Quantum fate of generic gravitational singularity

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OUTLINE

Introduction

- Friedmann's model (1922)
  - assumes isotropy and homogeneity of space
  - solution includes gravitational singularity
  - commonly used in astrophysics and cosmology
- Isotropy is unstable in the evolution towards singularity[1].
- In late 50-ties relativists (USSR, USA) started examination of models with homogeneous space (Bianchi-type models) in particular Bianchi VIII and IX.

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Belinskii-Khalatnikov-Lifshitz (BKL) conjecture

- Dynamics of BVIII and BIX were analyzed to get insight into the dynamics of spacetime near the cosmological spacelike singularity[2].
- BKL conjecture[3]:
  general relativity implies existence of generic solution that is singular (gravitational and matter fields invariants diverge)
  - corresponds to non-zero measure subset of all initial data
  - is stable against perturbation of initial data
  - depends on arbitrary functions of space

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BKL conjecture (cont)

- BKL in string theory[4]
  - appears in the low energy limits of bosonic sectors of all five types of superstring models
  - the Lorenzian hyperbolic Kac-Moody algebra underlies asymptotic structure of spacetime near cosmological singularity

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BKL conjecture and singularity theorems

- The Penrose-Hawking singularity theorems (of 60-ties) concern, among other things, possible existence of incomplete geodesics in spacetime, but incompleteness does not mean (in general) that the invariants diverge.
- These theorems say little about the dynamics of gravitational field near singularities so that are of little usefulness in the context of finding possible quantum dynamics.
- In what follows we focus our attention on the BKL treatment of singularities.

- The existence of generic singularities in solutions to Einstein's equations signal the existence of the limit of validity of GR.
- Expectation: Quantization of general relativity may lead to the theory that is devoid of singularities so that it could be used do explain observational data.

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Dynamics near the singularity

Near the singularity of BVIII and BIX, one can assume[5]:

- stress-energy tensor components can be ignored
- Ricci tensor components R_a^0 have negligible influence on the dynamics
- anisotropy of space may grow without bound

These assumptions lead to enormous simplification of the mathematical form of dynamics.

In particular, the dynamics can be well approximated by neglecting spatial derivatives in the field equations in comparison to time derivatives so that the system enters an ultra-local phase near the singularity.

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The BKL scenario

The asymptotic form (near the singularity) of the dynamical equations of general Bianchi VIII and IX models read

$$\frac{d^2 \ln a}{d\tau^2} = \frac{b}{a} - a^2, \quad \frac{d^2 \ln b}{d\tau^2} = a^2 - \frac{b}{a} + \frac{c}{b}, \quad \frac{d^2 \ln c}{d\tau^2} = a^2 - \frac{c}{b}, \qquad (1)$$

where $a = a(\tau),\ b = b(\tau),\ c = c(\tau)$
are effective directional scale factors.

The solutions to (1) must satisfy the constraint:

$$\frac{d \ln a}{d\tau}\frac{d \ln b}{d\tau} + \frac{d \ln a}{d\tau}\frac{d \ln c}{d\tau} + \frac{d \ln b}{d\tau}\frac{d \ln c}{d\tau} = a^2 + \frac{b}{a} + \frac{c}{b}. \qquad (2)$$

Eqs (1)–(2) represent the essence of the BKL scenario[6].

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The BKL scenario (cont)

BKL's theoretical arguing has numerical support:

- simulations of the dynamics near the singularity confirm the existence of the asymptotic dynamics[7]
- Kretschman's curvature invariant diverge in the evolution towards the singularity[8]

BKL dynamics

- different from commonly known mixmaster dynamics, proposed by C. Misner (1969); BKL results from considering general metric tensors of BVIII and BIX; Misner's from diagonal metric of BIX[9]
- has critical points of nonhyperbolic type; dynamics cannot be approximated by linearized equations; space of these points is related to gravitational singularity[10]

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Hamilton's dynamics underlying BKL scenario

Making use of the reduced phase space technique[11] enables rewriting the dynamics (1) - (2) in the form of the Hamiltonian system:

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\begin{aligned}
dq_1/dt &= \partial H/\partial p_1 = (p_2 - p_1 + t)/2F, && (3)\\
dq_2/dt &= \partial H/\partial p_2 = (p_1 - p_2 + t/2F, && (4)\\
dp_1/dt &= -\partial H/\partial q_1 = (2e^{2q_1} - e^{q_2 - q_1}/F, && (5)\\
dp_2/dt &= -\partial H/\partial q_2 = -1 + e^{q_2 - q_1}/F, && (6)
\end{aligned}
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where $H(q_1, q_2; p_1, p_2; t) := -q_2 - \ln F(q_1, q_2, p_1, p_2, t)$, and where

$$
F := -e^{2q_1} - e^{q_2 - q_1} - \frac{1}{4}(p_1^2 + p_2^2 + t^2) + \frac{1}{2}(p_1 p_2 + p_1 t + p_2 t) > 0. \quad (7)
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Hamiltonian depends explicitly on time, and is not of polynomial-type so that canonical quantization cannot be applied.

