

Theory of transverse emittance exchange by linear betatron coupling in circular accelerators

E. Métral

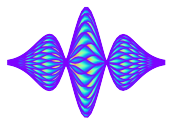
=> Follow-up of comment in recently published PRAB paper “Theory of emittance exchange through coupling resonance crossing” by Masamitsu Aiba and Jonas Kallestrup (<https://journals.aps.org/prab/pdf/10.1103/PhysRevAccelBeams.23.044003>) => Thanks to C. Hernalsteens for drawing my attention to this paper!

Theory of emittance exchange through coupling resonance crossing

Masamitsu Aiba^{id} and Jonas Kallestrup^{id}
Paul Scherrer Institut, 5232, Villigen, Switzerland

 (Received 11 December 2019; accepted 23 April 2020; published 30 April 2020)

In circular accelerators, transverse beam emittances can be exchanged by means of a coupling resonance crossing. It was first demonstrated experimentally at CERN accelerators, before a theoretical study revealed its mechanism. The existing theory of emittance exchange seems, however, to be deficient. In this study, we establish a theory of emittance exchange using the matrix formalism. Furthermore, we have found that the mechanism of emittance exchange is analogous to the transition dynamics of two-state quantum systems, thereby allowing the Landau-Zener formula to be applied for the prediction of transverse emittance values after nonadiabatic crossing.



However, the range of θ is not clearly defined. In the course of the derivation in [12], the parameters $|C|$ and Δ are related to θ as

$$\cos 2\theta = \cos \left(\arctan \frac{|C|}{\Delta} \right) = \frac{1}{\sqrt{1 + |C|^2/\Delta^2}}. \quad (4)$$

This clearly leads to an inequality of the form $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$, i.e., the emittance can be equalized but not exchanged whereas the final result [Eq. (34) in [12]] implicitly assumes a range of $0 \leq \theta \leq \frac{\pi}{2}$ or $-\frac{\pi}{2} \leq \theta \leq 0$.

[12] A. Franchi, E. Métral, and R. Tomás, Emittance sharing and exchange driven by linear betatron coupling in circular accelerator, *Phys. Rev. ST Accel. Beams* **10**, 064003 (2007).

Esther
12/08/2020

Answer to PEAR paper from Tamasina Atiba and Jonas Kalleberg (23.06.2020)
"Theory of emittance exchange through coupling resonance crossing"

- Reminder: $\text{ArcTan}(\text{Tan}(x)) = x$ if $x \in]-\frac{\pi}{2}; \frac{\pi}{2}[$
- In our case, $\text{Tan}(2\alpha) = \frac{|c|}{\Delta}$ and α will vary from 0 to $\frac{\pi}{2}$, i.e. 2α " " " " 0 to π .

$$\Rightarrow \text{ArcTan}(\text{Tan}(2\alpha)) = \begin{cases} 2\alpha & \text{if } 2\alpha \in]0; \frac{\pi}{2}[\text{ or } \alpha \in]0; \frac{\pi}{4}[\\ 2\alpha - \pi & \text{if } 2\alpha \in]\frac{\pi}{2}; \pi[\text{ or } \alpha \in]\frac{\pi}{4}; \frac{\pi}{2}[\end{cases}$$

- when $\alpha \in]0; \frac{\pi}{4}[$, same as in the paper.

when $\alpha \in]\frac{\pi}{4}; \frac{\pi}{2}[$ $\text{ArcTan}(\text{Tan}(2\alpha)) = 2\alpha - \pi$

