Self-gravitating tori around black holes: Bifurcation, ergoregions, and geometrical properties

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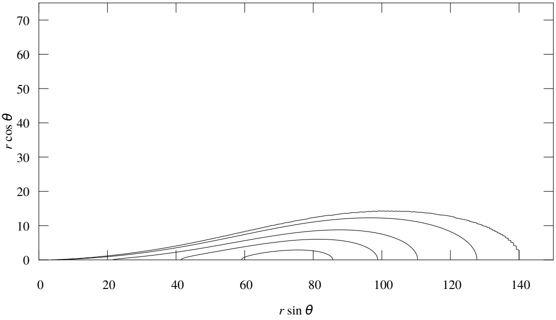
Literature

W. Dyba, W. Kulczycki, P. Mach, *Self-gravitating perfect-fluid tori around black holes: Bifurcation, ergoregions, and geometrical properties* Physical Review D 101, 044036 (2020)

- We model selfgravitating tori of prefect fluid fulfil polytropic equation of state in keplerian motion around a black hole
- We solve Einstein field equations and the Euler equation for the stationary axial symmetric spacetime



Main points of numerical method

Metric:

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 =$$

$$= -\alpha^2 dt^2 + r^2 \sin^2\theta\, \psi^4 (d\varphi + \beta dt)^2 + \psi^4 e^{2q}(dr^2 + r^2 d\theta^2). \quad (1)$$

Functions α, ψ, q, β depend only on r and θ.
The stress-momentum tensor:

$$T^{\alpha\beta} = \rho h u^\alpha u^\beta + p g^{\alpha\beta}, \quad (2)$$

ρ - baryonic rest mass density, h - specific enthalpy.
Barotropic equation of state:

$$p(\rho) = K\rho^\Gamma. \quad (3)$$

Rotation law

The 4-velocity $(u^\alpha) = (u^t, 0, 0, u^\varphi)$.
The coordinate angular velocity:

$$\Omega = \frac{u^\varphi}{u^t}. \tag{4}$$

The angular momentum per unit inertial mass ρh:

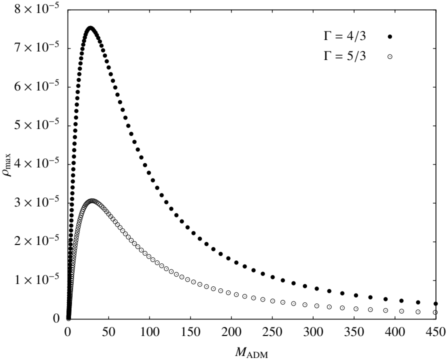
$$j \equiv u_\varphi u^t. \tag{5}$$

$$j(\Omega) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\Omega} \ln\left(1 - \left(\tilde{a}^2\Omega^2 + 3w^{4/3}\Omega^{2/3}(1-\tilde{a}\Omega)^{4/3}\right)\right). \tag{6}$$

w – a mass parameter. The spin parameter $\tilde{a}$ is equal to a (the spin parameter of black hole), in case of a massless disk (Kerr geometry).

Bifurcation

Solution parameters: Γ, r_1, r_2, ρ_{max}, a, m



The maximal density $\rho_{\rm max}$ versus the ADM mass $M_{\rm ADM}$. Geometric parameters of the solutions are $r_1 = 50$ and $r_2 = 100$, $a = 0$, $m = 1$.

Volume of the torus

The proper volume of the torus is computed as:

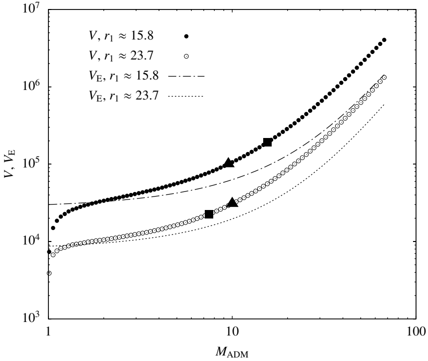
$$V = 2\pi \int dr \int d\theta r^2 \sin\theta \psi^6 e^{2q}, \tag{7}$$

In Euclidean geometry the volume of a proper torus (obtained by rotating a circle about an external axis) can be expressed as

$$V_{\mathrm{E}} = 2\pi \frac{r_1 + r_2}{2} \pi \left(\frac{r_2 - r_1}{2} \right)^2 = \frac{\pi^2}{4} (r_1 + r_2)(r_2 - r_1)^2, \tag{8}$$

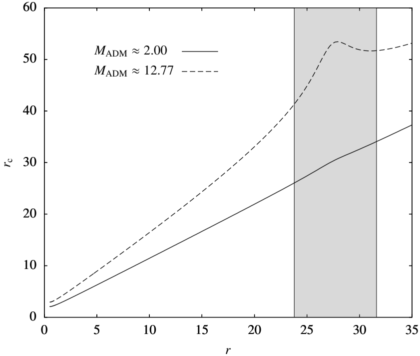
where r_1 and r_2 denote the torus inner and outer radius, respectively.

