Relativistic Kinematics

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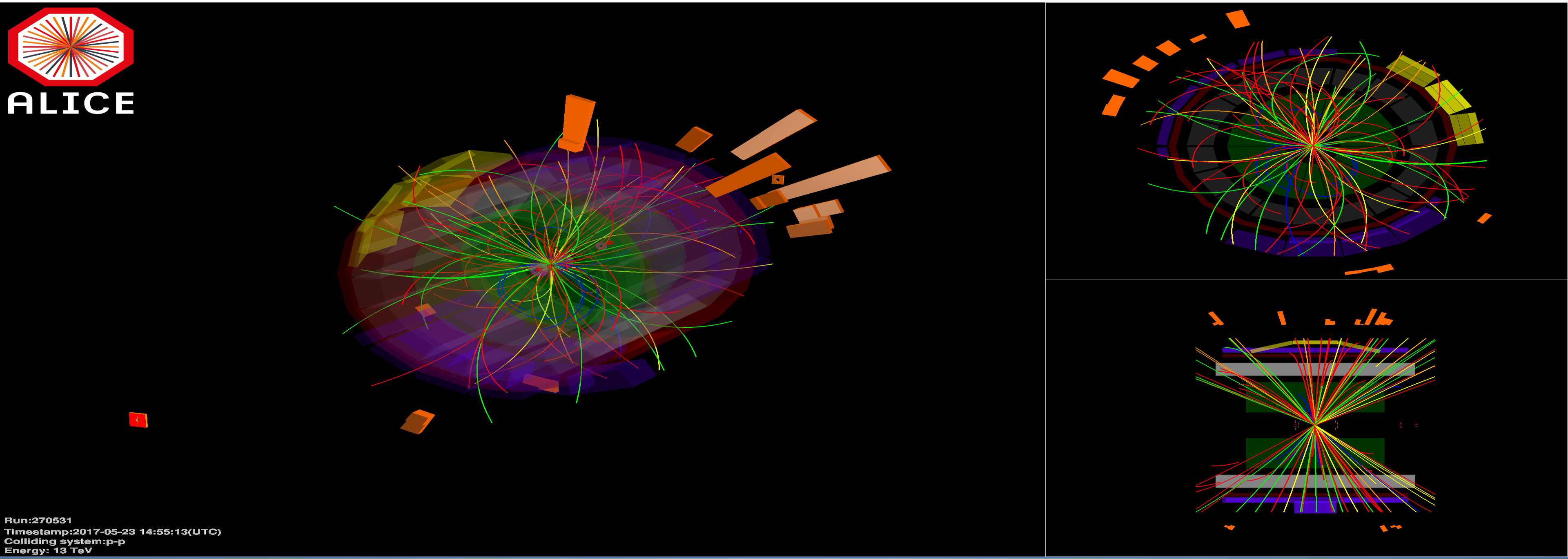
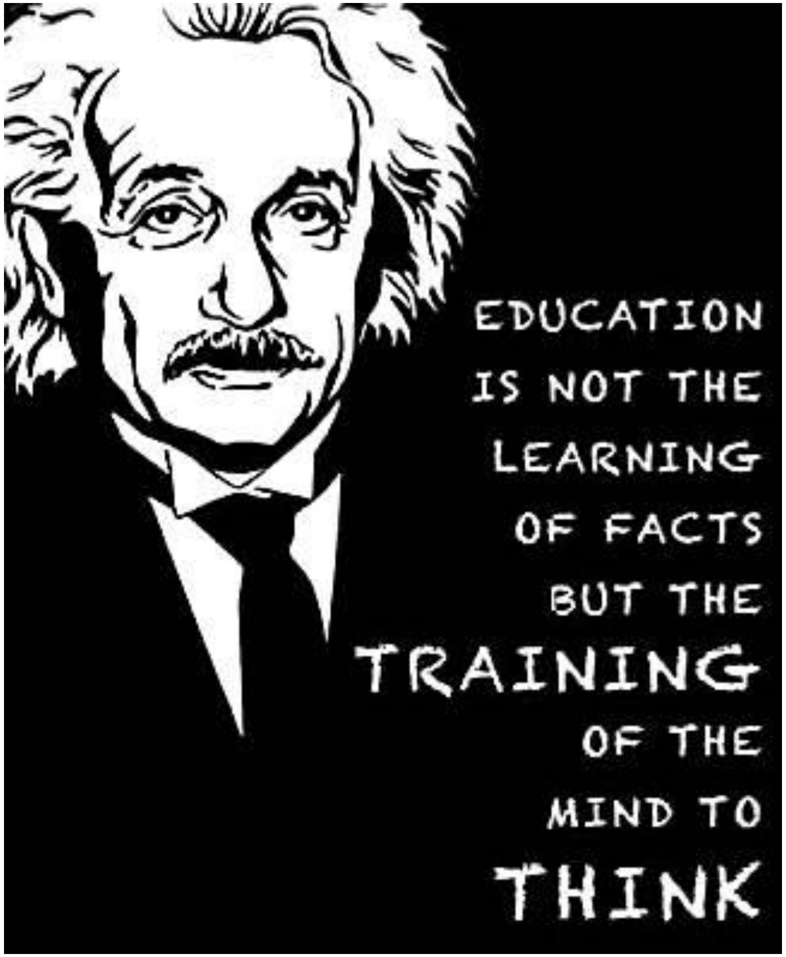




Human existence is based on two pillars: compassion and curiosity.

Compassion without curiosity is ineffective. Curiosity without compassion is inhuman.