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Quantization of BKL scenario

Roughly speaking, by quantization of classical system, represented by observables defined on phase space, I mean[12]

- ascribing to that system self-adjoint operators acting in Hilbert space
- ascribing to Hamilton's dynamics Schrödinger's dynamics
- examination of time dependance of probability amplitude

In what follows we make use of our recent results[13]

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[13] A. Góźdź, W.P., and G. Plewa, Eur. Phys. J. C **79**, 45 (2019);
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Quantization: choice of the Hilbert space

The physical phase space Π consists of the two half-planes:

$$\Pi = \Pi_1 \times \Pi_2 := \{(q_1, p_1) \in \mathbb{R} \times \mathbb{R}_+\} \times \{(q_2, p_2) \in \mathbb{R} \times \mathbb{R}_+\},$$

where $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x > 0\}$.
Separately, Π_1 and Π_2 can be identified with the affine group $\mathrm{Aff}(\mathbb{R})$.
This group has UIR realized in the Hilbert space $L^2(\mathbb{R}_+, d\nu(x))$, where $d\nu(x) = dx/x$, defined by

$$U(q, p)\psi(x) = e^{iqx}\psi(px).$$

This enables defining the continuous family of affine coherent states $|q, p\rangle \in L^2(\mathbb{R}_+, d\nu(x))$ as follows

$$|q, p\rangle = U(q, p)|\phi\rangle \ ,$$

where $|\phi\rangle \in L^2(\mathbb{R}_+, d\nu(x))$, is the so-called fiducial vector, which is a free parameter of this quantization scheme

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Quantization: defining quantum operators

The irreducibility of the representation leads (due to Schur' lemma) to the resolution of the unity in $L^2(\mathbb{R}_+, d\nu(x))$:

$$\frac{1}{A_\phi}\int_\Pi d\mu(q,p)|q,p\rangle\langle q,p| = \mathbb{I}\,, \tag{8}$$

where $d\mu(q,p) := dq\,dp/p^2$ is the left invariant measure on Π, and where $A_\phi := \int_0^\infty |\phi(x)|^2 \frac{dx}{x^2} < \infty$ is a constant.

Using (8), enables quantization of any observable $f : \Pi \to \mathbb{R}$

$$f \longrightarrow \hat{f} = \frac{1}{A_\phi}\int_\Pi d\mu(q,p)|q,p\rangle f(q,p)\langle q,p|\,. \tag{9}$$

The operator $\hat{f}$ is symmetric (Hermitian) by construction.
No ordering ambiguity occurs (disaster of canonical quantization).

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Quantum dynamics

If $\hat{H}$ is bounded on $\mathcal{H}$, it is self-adjoint on $\mathcal{H}$. We can define the quantum evolution using the Schrödinger equation:

$$i\frac{\partial}{\partial\tau}|\psi(\tau)\rangle = \hat{H}(t)|\psi(\tau)\rangle \;, \tag{10}$$

where $|\psi\rangle \in \mathcal{H}$, and where τ is an evolution parameter at the quantum level.
In general, the parameters t and τ are quite different. To get the consistency between the classical and quantum levels we assume that $t = \tau$, which defines the time parameter at both levels.

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Resolution of singularity: dynamics near singularity

Near the gravitational singularity, the classical Hamiltonian simplifies so that the Schrödinger equation takes the form

$$i\frac{\partial}{\partial t}\Psi(t,x_1,x_2) = \left(i\frac{\partial}{\partial x_2} - \frac{i}{2x_2} - K(t,x_1,x_2)\right)\Psi(t,x_1,x_2)\,, \tag{11}$$

where

$$K = \frac{1}{A_{\Phi_1}A_{\Phi_2}} \int_0^\infty \frac{dp_1}{p_1^2}\int_0^\infty \frac{dp_2}{p_2^2}\ \ln\left(F_0(t,\frac{p_1}{x_1},\frac{p_2}{x_2})\right)|\Phi_1(x_1/p_1)|^2|\Phi_2(x_2/p_2)|^2 \tag{12}$$

and where

$$F_0(t,p_1,p_2) := p_1p_2 - \frac{1}{4}(t-p_1-p_2)^2\,. \tag{13}$$

Resolution of singularity: general solution

The general solution to our Schrödinger equation (11) reads

$$\Psi = \eta(x_1, x_2 + t - t_0) \sqrt{\frac{x_2}{x_2 + t - t_0}} \exp\left(i \int_{t_0}^{t} K(t', x_1, x_2 + t - t') \, dt' \right) , \tag{14}$$

where $t \geq t_0 > 0$, and where $\eta(x_1, x_2) := \Psi(t_0, x_1, x_2)$ is the initial state satisfying the condition

$$\eta(x_1, x_2) = 0 \quad \text{for} \quad x_2 < t_H , \tag{15}$$

with $t_H > 0$ being the parameter of our model.

