# The small sphere limits of quasilocal masses in higher dimensions

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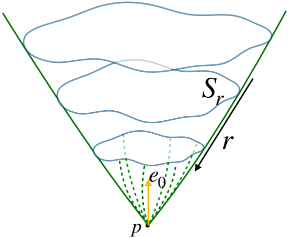
# Quasilocal mass

- In GR, purely gravitational energy cannot be covariantly represented as a density. Quasilocal mass (QLM) characterises the gravitational energy quasilocally. (c.f. Xie's talk earlier)
- It is a functional that assigns a positive value to a codimension-two topological sphere $S$ using the data associated with $S$.
- **Hawking mass**, Hayward mass, Penrose mass, Bartnik mass, **Brown-York mass**, **Liu-Yau mass**, Wang-Yau mass and etc.
- It should satisfy: positivity, rigidity, monotonicity, the global asymptotics and the local asymptotics, i.e. **the small sphere limit (SSL)**. Christodoulou& Yau '88, Szabados '09
- We shall focus on the SSL, which effectively probes the local gravitational energy.

# Small sphere limit (SSL)

- There is a conventional way to take the SSL due to Horowitz and Schmidt, known as the light-cone cuts. Horowitz & Schmidt '82
- Given $(p, e_0)$, one defines a one-parameter family of spheres $S_r$ as light-cone cuts w.r.t. the affine parameter $r$ normalised with $e_0$.




- In 4D non-vacuum (with matter), the small sphere limit should yield the stress tensor component: $\lim_{r\to 0} r^{-(n-1)} M(S_r) \propto T(e_0, e_0)$.
- In 4D vacuum, the small sphere limit should yield the **Bel-Robsinson tensor** component: $\lim_{r\to 0} r^{-(n+1)} M(S_r) \propto Q(e_0, e_0, e_0, e_0)$. Szabados '09

# The Bel-Robinson (BR) superenergy

- The BR tensor is the "square" of the Weyl tensor that is dominant, symmetric, traceless and divergenceless in vacuum. Bel '59, Robinson '97
- We are interested in the $Q = Q(e_0, e_0, e_0, e_0)$ component and it can be conveniently written in terms of the electromagnetic decomposition of the Weyl tensor. $Q = E^2 + B^2$. This is sometimes refered as the Bel-Robinson (BR) superenergy, for its wrong dimension $[T] = -2, [Q] = -4$.
- In higher dimensions, the BR tensor has a **unique** generalisation by assuming it is square in Weyl and dominant, where $E, H, D$ are electro-magnetic decompositions of the Weyl tensor in higher dimensions. Senovilla '00

$$Q = \frac{1}{2}\left[E^2 + H^2 + \frac{1}{4}D^2\right]. \quad (1)$$

# QLM in higher dimensions

- We'd like to compute the SSL's in higher dimensions.
- Most investigations of QLM's in the literature are limited to 4D.
- It is difficult to come up with a good definition that works for all dimensions (except maybe the Bartnik mass).
- We make an attempt here to extend the 4D definitions of the Hawking mass Hawking '68, the BY mass Brown & York '93 and the LY mass Liu & Yau '03 to higher dimensions and put sufficient assumptions that allows us to evaluate the SSL. (c.f. Miao, Tam & Xie '17)

# The Hawking mass

## Definition 1

*For a spacelike codimension-2 topological sphere $S$ in an n-dimensional spacetime, the Hawking type mass is defined as*

$$M(S) = \frac{\left(\frac{Vol(S)}{\Omega_{n-2}}\right)^{\frac{1}{n-2}}}{(n-2)(n-3)\Omega_{n-2}} \int_S \left(\frac{\mathcal{R}}{2} + \frac{n-3}{n-2}\theta^-\theta^+\right) \mathrm{d}\sigma \tag{2}$$

*where $\mathcal{R}$ is the intrinsic scalar curvature on $S$, and $\theta^\pm$ are the null expansions.*

## Remark 1

*It reduces to the Misner-Sharp mass for round spheres. This generalisation is also used by Miao, Tam & Xie to study the global asymptotics.*

# The SSL of the Hawking mass

## Theorem 1

*Let $S_r$ be the family of surfaces shrinking towards $p$ along light-cone cuts defined with respect to $(p, e_0)$ in an $n$-dimensional spacetime, the limits of the Hawking mass as $r$ goes to 0 are*