-Victor F. Weisskopf

References

- Relativistic Kinematics for Beginners (With Monte Carlo)- Raghunath Sahoo and Basanta K. Nandi (CRC Press- 2020)
- Relativistic Kinematics- Raghunath Sahoo (arXiv:1604.02651)
- Introduction to High Energy Heavy-Ion Collisions- C.Y. Wong
- Ultra-relativistic Heavy-Ion Collisions- R. Vogt
- Introduction to High Energy Physics- D.H. Perkins
- Introduction to Elementary Particles- D.J. Griffiths
- Relativistic Kinematics- R. Hagedorn
- Lecture Notes in Physics (Global Properties in Heavy-Ion Collisions)- M. Kleimat, R. Sahoo, T. Schuster and R. Stock
- Hadrons and Quark-Gluon Plasma- Rafelski and Latessier
- Introduction to Heavy-Ion Collisions- L.P. Tsernai

Disclaimer/Acknowledgement

- This presentation is purely for academic training purpose
- Most of the slides are based on our upcoming book, given in reference-1
- Many plots and ideas come from the excellent textbooks listed on the previous slide
- Lots of good material can nowadays be found by just *Googling*. The authors are then not always recognizable/acknowledgeable.


Welcome
to
Geneva, Switzerland

CERN


World's Most Powerful Accelerator:
The Large Hadron Collider
27 km circumference
~ 100 m underground
Design Energy:
14 TeV (pp), 5.5 TeV (Pb-Pb)
CERN
Lake Geneva
Jura mountains
CMS
LHCb
Studying Heavy Ions
ALICE
ATLAS

Why High Energies?

- Particle physics deals with "elementary particles".
- Elementary means the particles have no substructure or they are point-like objects.

Note: Elementariness depends on the resolution of the probe used to investigate the structure/sub-structure.

Note: Once "atom" was elementary

The word *atom* is derived from the Greek word *atomos*, which means "uncuttable" or Indivisible – no substructure- elementary !

Assuming the probing beams consist of point-like particles the resolution is limited by the de Broglie wavelength of these beam particles given by:

$$\lambda = \frac{h}{p}$$

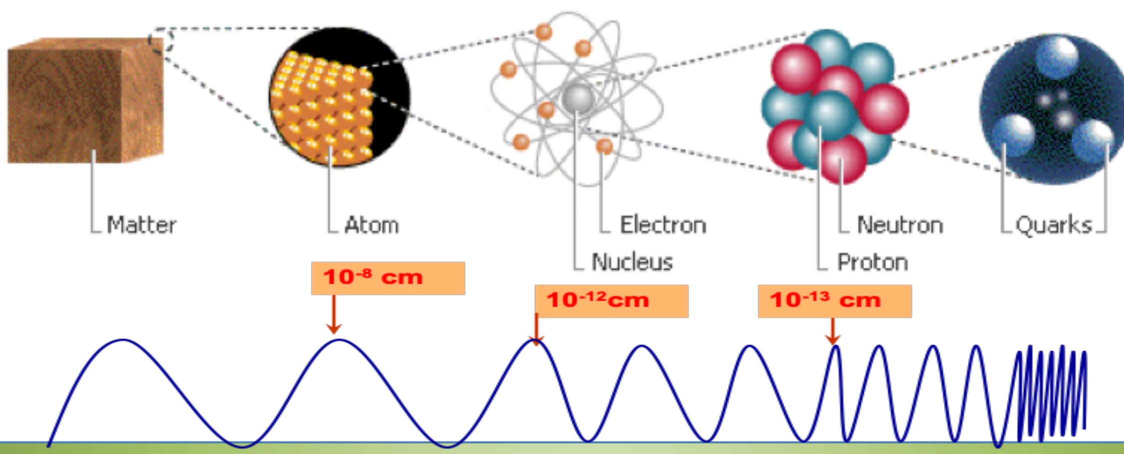
where, p = beam momentum and h = Planck's constant.

Why High Energies?

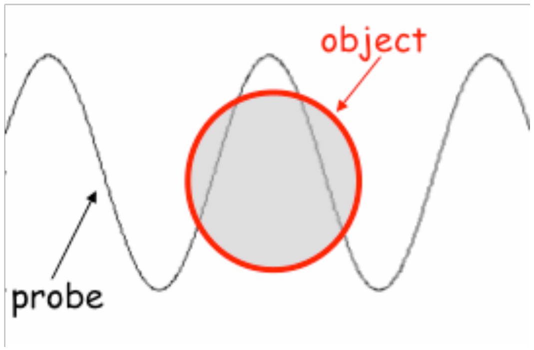
$$\lambda = \frac{h}{p}$$ where, p = beam momentum and h = Planck's constant.

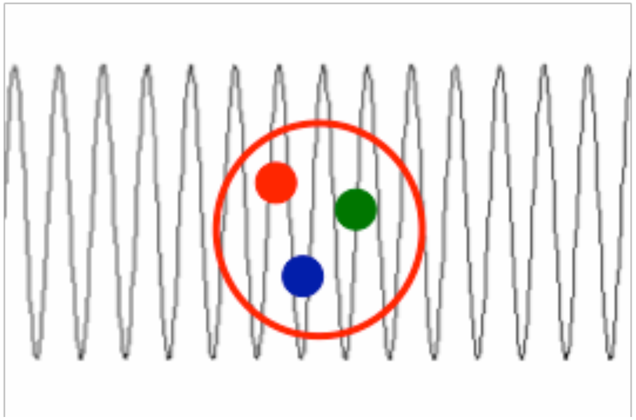
Beams of high momentum have shorter de Broglie wavelength and can thus have higher resolution.

Example: To probe a dimension of 1 Fermi (= 10^{-15} meters) (say the inner structure of a nucleon, the charge radius of proton being ~ *0.843 fm*), we need to use a beam momentum of around 1.47 GeV.



Why High Energies?





Low energy probe: larger wavelength

High energy probe: smaller wavelength

In addition, remember Einstein's formula: $E = mc^2$

High energy helps us to produce particles of higher masses (massive gauge bosons, Higgs, other exotic particles/resonances) in nature's way !