Volume of the torus versus ADM mass



Other parameters of the solutions are: $r_2 \approx 31.6$, $a = 0$, $m = 1$, and $\Gamma = 4/3$. Squares represent solutions with the lowest mass for which a nonmonotonic behavior of r_c with respect to r has been observed. Triangles denote solutions with maximal values of $\rho_{\max}$.

Nonmonotonic behavior of circumferential radius

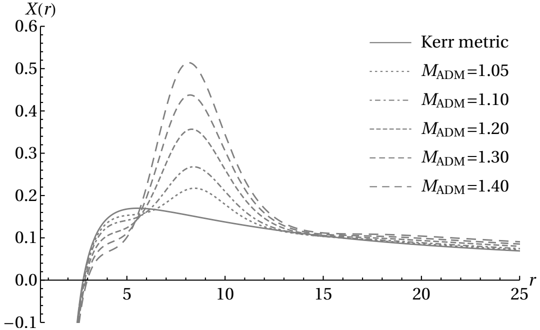


Other parameters of the solutions are: $r_1 \approx 23.7$, $r_2 \approx 31.6$, $a = 0$, $m = 1$, and $\Gamma = 4/3$. The area between r_1 and r_2 is marked in grey.

Location of ISCO in the torus-black hole spacetime

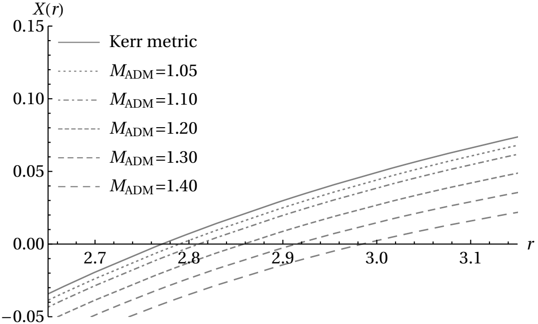
Consider a circular geodesic at the equatorial plane $\theta = \pi/2$ for the metric given by (1). We introduce a quantity $X(r)$, controlling the radial stability of a geodesic. $X(r) > 0$ – stable orbits. $X(r) < 0$ – unstable orbits.

$X(r)$:Kerr solution and I1-I5 solutions $(a = 0.6)$



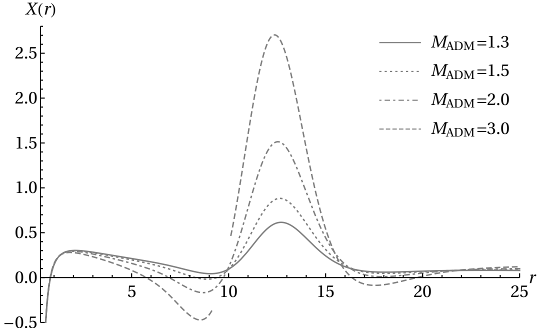
$\Gamma = 4/3,\ m = 1,\ a = 0.6,\ r_1 = 2.9,\ r_2 = 18.1$

$X(r)$:Kerr solution and I1-I5 solutions $(a = 0.6)$ – Near ISCO



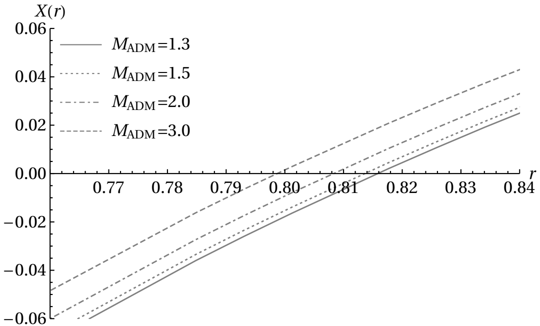
$$\Gamma = 4/3,\ m = 1,\ a = 0.6,\ r_1 = 2.9,\ r_2 = 18.1$$

$X(r)$: I9-I12 solutions ($a = 0.96$)