1. *In non-vacuum,*

$$\lim_{r \to 0} r^{-(n-1)} M_H(S_r) = \frac{\Omega_{n-2}}{n-1} T(e_0, e_0)|_p. \tag{3}$$

2. *In vacuum or the stress tensor $T$ vanishes in an open set containing $p$,*

$$\lim_{r \to 0} r^{-(n+1)} M_H(S_r) = \mathcal{W}|_p \, (\neq Q). \tag{4}$$

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# The Brown-York mass

## Definition 2

*Given a spacelike codimension-two topological sphere $S$ embedded on $\Sigma$ in an $n$-dimensional spacetime and the reference isometric embedding $\tilde{S}$ in $\tilde{\Sigma}$, then the Brown-York type mass is defined as*

$$M_{BY}(S,\Sigma) := \frac{1}{\Omega_{n-2}(n-2)} \int_S \tilde{H} - H \, \mathrm{d}\sigma \tag{5}$$

*where $\sigma$ is the induced volume form on $S$, $H$ is the mean curvature of $S$ in $\Sigma$ and $\tilde{H}$ is the mean curvature of $\tilde{S}$ as embedded in $\tilde{\Sigma}$.*

## Remark 2

*The mass entails a reference zero point energy and one has to make a choice. We use the lightcone embedding as used by* Brown, Lau& York '98 *in calculating the SSL in 4D.*

# The SSL of the BY mass

## Theorem 2

*Let $S_r$ be the family of conformally flat light-cone cuts shrinking towards $p$ defined with respect to $(p, e_0)$ in an $n$-dimensional spacetime, the limits of the Brown-York mass as $r$ goes to 0 are*

1. *In non-vacuum,*

$$\lim_{r\to 0} r^{-(n-1)} M_{BY}(S_r) = \frac{\Omega_{n-2}}{n-1} T(e_0, e_0)|_p. \tag{6}$$

2. *In vacuum or the stress tensor $T$ vanishes in an open set containing $p$,*

$$\lim_{r\to 0} r^{-(n+1)} M_{BY}(S_r) = \mathcal{W}|_p. \tag{7}$$

JZW '20

# The vacuum limit $\mathcal{W}$

Without further details, the SSL of the **Liu-Yau mass** is also characterized by the same quantity $\mathcal{W}$. Therefore, quite generally the SSL in arbitrary spacetime dimensions $n$ is characterized by

$$\mathcal{W} := \frac{(6n^2 - 20n + 8)E^2 + 6(n-3)H^2 - 3D^2}{36(n-3)(n-2)(n^2-1)}, \tag{8}$$

whereas the BR superenergy is

$$Q = \frac{1}{2}\left[E^2 + H^2 + \frac{1}{4}D^2\right]. \tag{9}$$

- In 4D, $D^2 = 4E^2, H^2 = 2B^2$, one recovers the BR superenergy $\mathcal{W} = Q = E^2 + B^2$.
- Our results match the earlier 4D results by Horowitz & Schmidt '82, Bergqvist '94, Brown, Lau & York '98, P.P. Yu '07
- $\mathcal{W}$ is **not** positive in general!

# Conclusion

- The BR superenergy $Q$ **does not** characterise the SSL of QLM's in dimensions $n > 4$.
- $Q$ is replaced by $\mathcal{W}$ which is not always positive.
- It is worth to investigate more on the seemingly universal quantity $\mathcal{W}$ to clarify its physical meaning.
- It is also plausible that these QLM's are not sensible notions in higher dimensions. To identify a proper notion of QLM, we need more physical insights rather than just geometry.

Thank you for your attention!

# Isometric embedding

- In higher dimensions, isometric embedding of codimension-two surfaces into Minkowski spacetime may not exist.
- Following BLY, we use light-cone embedding as the references. We would like to isometrically embed our surface to a light-cone $\tilde{N}_p$ in Minkowski spacetime $\tilde{M}^n$. Brown, Lau& York '98
- We assume conformal flatness to evaluate the SSL. The light-cone embedding exists iff the surface is conformally flat. Brinkmann '23
- The uniqueness is fixed by explicit choices (BLY) and the Gauss-Codazzi equation.