The essential mathematic tool in high energy physics:

Special Theory of Relativity.

Special Theory of Relativity

Why special theory of relativity in high energy physics?

In HEP, particles are treated relativistically, because $E \approx pc \gg mc^2$

Relativistic factor, γ = E/m

Note, for a particle at rest, γ =1

- when close to the speed of light (v ~ c), there is no use to talk about velocities
- two particles of similar velocities may have very different γ, e.g. a pion and electron of 1 GeV/c

Special Theory of Relativity

Caveats:

- ✓ Space and time can't be treated independently as is done in Newtonian mechanics.
- ✓ Physical objects that were treated as an independent 3-component vector and a scalar in non-relativistic physics mix in high energy phenomena *e.g.* Energy and Momentum, Time and Space Coordinates.
- ✓ Combined to form a 4-component Lorentz vector that transforms like a time and space coordinate.

For a consistent and unified treatment, we rely on Einstein's Special Theory of Relativity (1905) having the following two underlying principles..

Special Theory of Relativity

Principle-1: Invariance of velocity of light:

Velocity of light always remains as the constant c in any *inertial frame*.

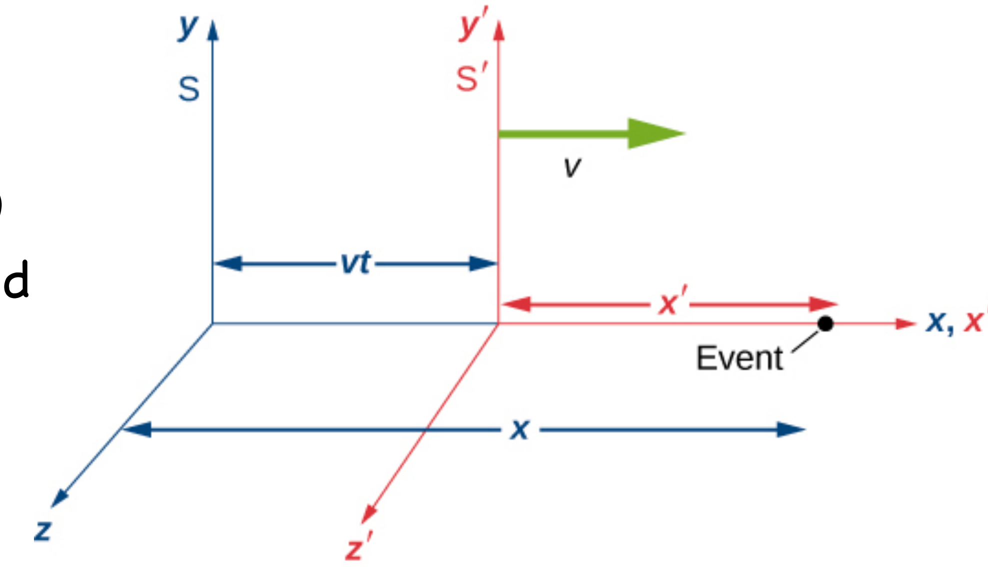
Principle-2: Relativity Principle:

This requires covariance of the equations: physical law should keep its form invariant in any inertial frame of reference.

Note:

- The principle of relativity applies to Galilean transformation and is valid in Newtonian mechanics as well.
- The invariance of velocity of light necessitates Lorentz transformation in changing from one inertial system to another that are moving relative to each other with constant speed.

Lorentz Transformation



Consider a Lorentz boost in x-direction. A particle at (t, x, y, z) in a coordinate frame S is boosted to (t', x', y', z') with velocity v.

This is equivalent to changing to another coordinate frame S' which is moving in the x-direction at velocity $-v$.

S' is assumed to coincide with S at $t = t' = 0$. The the two coordinates are related by the following equations:

Lorentz Transformation

$$t \rightarrow t' = \frac{t + \left(v/c^2\right)x}{\sqrt{1-\left(v/c\right)^2}} \Rightarrow x^{0'} = \gamma\left(x^0 + \beta x\right),$$

$$x \rightarrow x' = \frac{x + vt}{\sqrt{1-\left(v/c\right)^2}} \Rightarrow x^{1'} = \gamma\left(\beta x^0 + x\right),$$

$$x^{2'} = x^2$$

$$x^{3'} = x^3$$

where,

$$\beta = v/c, \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

In matrix form:

$$\begin{bmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{bmatrix} = \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}$$

Lorentz Boost Factor

$$\beta = v/c, \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Relativity Equations