$\Gamma = 4/3,\ m = 1,\ a = 0.96,\ r_1 = 6.7,\ r_2 = 20.8$

$X(r)$: I9-I12 solutions ($a = 0.96$) – Near ISCO



$\Gamma = 4/3,\ m = 1,\ a = 0.96,\ r_1 = 6.7,\ r_2 = 20.8$

ISCO Summary

TABLE III. Locations of the Innermost Stable Circular Orbit (ISCO) for a collection of numerical solutions. From left to right the columns report: solution number, the polytropic exponent Γ, the black-hole spin parameter a, the mass of the black hole M_{BH}, the inner coordinate radius of the torus r_1, the inner circumferential radius of the torus $r_{c,1}$, the outer coordinate radius of the torus r_2, the outer circumferential radius of the torus $r_{c,2}$, the coordinate radius of the ISCO r_{ISCO}, the circumferential radius of the ISCO $r_{c,ISCO}$. All solutions were obtained assuming $m = 1$. For solutions I1–I8, the spatial grid resolution around the ISCO is $\Delta r \approx 0.015$; for solutions I9–I12 it is $\Delta r \approx 0.0063$, and in the case of solutions I13–I20 it is $\Delta r \approx 0.022$. For reference: the circumferential radius of the ISCO in the Kerr spacetime with $m = 1$, $a = 0.6$ is $r_{c,ISCO} \approx 3.90$; for $m = 1$, $a = 0.96$ it is $r_{c,ISCO} \approx 2.31$, and for $m = 1$, $a = 0.1$ it is $r_{c,ISCO} \approx 5.67$.

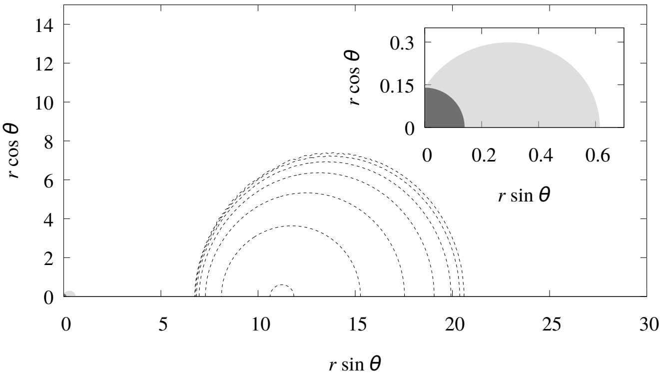
No.	Γ	a	M_{BH}	M_{ADM}	r_1	$r_{c,1}$	r_2	$r_{c,2}$	r_{ISCO}	$r_{c,ISCO}$
I1	4/3	0.60	1.003	1.050	2.90	4.05	18.10	19.18	2.80	3.95
I2	4/3	0.60	1.005	1.100	2.90	4.07	18.10	19.23	2.81	3.98
I3	4/3	0.60	1.011	1.200	2.90	4.12	18.10	19.35	2.86	4.07
I4	4/3	0.60	1.017	1.300	2.90	4.16	18.10	19.46	2.92	4.18
I5	4/3	0.60	1.023	1.400	2.90	4.21	18.10	19.58	3.00	4.31
I6	4/3	0.60	1.029	1.500	2.90	4.26	18.10	19.69	3.09	4.46
I7	4/3	0.60	1.011	1.200	3.01	4.22	18.10	19.35	2.86	4.07
I8	4/3	0.60	1.010	1.200	3.51	4.72	18.10	19.35	2.84	4.05
I9	4/3	0.96	1.001	1.300	6.70	7.97	20.80	22.18	0.82	2.32
I10	4/3	0.96	1.001	1.500	6.70	8.10	20.80	22.42	0.81	2.32
I11	4/3	0.96	1.003	2.000	6.70	8.43	20.80	23.02	0.81	2.34
I12	4/3	0.96	1.005	3.000	6.70	9.14	20.80	24.25	0.80	2.39
I13	4/3	0.10	1.004	1.050	4.81	5.89	20.80	21.87	4.65	5.73
I14	4/3	0.10	1.009	1.100	4.81	5.91	20.80	21.93	4.70	5.81
I15	4/3	0.10	1.018	1.200	4.81	5.97	20.80	22.04	4.81	5.97
I16	4/3	0.10	1.027	1.300	4.81	6.03	20.80	22.16	4.96	6.19
I17	5/3	0.10	1.004	1.050	4.81	5.88	20.80	21.87	4.65	5.73
I18	5/3	0.10	1.008	1.100	4.81	5.91	20.80	21.93	4.70	5.80
I19	5/3	0.10	1.016	1.200	4.81	5.96	20.80	22.05	4.76	5.91
I20	5/3	0.10	1.024	1.300	4.81	6.01	20.80	22.17	4.87	6.08

