Theory of transverse emittance exchange by linear betatron coupling in circular accelerators

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=> Follow-up of comment in recently published PRAB paper "Theory of emittance exchange through coupling resonance crossing" by Masamitsu Aiba and Jonas Kallestrup (https://journals.aps.org/prab/pdf/10.1103/PhysRevAccelBeams.23.044003) => Thanks to C. Hernalsteens for drawing my attention to this paper!

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Theory of emittance exchange through coupling resonance crossing

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In circular accelerators, transverse beam emittances can be exchanged by means of a coupling resonance crossing. It was first demonstrated experimentally at CERN accelerators, before a theoretical study revealed its mechanism. The existing theory of emittance exchange seems, however, to be deficient. In this study, we establish a theory of emittance exchange using the matrix formalism. Furthermore, we have found that the mechanism of emittance exchange is analogous to the transition dynamics of two-state quantum systems, thereby allowing the Landau-Zener formula to be applied for the prediction of transverse emittance values after nonadiabatic crossing.

However, the range of θ is not clearly defined. In the course of the derivation in [12], the parameters $|C|$ and Δ are related to θ as

$$\cos 2\theta = \cos\left(\arctan\frac{|C|}{\Delta}\right) = \frac{1}{\sqrt{1 + |C|^2/\Delta^2}}. \quad (4)$$

This clearly leads to an inequality of the form $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$, i.e., the emittance can be equalized but not exchanged whereas the final result [Eq. (34) in [12]] implicitly assumes a range of $0 \leq \theta \leq \frac{\pi}{2}$ or $-\frac{\pi}{2} \leq \theta \leq 0$.

[12] A. Franchi, E. Metral, and R. Tomás, Emittance sharing and exchange driven by linear betatron coupling in circular accelerator, Phys. Rev. ST Accel. Beams **10**, 064003 (2007).

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Answer to PRAB paper from Masamitsu Aiba and Jonas Kallestrup (23, 044003 (2020))
"Theory of emittance exchange through coupling resonance crossing"

- Reminder: $\text{ArcTan}(\text{Tan}(x)) = x$ if $x \in]-\frac{\pi}{2}; \frac{\pi}{2}[$
- In our case, $\text{Tan}(2\alpha) = \frac{|c|}{\Delta}$ and α will vary from 0 to $\frac{\pi}{2}$, i.e. 2α " 0 to π.

$$\Rightarrow \text{ArcTan}(\text{Tan}(2\alpha)) = 2\alpha \text{ if } 2\alpha \in]0; \tfrac{\pi}{2}[\text{ (or } \alpha \in]0; \tfrac{\pi}{4}[)$$
$$= 2\alpha - \pi \text{ if } 2\alpha \in]\tfrac{\pi}{2}; \pi[\text{ (or } \alpha \in]\tfrac{\pi}{4}; \tfrac{\pi}{2}[)$$

- when $\alpha \in]0; \frac{\pi}{4}[$, Same as in the paper.
- when $\alpha \in]\frac{\pi}{4}; \frac{\pi}{2}[$ (i.e. when $\Delta < 0$): $\text{ArcTan}(\text{Tan}(2\alpha)) = 2\alpha - \pi$

$$\Rightarrow \text{Cos}(\text{ArcTan}(\text{Tan}(2\alpha))) = \text{Cos}(2\alpha - \pi) = -\text{Cos}(2\alpha)$$

$$\| \quad \frac{1}{\sqrt{1+\text{Tan}^2(2\alpha)}} = \frac{1}{\sqrt{1+\frac{|c|^2}{\Delta^2}}}$$

$$\Rightarrow \boxed{\text{Cos}(2\alpha) = -\frac{1}{\sqrt{1+\frac{|c|^2}{\Delta^2}}}}$$

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$$\sin^2\alpha = \frac{1-\cos(2\alpha)}{2}$$
$$= \frac{1}{2}\left[1 + \frac{1}{\sqrt{1+\frac{|c|^2}{\Delta^2}}}\right]$$
$$= \frac{1}{2}\left[\frac{\sqrt{1+\frac{|c|^2}{\Delta^2}}+1}{\sqrt{1+\frac{|c|^2}{\Delta^2}}}\right]$$

We multiply top/bottom by $\sqrt{1+\frac{|c|^2}{\Delta^2}} - 1$

$$= \frac{1}{2}\left[\frac{\frac{|c|^2}{\Delta^2}}{1+\frac{|c|^2}{\Delta^2} - \sqrt{1+\frac{|c|^2}{\Delta^2}}}\right]$$
$$= \frac{|c|^2/2}{\Delta^2 + |c|^2 - \frac{\Delta^2}{\sqrt{\Delta^2}}\sqrt{\Delta^2+|c|^2}}$$

or $\sqrt{\Delta^2} = -\Delta$ as $\Delta < 0$

$$\Rightarrow \boxed{\sin^2\alpha = \frac{|c|^2/2}{\Delta^2+|c|^2+\Delta\sqrt{\Delta^2+|c|^2}}}$$

It is the same equation as Eq. (36) of my paper from 2001 (CERN/PS 2001-066 (AE))

⇒ Conclusion: the same final result is obtained but indeed to be correct, I should have said: "Using the fact that $\cos(2\alpha) = \cos\left(\text{ArcTan}\left(\frac{|c|}{\Delta}\right)\right) = \frac{1}{\sqrt{1+\frac{|c|^2}{\Delta^2}}}$ for $\alpha \in]0; \frac{\pi}{4}[$ AND $\cos(2\alpha) = -\frac{1}{\sqrt{1+\frac{|c|^2}{\Delta^2}}}$ for $\alpha \in]\frac{\pi}{4}; \frac{\pi}{2}[$, one can show that..."

Adiabatic vs. nonadiabatic crossing

- Reminder: the theory described in https://cds.cern.ch/record/529690/files/ps-2001-066.pdf is valid for the adiabatic crossing only => See comment there:

> Note that on the coupling resonance, $Q_v - Q_u = |C|$. The oscillation period of the cosine term in Eqs. (25) and (26) is thus $T_\phi = 2\pi / |C|$. If $|C|$ is infinitely small, then an infinitely long time is needed to cross the resonance to average this term to 0. The second average yields

- *N.B.: I could check to see in the future what I would obtain without making this assumption and compare to this new paper/theory*

Conclusion

- The same final formulae for the coupled emittances (for adiabatic crossing) are obtained and Masamitsu&Jonas agree with the previous slides and conclusions

$$\varepsilon_x = \varepsilon_{x0} - \left(\varepsilon_{x0} - \varepsilon_{y0}\right) \frac{|C|^2 / 2}{\Delta^2 + |C|^2 + \Delta\sqrt{\Delta^2 + |C|^2}}$$

$$\varepsilon_y = \varepsilon_{y0} + \left(\varepsilon_{x0} - \varepsilon_{y0}\right) \frac{|C|^2 / 2}{\Delta^2 + |C|^2 + \Delta\sqrt{\Delta^2 + |C|^2}}$$

- In addition, Masamitsu&Jonas obtained a new very interesting result as they can predict the transverse emittance values after nonadiabatic crossing by applying the Landau-Zener formula with the relevant accelerator parameters. The theoretical prediction is confirmed by numerical tracking
- Personal remark: In some past LHC MD (L. Carver et al.) it was not possible to obtain the emittance exchange as we clearly saw it in the past, e.g. in the PS => Still to fully understand why (MD to be redone and detailed analysis of this new theory / nonadiabaticity)…