# Light-cone isometric embedding

The BY mass depends on the hypersurface $\Sigma \supset S$. To fix $\Sigma$, we follow the proposal by Brown and York: Brown, Lau & York '98

- Set $H = \theta^+/2 - \theta^-, k = \theta^+/2 + \theta^-$.
- For the reference, we set $\tilde{\theta^+} := \tilde{H} + \tilde{k} = \theta^+$.
- The Gauss-Codazzi equation $\tilde{H}^2 - (\theta^+ - \tilde{H})^2 = \frac{n-2}{n-3}\mathcal{R}$ fixes $\tilde{H}$.

The definition of the LY mass is similar to the BY mass, but the $H$ is replaced by the norm of the mean curvature vector $|\mathbf{H}|$. Therefore, unlike the BY mass, LY mass is a covariantly defined QLM, so we do not need to fix any hypersurface $\Sigma$ or $\tilde{\Sigma}$ a priori. Such a surface is shear-free, so the Gauss-Codazzi equation implies

$$|\tilde{\mathbf{H}}|^2 = \frac{n-2}{n-3}\mathcal{R}. \tag{10}$$

# The Liu-Yau mass

**Definition 3**

*Given a spacelike codimension-two topological sphere $S$ in an n-dimensional spacetime with spacelike mean curvature vector, the Liu-Yau type mass is defined as*

$$M_{LY}(S) := \frac{1}{\Omega_{n-2}(n-2)} \int_S |\tilde{\mathbf{H}}| - |\mathbf{H}| \, \mathrm{d}\sigma \tag{11}$$

*where $\sigma$ is the induced volume form on $S$, $\tilde{\mathbf{H}}$ is the mean curvature vector of $S$ as embedded in $\tilde{M}^n$.*

# The SSL of the LY mass

## Theorem 3

*Let $S_r$ be the family of conformally flat light-cone cuts shrinking towards $p$ defined with respect to $(p, e_0)$ in an $n$-dimensional spacetime, the limits of the Liu-Yau mass as $r$ goes to $0$ are*

1. *In non-vacuum,*

$$\lim_{r \to 0} r^{-(n-1)} M_{LY}(S_r) = \frac{\Omega_{n-2}}{n-1} T(e_0, e_0)|_p. \tag{12}$$

2. *In vacuum or the stress tensor $T$ vanishes in an open set containing $p$,*

$$\lim_{r \to 0} r^{-(n+1)} M_{LY}(S_r) = \mathcal{W}|_p - \frac{(n^2-n-3)E^2|_p}{12(n^2-1)(n-2)^2(n-3)^2}. \tag{13}$$

## Remark 3

*In 4D, the vacuum SSL's of the Liu-Yau mass and the Wang-Yau mass have the same form.* *P.P.Yu '07, Chen, Wang & Yau '15*

# Gravitoelectromagnetism

- In vacuum ($Ric = 0$), the Riemann tensor equals to the Weyl tensor which governs the pure gravitational field behaviors.
- Given $(p, e_0)$, the Weyl tensor $C$ in four dimensions can be decomposed into electro-magnetic components.

$$E := C(\cdot, e_0, \cdot, e_0) \qquad B := \star C(\cdot, U, \cdot, U) \qquad \left( \star C_{abcd} = \frac{1}{2} \epsilon_{ab}{}^{ef} C_{efcd} \right)$$

- Physically, $E$ is responsible for the gravitational induction phenomena like tidal distortions, and $B$ together with $E$ is responsible for gravitational radiation, in particular upon a FLRW background.
- In higher dimensions, similar decompositions can be made. One obtains the electro-electro part $E$, the electric-magnetic part $H$ and the magnetic-magnetic part $D$.

# Gravitoelectromagnetism

- The Bel Robinson tensor is defined with the Weyl tensor $C$

$$Q_{abcd} = C_{aecf} C_b{}^e{}_d{}^f + \star C_{aecf} \star C_b{}^e{}_d{}^f \quad (n = 4)$$

- It is the unique tensor with the dominant property and quadratic in the Weyl tensor. It is also totally symmetric, traceless and divergence free.
- The BR superenergy density w.r.t. a timelike vector $U$ is defined as $Q := Q(U, U, U, U) \geq 0$.
- In analogy to electrodynamics, $Q = E^2 + B^2$ and $Q(\cdot, U, U, U)$ is the super-Poynting vector.
- Imposing the dominant property, $Q$ admits a **unique** generalisation in arbitrary dimensions.

$$Q = \frac{1}{2}E^2 + \frac{1}{2}H^2 + \frac{1}{8}D^2.$$