An ergoregion (or traditionally an ergosphere) is defined as a region outside the black hole horizon, where the Killing vector ξ^μ, which is asymptotically timelike (i.e., $\xi_\mu \xi^\mu < 0$), becomes spacelike ($\xi_\mu \xi^\mu > 0$). For the Killing vector $\xi^\mu = (1, 0, 0, 0)$ and metrics (1) this actually means that

$$g_{\mu\nu} \xi^\mu \xi^\nu = g_{tt} = -\alpha^2 + \psi^4 r^2 \sin^2 \theta \beta^2 > 0. \tag{9}$$

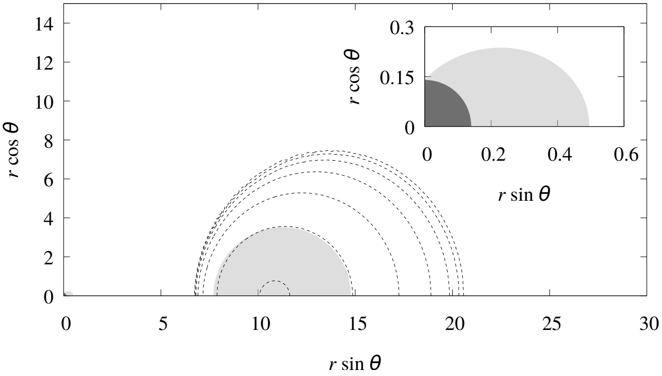
A surface defined by the condition $\xi_\mu \xi^\mu = 0$ is usually called an ergosurface.

E1 – Single ergoregion around black hole



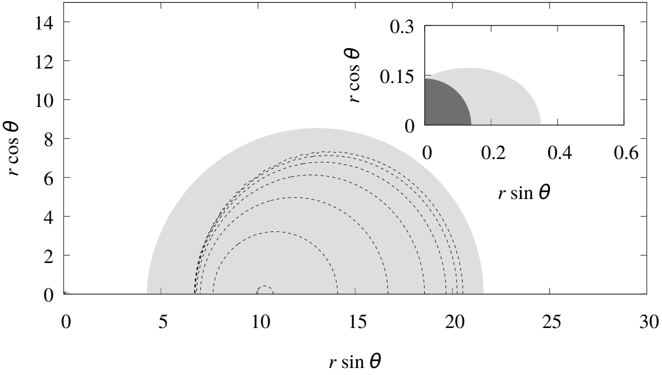
The ergoregion is marked in grey. Black color marks the region inside the horizon. Broken density isolines correspond to $\rho = 8 \times 10^{i}$, $i = -10, -9, \ldots, -4$.

E2 – Double ergoregion: one around black hole and toroidal one insaid the torus (disk)



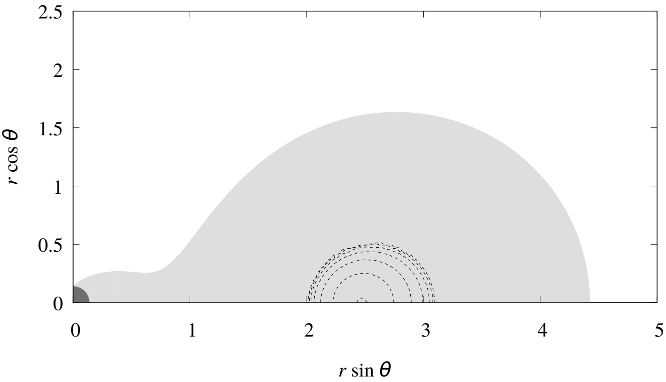
The ergoregions are marked in grey. Black color marks the region inside the horizon. Broken density isolines correspond to $\rho = 6 \times 10^i$, $i = -10, -9, \ldots, -4$.

E3 – Double ergoregion: one around black hole and toroidal one inclusive torus



The ergoregions are marked in grey. Black color marks the region inside the horizon. Broken density isolines correspond to $\rho = 4 \times 10^i$, $i = -10, -9, \ldots, -4$.

E4 – Single ergoregion containing both: black hole and torus



The ergoregion is marked in grey. Black color marks the region inside the horizon. Broken density isolines correspond to $\rho = 5 \times 10^{i}$, $i = -8, -7, \ldots, -2$.

Linear stability: Seguin's Condition

- We apply Seguin's condition (necessary condition for the dynamical linear stability against axially symmetric perturbations) to ours solutions.
- The results support a suggestion that solutions belonging to the massive bifurcation branches are dynamically unstable.

Conclusion

- In the space of solutions the parametric bifurcation occurring, for given maximal density there exist two solution with different ADM mass.
- The Presence of a torus can influence location of the ISCO. Circumferential radius (geometrical radius) of the ISCO grows with the increasing ADM mass of the system.
- A massive disc can have its own toroidal ergoregion. However such configurations are probably unstable (all of the obtained solutions of this type are unstable according to Seguin's linear stability criterion).