Change in Length

$$l = l_0 \sqrt{1 - \frac{v^2}{c^2}}$$

v = velocity of object
c= velocity of light

Where l = observed length
l_0 = rest length

Change in Time

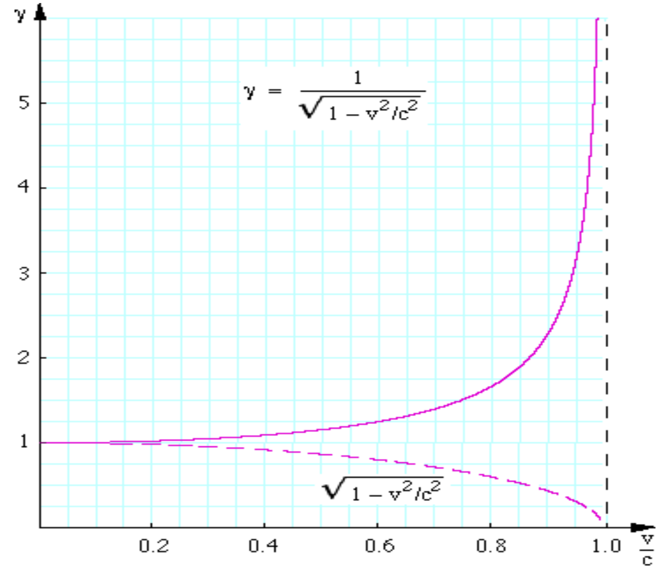
$$t = t_0 \sqrt{1 - \frac{v^2}{c^2}}$$

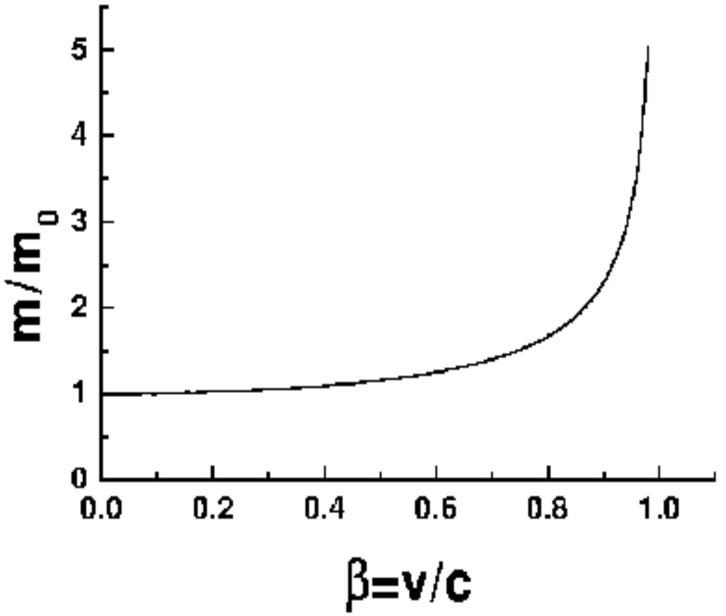
Where t = observed time
t_0 = rest time

Change in Mass

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Where m = observed mass
m_0 = rest mass

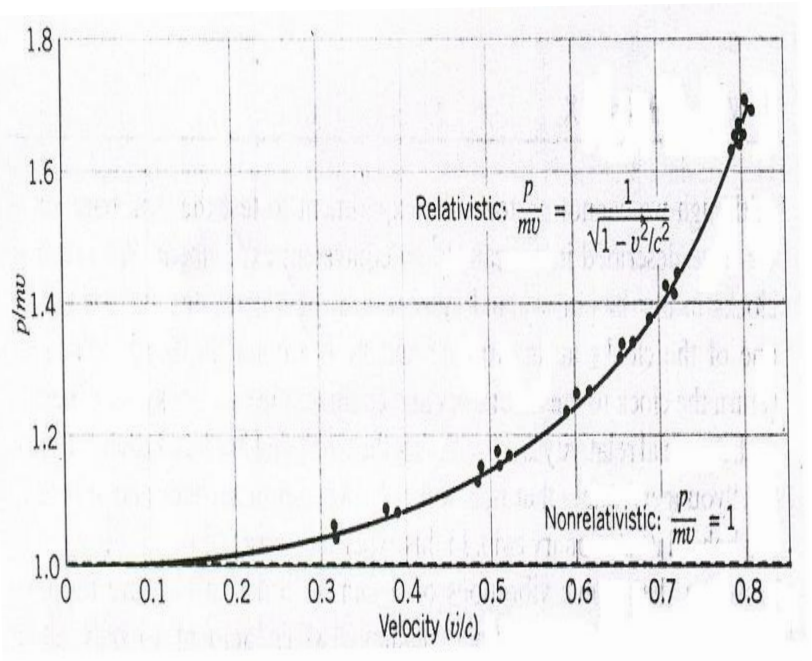


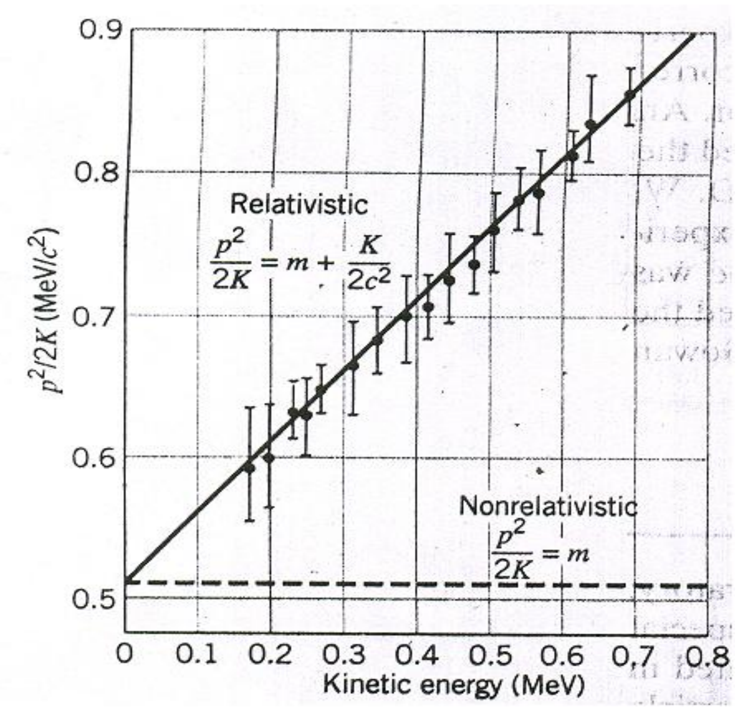


Why should I get convinced on STR?

Experimental Proof of STR

Measurement of velocity and momentum of electron in Nuclear Radioactive decay.