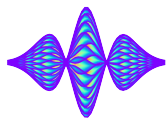
i.e. when $\Delta < 0$ $\Rightarrow \cos(\text{ArcTan}(\text{Tan}(2\alpha))) = \cos(2\alpha - \pi) = -\cos(2\alpha)$

$$\frac{1}{\sqrt{1 + \text{Tan}^2(2\alpha)}}$$

$$= \frac{1}{\sqrt{1 + \frac{|c|^2}{\Delta^2}}}$$

$$\Rightarrow \cos(2\alpha) = -\frac{1}{\sqrt{1 + \frac{|c|^2}{\Delta^2}}}$$

(1)



$$\sin^2 \alpha = \frac{1 - \cos(2\alpha)}{2}$$

$$= \frac{1}{2} \left[1 + \frac{1}{\sqrt{1 + \frac{|c|^2}{\Delta^2}}} \right]$$

$$= \frac{1}{2} \left[\frac{\sqrt{1 + \frac{|c|^2}{\Delta^2}} + 1}{\sqrt{1 + \frac{|c|^2}{\Delta^2}}} \right]$$

we multiply by $\frac{1}{\sqrt{1 + \frac{|c|^2}{\Delta^2}} - 1}$

$$= \frac{1}{2} \left[\frac{\frac{|c|^2}{\Delta^2}}{1 + \frac{|c|^2}{\Delta^2} - \sqrt{1 + \frac{|c|^2}{\Delta^2}}} \right]$$

$$= \frac{|c|^2/2}{\Delta^2 + |c|^2 - \frac{\Delta^2}{\sqrt{\Delta^2 + |c|^2}}}$$

or $\sqrt{\Delta^2} = -\Delta$ can $\Delta < 0$

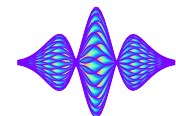
$$\Rightarrow \sin^2 \alpha = \frac{|c|^2/2}{\Delta^2 + |c|^2 + \Delta \sqrt{\Delta^2 + |c|^2}}$$

It is the same equation as Eq. (36) of my paper from 2001 (CERN/PS 2001-066(HE))

Conclusion: the same final result is obtained but indeed to be correct, I should have said: "Using the fact that $\cos(2\alpha) = \cos(\text{ArcTan}(\frac{|c|}{\Delta})) = \frac{1}{\sqrt{1 + \frac{|c|^2}{\Delta^2}}}$ for $\alpha \in]0; \frac{\pi}{4}[$ AND $\cos(2\alpha) = -\frac{1}{\sqrt{1 + \frac{|c|^2}{\Delta^2}}}$ for $\alpha \in]\frac{\pi}{4}; \frac{\pi}{2}[$ one can show that..."



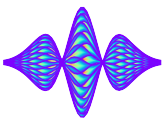
Adiabatic vs. nonadiabatic crossing



- ◆ Reminder: the theory described in <https://cds.cern.ch/record/529690/files/ps-2001-066.pdf> is valid for the adiabatic crossing only => See comment there:

Note that on the coupling resonance, $Q_v - Q_u = |C|$. The oscillation period of the cosine term in Eqs. (25) and (26) is thus $T_\phi = 2\pi / |C|$. If $|C|$ is infinitely small, then an infinitely long time is needed to cross the resonance to average this term to 0. The second average yields

- ◆ *N.B.: I could check to see in the future what I would obtain without making this assumption and compare to this new paper/theory*



- ◆ The same final formulae for the coupled emittances (for adiabatic crossing) are obtained and Masamitsu&Jonas agree with the previous slides and conclusions

$$\varepsilon_x = \varepsilon_{x0} - (\varepsilon_{x0} - \varepsilon_{y0}) \frac{|C|^2 / 2}{\Delta^2 + |C|^2 + \Delta \sqrt{\Delta^2 + |C|^2}}$$
$$\varepsilon_y = \varepsilon_{y0} + (\varepsilon_{x0} - \varepsilon_{y0}) \frac{|C|^2 / 2}{\Delta^2 + |C|^2 + \Delta \sqrt{\Delta^2 + |C|^2}}$$

- ◆ In addition, Masamitsu&Jonas obtained a new very interesting result as they can predict the transverse emittance values after nonadiabatic crossing by applying the Landau-Zener formula with the relevant accelerator parameters. The theoretical prediction is confirmed by numerical tracking
- ◆ Personal remark: In some past LHC MD (L. Carver et al.) it was not possible to obtain the emittance exchange as we clearly saw it in the past, e.g. in the PS => Still to fully understand why (MD to be redone and detailed analysis of this new theory / nonadiabaticity)...