For $t < t_H$ we get

$$\langle \Psi(t) | \Psi(t) \rangle = \int_0^\infty \frac{dx_1}{x_1} \int_{t_H}^\infty \frac{dx_2}{x_2} |\eta(x_1, x_2)|^2 , \tag{16}$$

so that the inner product is time independent, which implies that the quantum evolution is unitary.

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where $t \geq t_0 > 0$, and where $\eta(x_1, x_2) := \Psi(t_0, x_1, x_2)$ is the initial state satisfying the condition

$$\eta(x_1, x_2) = 0 \quad \text{for} \quad x_2 < t_H\,, \tag{15}$$

with $t_H > 0$ being the parameter of our model.
For $t < t_H$ we get

$$\langle\Psi(t)|\Psi(t)\rangle = \int_0^\infty \frac{dx_1}{x_1}\int_{t_H}^\infty \frac{dx_2}{x_2}\,|\eta(x_1, x_2)|^2\,, \tag{16}$$

so that the inner product is time independent, which implies that the quantum evolution is unitary.

Quantum bounce: time reversal invariance

The operator of the time reversal, $\hat{T} : \mathcal{H} \rightarrow \mathcal{H}$, is defined to be

$$\hat{T}\,\psi(t, x_1, x_2) = \tilde{\psi}(t, x_1, x_2) := \psi(-t, x_1, x_2)^*, \quad \text{where} \quad \psi \in \mathcal{H}\,. \tag{17}$$

Due to (11), the Schrödinger equation for $\tilde{\psi}$ reads

$$i\frac{\partial}{\partial t}\tilde{\Psi}(t, x_1, x_2) = \left(-i\frac{\partial}{\partial x_2} + \frac{i}{2x_2} - K(-t, x_1, x_2)\right)\tilde{\Psi}(t, x_1, x_2)\,. \tag{18}$$

The general solution to (18), for $t < 0$, is found to be

$$\tilde{\Psi} = \eta(x_1, x_2+|t|-|t_0|)\sqrt{\frac{x_2}{x_2 + |t| - |t_0|}}\,\exp\left(i\int_{t_0}^{t} K(-t', x_1, x_2 - t + t')\,dt'\right) \tag{19}$$

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Quantum bounce: continuity of probability density

The unitarity of the evolution (with $t_0 = 0$) can be obtained again if

$$\eta(x_1, x_2) = 0 \quad \text{for} \quad x_2 < |t_H| \,, \tag{20}$$

which corresponds to the condition (15).

Since the solutions (14) and (19) differ only by the corresponding phases, the probability density is continuous at $t = 0$, which means that we are dealing with quantum bounce at $t = 0$ (that marks the classical singularity).

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Conclusions

- BKL scenario concerns generic singularity of general relativity so that its resolution at quantum level strongly suggests that it is fairly probable that quantum GR, to be constructed, would be free from singularities.
- It makes sense applying preliminary versions of quantum gravity to address the issues of cosmological and BHs singularities.

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Thank you!

Appendix A

Quantization of interior of black hole:

Cosmological model can be used to describe black hole after imposing the condition that one deals with isolated object. This may be reduced to the problem of merging finite region of specific spacetime with the Schwarzschild (Sch) spacetime.

- spherically symmetric static BHs
  - shell model[14]: Minkowski+shell+Sch; quantization within ACS method, in progress
  - FRW+Sch, LTB+Sch; classical formulation is done[15]
  - quantum FRW+Sch (done within ACS method)[16]
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[14] J. Kijowski, G. Magli, and D. Malafarina, Phys.Rev D **74**, 084017 (2006).

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Challenges:

- anisotropic static BHs: Bianchi (inside) + (outside?);
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  it is expected that BIX BH would radiate gravitational waves
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- BKL (having no symmetry) scenario (inside) + (outside?)
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Applications:

In case of astrophysics, quantum bounce (black to white hole transition), may lead to small bang (analogy with cosmological Big Bang).

Quantum gravity may be used to get insight into the origin of numerous highly energetic explosions in distant galaxies, and vice versa.