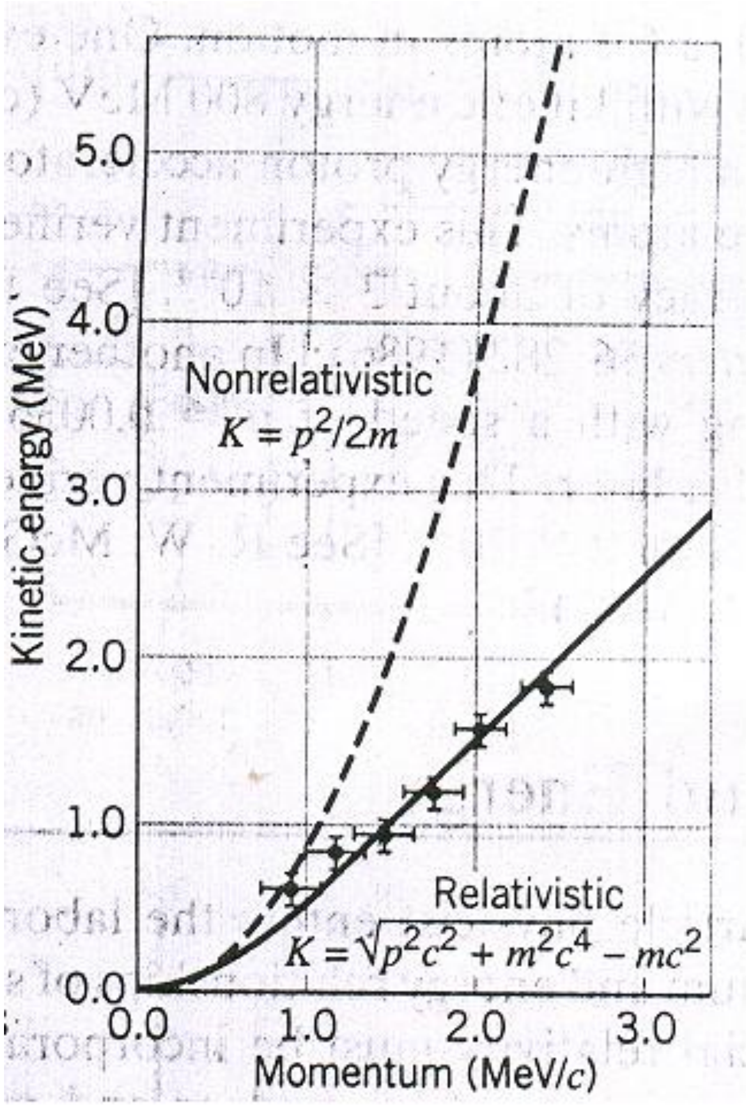


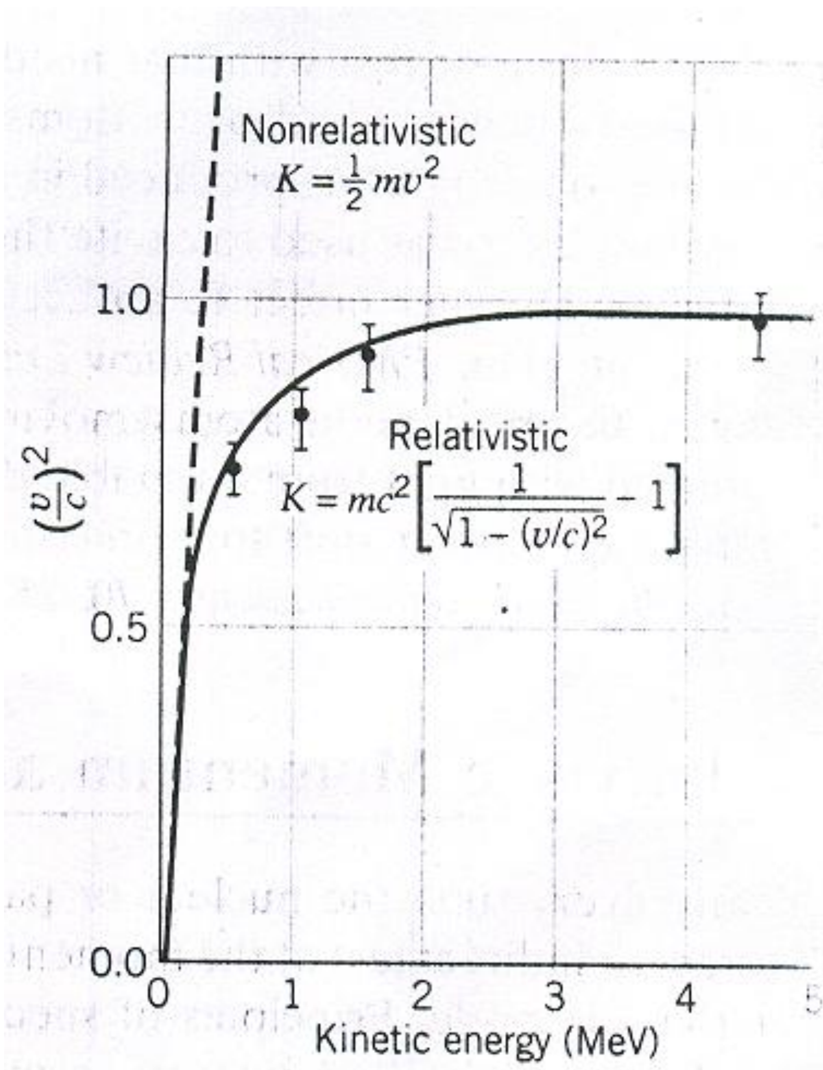


$$E = K + mc^2$$
$$\Rightarrow \sqrt{p^2c^2 + m^2c^4} = K + mc^2$$
$$\Rightarrow p^2c^2 + m^2c^4 = (K + mc^2)^2$$
$$\Rightarrow p^2c^2 + m^2c^4 = K^2 + m^2c^4 + 2Kmc^2$$
$$\Rightarrow p^2c^2 = K^2 + 2Kmc^2$$
$$\Rightarrow \frac{p^2}{2K} = m + \frac{K}{2c^2}$$

Why should I get convinced on STR?

