (B_S^\alpha)} = -\frac{g_D}{16\pi m^2} m (m^2 g_B^{\mu\nu} + m^2 g_B^{\sigma\tau}) F_B^{\sigma\mu}$$

L273-4 as L134 with  $A \rightarrow B$ ,  $B \rightarrow B_D$ ,  $g \rightarrow g_S$  (see p. 4)

$$g_S \frac{\partial L_{2734}}{\partial (B_S^\alpha)} = \frac{g_D m}{16\pi m^2} \left[ m (\epsilon^{\mu\nu\sigma} \epsilon^{\nu\rho\sigma}) \partial_S F_{\alpha\rho}^{AD} - (m^2 \epsilon^{\mu\nu\sigma} - m^2 \epsilon^{\mu\nu\rho}) \partial_S F_{\alpha\sigma}^{AB} \right]$$

$$\begin{aligned} g_S \frac{\partial L_{1374}}{\partial (B_S^\alpha)} &= \partial_S \frac{\partial}{\partial S B_D^\beta} - \frac{e c e}{m^2} m \epsilon^{\mu\nu\sigma} B^\nu F_{\mu\nu}^B (\partial_S B_D^\beta - \partial_S B_\alpha) \\ &= -\frac{e c e}{m^2} \partial_S m \epsilon^{\mu\nu\sigma} B^\nu F_{\mu\nu}^B (\delta_{\alpha}^{\beta} \delta_{\mu}^{\sigma} - \delta_{\alpha}^{\sigma} \delta_{\mu}^{\beta}) = -\frac{e c e}{m^2} \partial_S (m \epsilon^{\mu\nu\sigma} B_\mu - m \epsilon^{\mu\nu\sigma} F_B^{\mu\sigma}) \\ &= \frac{e c e}{m^2} \epsilon_{\mu\nu\sigma} (m^2 \partial_S B^\mu - m^2 \partial_S F_B^{\mu\sigma}) \end{aligned}$$

Same as L374 for  $B$  with  $B_D \rightarrow B$  p. 4

$\mathcal{E}-L$  for  $B_D$ :

$$\begin{aligned} \frac{g_D}{4\pi m^2} m_\mu (m^2 g_B^{\mu\nu} + m^2 g_B^{\sigma\tau}) F_B^{\sigma\mu} &+ \frac{g_D m_\mu}{16\pi m^2} \left[ m^2 (e^{\mu\nu\sigma} - e^{\mu\nu\tau}) \partial_S F_{\sigma\tau}^{AD} + (m^2 \epsilon^{\mu\nu\sigma} - m^2 \epsilon^{\mu\nu\tau}) \partial_S F_{\sigma\tau}^{AB} \right] \\ &+ \frac{e c e m}{m^2} (m^2 g_B^{\mu\nu} - m^2 g_B^{\sigma\tau}) F_B^{\mu\sigma} = g_D K_D^\sigma - m^2 B_D^\sigma \end{aligned}$$

$\delta$ 

0 Note that one can reduce the last term of p.3 Eq. 1,  $-\frac{g_{\mu\nu}}{4\pi m^2} \epsilon^{\mu\nu\alpha\beta} \partial_\nu F_{\alpha\beta}^B$ , by noting

$$\epsilon^{\mu\nu\alpha\beta} (\partial_\nu \gamma_B - \partial_\nu \gamma_B) = \epsilon^{\mu\nu\alpha\beta} \partial_\nu \gamma_B, \text{ to } \boxed{\frac{-g_{\mu\nu}}{4\pi m^2} \epsilon^{\mu\nu\alpha\beta} \partial_\nu \gamma_B} \quad (1)$$

symm.  $\Rightarrow 0$

1 Now consider, bottom of p.2  $\frac{g_{\mu\nu}}{8\pi m^2} \epsilon^{\mu\nu\alpha\beta} \partial_\beta F_{\alpha\gamma}^B = \frac{g_{\mu\nu}}{8\pi m^2} \epsilon^{\mu\nu\alpha\beta} \partial_\beta \underbrace{\partial_\alpha \gamma_B}_{\partial_\alpha \gamma_B - \partial_\alpha \gamma_B} \underbrace{\partial_\beta \gamma_B}_{\partial_\beta \gamma_B - \partial_\beta \gamma_B}$

Rename ( $\mu \rightarrow \delta, \nu \rightarrow \beta, \alpha \rightarrow \mu, \beta \rightarrow \nu$ ) and use  $\epsilon^{\mu\nu\alpha\beta} = +\epsilon^{\mu\nu\beta\alpha}$ ,  $\frac{g_{\mu\nu}}{8\pi m^2} \epsilon^{\mu\nu\alpha\beta} \partial_\beta \gamma_B = \frac{g_{\mu\nu}}{8\pi m^2} \epsilon^{\mu\nu\beta\alpha} \partial_\beta \gamma_B$   
 which is  $\boxed{-\frac{1}{2} \epsilon^{\delta\beta\mu\alpha}}$

2 THE LAST TERM OF P.2 line 1 &  $m^2 \epsilon^{\mu\nu\delta\beta} \partial_\beta F_{\delta\delta}^B = m^2 \epsilon^{\mu\nu\delta\beta} (\sum_{\alpha} \partial_\beta \gamma_\delta - \sum_{\beta} \partial_\delta \gamma_\beta) = \boxed{0}$

3 The remaining term is  $\frac{g_{\mu\nu}}{16\pi m^2} \epsilon^{\mu\nu\delta\beta} F_{\delta\beta}^B$ , which we hope is  $-\frac{1}{2} \times \text{Eq. (1)}$ .  
 Rename ( $\mu \rightarrow \nu, \nu \rightarrow \delta, \delta \rightarrow \beta$ ) and observe that  $\epsilon^{\mu\nu\delta\beta} = \epsilon^{\mu\nu\beta\delta}$ , the expression becomes  $\frac{g_{\mu\nu}}{16\pi m^2} \epsilon^{\mu\nu\beta\delta} (\partial_\delta \gamma_\nu - \partial_\nu \gamma_\delta)$ .

I do not have a general result, but for instance,  $\epsilon^{\mu\nu\beta\delta} (\gamma_\nu - \gamma_\delta \gamma_\beta) = -2F_{\nu\beta}$   
 which with Eq.  $\epsilon^{\mu\nu\beta\delta} \gamma_\beta = -F_{\nu\beta}$  which suggests the last term is  $+\frac{1}{2} \text{ Eq. (1)}$