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Appendix B

The general form of a line element of the Bianchi IX model, in the synchronous reference system, reads:

$$ds^2 = dt^2 - \gamma_{ab}(t) e^a_\alpha e^b_\beta dx^\alpha dx^\beta, \tag{21}$$

where $a, b, \ldots$ run from 1 to 3 and label frame vectors; $\alpha, \beta, \ldots$ take values $1, 2, 3$ and concern space coordinates, and where γ_{ab} is a spatial metric.
The homogeneity of the Bianchi IX model means that the three independent differential 1-forms $e^a_\alpha dx^\alpha$ are invariant under the transformations of the isometry group of the Bianchi IX model.
The cosmological time variable t is redefined as follows:

$$dt = \sqrt{\gamma}\, d\tau, \qquad \gamma := det[\gamma_{ab}] \tag{22}$$

where γ is the volume density, and $\gamma \to 0$ denotes the singularity.

Appendix C

The phase space $\Pi = \mathbb{R} \times \mathbb{R}_+$ can be identified with the affine group $G = \mathrm{Aff}(\mathbb{R})$ by defining the multiplication law as follows[17]

$$(q', p') \cdot (q, p) = (p'q + q', p'p), \tag{23}$$

with the unity $(0, 1)$ and the inverse

$$(q', p')^{-1} = (-\frac{q'}{p'}, \frac{1}{p'}). \tag{24}$$

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Appendix B (cont)

The affine group has two, nontrivial, inequivalent irreducible unitary representations. Both are realized in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}_+, d\nu(x))$, where $d\nu(x) = dx/x$ is the invariant measure on the multiplicative group $(\mathbb{R}_+, \cdot)$.
In what follows we choose the one defined by

$$U(q,p)\psi(x) = e^{iqx}\psi(px)\,. \tag{25}$$

Integration over the affine group reads

$$\int_G d\mu(p,q) := \int_{-\infty}^{\infty} dp \int_0^{\infty} dq/q^2\,, \tag{26}$$

where the measure $d\mu(p,q)$ is left invariant.

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Fixing the normalized vector $|\Phi\rangle \in L^2(\mathbb{R}_+, d\nu(x))$, called the fiducial vector, one can define a continuous family of affine coherent states $|q,p\rangle \in L^2(\mathbb{R}_+, d\nu(x))$ as follows

$$|q,p\rangle = U(q,p)|\Phi\rangle. \tag{27}$$

The irreducibility of the representation, used to define the coherent states (27), enables making use of Schur's lemma, which leads to the resolution of the unity in $L^2(\mathbb{R}_+, d\nu(x))$:

$$\int_{\mathrm{G}} d\mu(q,p)|q,p\rangle\langle q,p| = A_\Phi \, \hat{\mathbb{I}} \,, \tag{28}$$

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$$f \longrightarrow \hat{f} := \frac{1}{A_\Phi} \int_{\mathrm{G}} d\mu(q,p) |q,p\rangle f(q,p) \langle q,p| \,, \tag{30}$$

where $\hat{f} : \mathcal{H} \to \mathcal{H}$ is the corresponding quantum observable.
The mapping (30) is covariant in the sense that one has

$$U(\xi_0)\hat{f}U^\dagger(\xi_0) = \frac{1}{A_\Phi} \int_{\mathrm{G}_1} d\mu_L(\xi) |\xi\rangle f(\xi_0^{-1} \cdot \xi) \langle \xi| == \widehat{\mathcal{L}^L_{\xi_0} f} \,, \tag{31}$$

where $\mathcal{L}^L_{\xi_0} f(\xi) = f(\xi_0^{-1} \cdot \xi)$ is the left shift operation, and where $\xi_0^{-1} \cdot \xi = (q_0, p_0)^{-1} \cdot (q,p) = (\frac{q-q_0}{p_0}, \frac{p}{p_0})$, with $\xi := (q,p)$.

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Eq. (30) defines a linear mapping and the observable $\hat{f}$ is a symmetric operator. Let us evaluate the norm of this operator:

$$\|\hat{f}\| \leq \frac{1}{A_\Phi} \int_{\mathrm{G}} d\mu(q,p)|f(q,p)|\||q,p\rangle\langle q,p| = \frac{1}{A_\Phi} \int_{\mathrm{G}} d\mu(q,p)|f(q,p)|\,. \tag{32}$$

This implies that, if the classical function f belongs to the space of integrable functions $L^1(\mathrm{G}, d\mu_L(q,p))$, the operator $\hat{f}$ is bounded so that it is a self-adjoint operator. Otherwise, it is defined on a dense subspace of $L^2(\mathbb{R}_+, d\nu(x))$ and its possible self-adjointness becomes an open problem.

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