Experimental Proof of STR





The Proper Time (τ)

It is the time an observer feels/records in its own rest frame.

The proper time $d\tau \equiv dt\sqrt{1-\beta^2}$ is a Lorentz invariant scalar.

Proof:

$$ds^2 = (cdt)^2 - dx^2 - dy^2 - dz^2$$

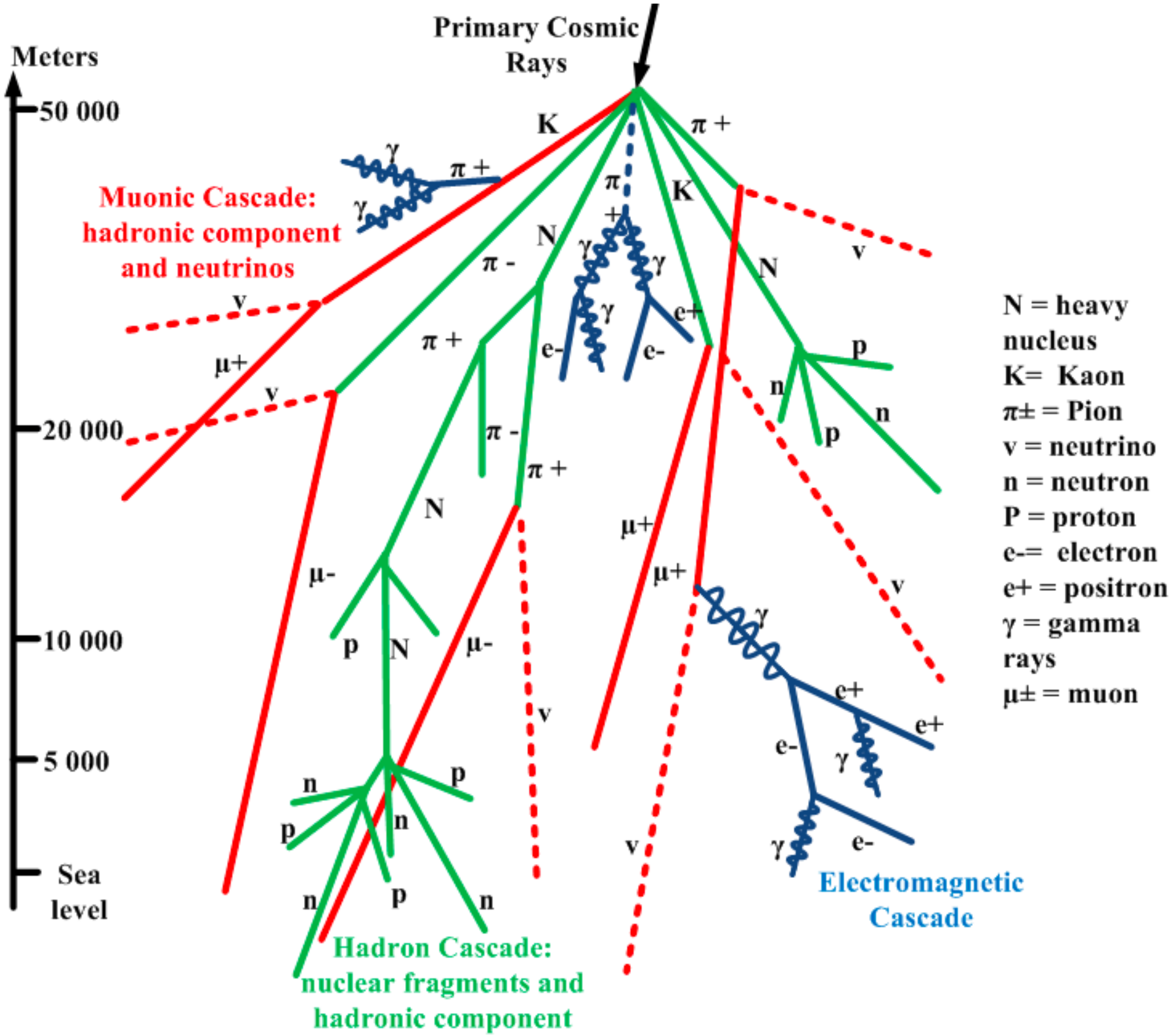
$$= c^2 dt^2 \left[1 - \frac{dx^2}{dt^2} - \frac{dy^2}{dt^2} - \frac{dz^2}{dt^2}\right]$$

$$= c^2 dt^2 \left(1-\beta^2\right)$$

$$= (cd\tau)^2$$

is Lorentz invariant by definition.

The STR and Cosmic Muons

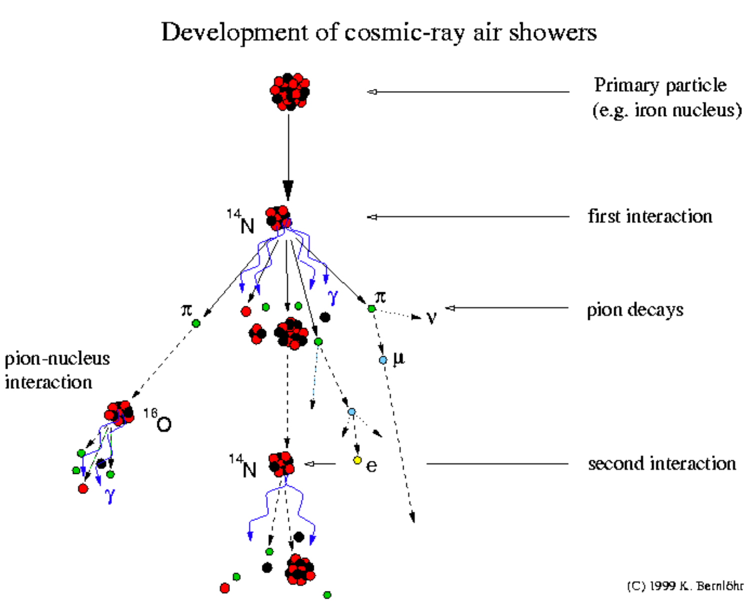


The STR and Cosmic Muons

Problem-1.0: Cosmic ray muons are produced high in the atmosphere (at 50 Km, say) and travel towards the earth at a speed nearly equal to light (0.99998c say) without colliding with anything on the way down.

(a) Given the lifetime of the muon (2.2×10^{-6} sec), how far would it go before disintegrating, according to pre-relativistic physics? Would the muons make it to the ground level?

(b) Now answer the same question using relativistic physics. (Because of time dilation, the muons last longer, so they travel farther.)



Home Assignment-1.0: $\pi^- \rightarrow \mu^- + \bar{\nu}_\mu$

Lifetime of pion is much shorter (2.6×10^{-8} sec). Assuming that the pions have the same speed (0.998c), will they reach the ground level?

The STR and Cosmic Muons

Solution to Problem-1.0: $\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu$

Pre-relativistic:

The distance travelled by muon: d = vt ≈ (3 x 10^8 m/sec.) x (2x10^{-6} sec)
= 600 meter

This is less than 50 km

⟹ muons don't reach the earth before decaying.

Relativistic: The muons live longer in the earth frame by a factor γ:

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \simeq 160$$

The correct distance travelled in earth frame is: d = v (γT) ≈ 100 km.

[The proper time interval (in observer's own rest frame): $\Delta\tau = \Delta t/\gamma$]

The fact that muons reach the laboratory in predicted abundances (while the naïve *d = vT* reasoning would predict that we shouldn't see any) is one of the many experimental tests that support the special theory of relativity.

The Choice of Units in High Energy Physics

We know: $$x^2 = c^2t^2 - \vec{x}^2$$

$$p^2 = m_0^2 c^2$$

Note that velocity of light (c) appears directly in these and many formulae.

The de Broglie relation between 4-momentum and wave vector of a particle is: $E = \hbar\omega$

In 4-vector notation, $P = \hbar K$, $where\ \ P = \left\{\frac{E}{c}, p\right\}, K = \left\{\frac{\omega}{c}, k\right\}$

Choose a system of unit in which, $\boxed{\hbar = c = 1}$

Now the relativistic formula for energy, $E^2 = p^2c^2 + {m_0}^2c^4$

in the new system of units becomes, $\boxed{E^2 = p^2 + {m_0}^2}$

In particle physics, the unit of energy is GeV *(1 GeV = 10⁹ eV)*.
This is motivated by the fact that the rest mass of proton is m_p~ 1 GeV.

The Choice of Units in High Energy Physics

This gives rise to mass (m), momentum (mc), energy (mc^2) in GeV.

Length ($\frac{\hbar}{mc}$) and time ($\frac{\hbar}{mc^2}$) in GeV^{-1}.

In this system of units, we can obtain

$$\boxed{1\ sec = 1.52 \times 10^{24}\ GeV^{-1}} \boxed{1\ meter = 5.07 \times 10^{15}\ GeV^{-1}}$$

$$\boxed{1\ fermi \equiv 1\ fm = 10^{-15} m\ =\ 5.07\ GeV^{-1}}$$

$$\boxed{1\ fm = 3.33 \times 10^{-23}\ sec}$$

$$\boxed{197\ MeV = 1\ fm^{-1}}$$

The advantage of using natural unit in particle physics:

- We deal with strong interaction
- Lifetime is ~ 10^{-23} *sec*
- Decay length of particles can be better expressed in terms of *fermi*
- Energy scale is GeV/TeV

Four Vectors

The position-time 4-vector: x^{μ}, $\mu = 0, 1, 2, 3$; with
$x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$.

$$I \equiv \left(x^0\right)^2 - \left(x^1\right)^2 - \left(x^2\right)^2 - \left(x^3\right)^2$$

$$= \left(x^{0'}\right)^2 - \left(x^{1'}\right)^2 - \left(x^{2'}\right)^2 - \left(x^{3'}\right)^2$$

I is called the 4-dim length element, which is Lorentz Invariant (LI):
A quantity having the same value in all inertial frame.

This is like $r^2 = x^2 + y^2 + z^2$ being invariant under spatial rotation.

Four Vectors

I could be written in the form of a sum: $I = \sum_{\mu=0}^{3} x^\mu x^\mu$

To take care of the negative signs, let's define a metric $g_{\mu\nu}$ such that

$$g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Temporal component is positive and spatial component is negative: convention, can be vice-versa as well.

Now $I = g_{\mu\nu} x^\mu x^\mu$

Define covariant 4-vector x_μ (index down): $x_\mu = g_{\mu\nu}\, x^\nu$

x^μ (index up) is called *"contravariant 4-vector"*

With the above definitions, now $I = x_\mu x^\mu = x^\mu x_\mu$

Four Vectors

$$I = x_\mu x^\mu = x^\mu x_\mu$$

To each contravariant 4-vector a^μ, a covariant 4-vector could be assigned and vice-versa.

$$a^\mu = g^{\mu\nu} a_\nu$$

$$a_\mu = g_{\mu\nu} a^\nu$$

$g^{\mu\nu}$ are the elements in g^{-1}, since $g^{-1} = g$, $g^{\mu\nu} = g_{\mu\nu}$

Given any two 4-vectors, a^μ and b^μ,

$$a^\mu b_\mu = a_\mu b^\mu = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3$$

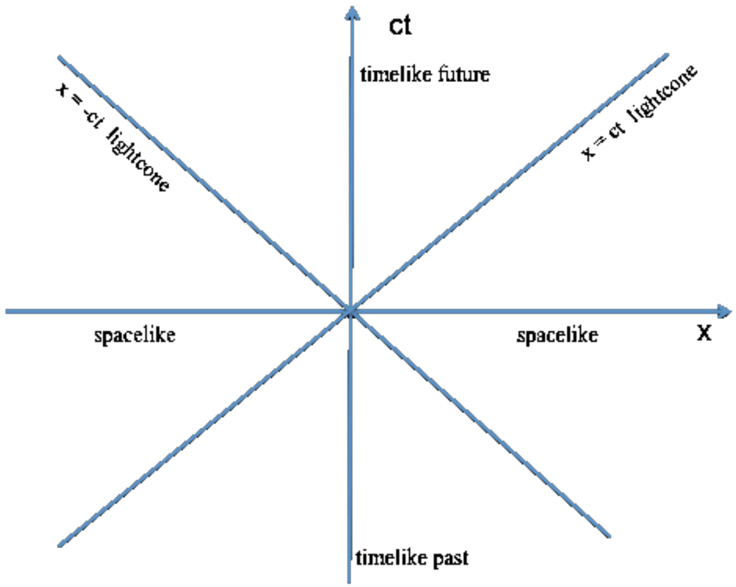
is LI.

This is called "4-vector scalar product".

Four Vectors

Remember Einstein's summation convention (repeated Greek indices are to be summed),

$$a.b \equiv a_\mu b^\mu$$
$$= a^0 b^0 - \vec{a}.\vec{b}$$



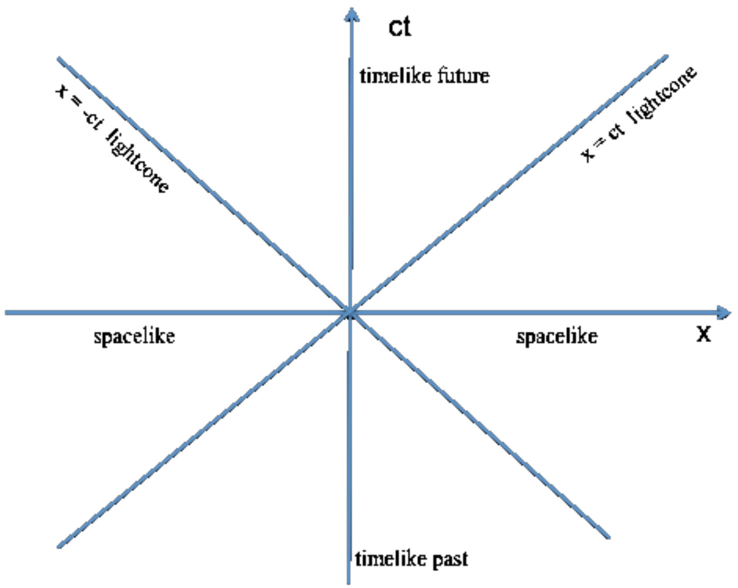
$$a^2 \equiv a.a = (a^0)^2 - \vec{a}^2$$

temporal component spatial component

$a^2 > 0$: a^μ is called *time-like*

- Events are in the forward light-cone
- They appear later than the origin *O*
- Events in the backward light-cone appear earlier than O
- Only events in the backward light-cone can influence *O*.
- And *O* can have influence only on the events in forward cone

Four Vectors



$$a^2 \equiv a.a = (a^0)^2 - \vec{a}^2$$

temporal component　　spatial component

$a^2 < 0$: a^μ is called *space-like*

- Events are called space-like events and there is no interaction with *O*

$a^2 = 0$: a^μ is called *light-like*

- Connects all those events with the origin which can be reached by a light signal

Four-Velocity

The velocity of a particle is given by:

$$\vec{v} = \frac{d\vec{x}}{dt}$$

$d\vec{x}$: distance travelled in the lab. frame
dt : time measured in the same frame

Proper velocity of the particle is:

$$\vec{\eta} = \frac{d\vec{x}}{d\tau}$$

$d\vec{x}$: distance travelled in the lab. frame
$d\tau$: is the proper time

One can show that: $\boxed{\vec{\eta} = \gamma\vec{v}}$

Note that it is easy to work with the proper velocity as only $d\vec{x}$ transforms under LT.

Now $$\eta^\mu = \frac{dx^\mu}{d\tau} \Rightarrow \eta^0 = \frac{dx^0}{d\tau} = \frac{d(ct)}{\frac{1}{\gamma}dt} = \gamma c$$

Four-Velocity

Hence,

$$\boxed{\eta^{\mu} = \gamma(c,\ v_x,\ v_y,\ v_z)}$$ This is called the *proper velocity 4-vector.*

→As per our convention, spatial component brings up negative sign for covariant tensors.

We can show that: $\eta^{\mu}\eta_{\mu} = c^2$

which is L.I. and also proves that 4-vector scalar product is L.I.

Energy-Momentum Four Vector

We know: *momentum = (mass) x (velocity)*

Velocity could be *"ordinary velocity"* or the *"proper velocity"*.

Classically, both are same.

In relativity, momentum is the product of mass and proper velocity *i.e.*

$$\vec{p} \equiv m\vec{\eta}$$

$$p^{\mu} = m\eta^{\mu}$$

The spatial component of p^{μ} constitutes the (relativistic) momentum vector:

$$\vec{p} = \gamma m\vec{v} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}} \qquad p^0 = \gamma mc$$

Relativistic energy, E: $$E \equiv \gamma mc^2 = \frac{mc^2}{\sqrt{1-v^2/c^2}}$$

Hence, $$p^0 = \frac{E}{c}$$

Energy-Momentum Four Vector

The energy-momentum 4-vector: $\boxed{p^\mu = \left(\frac{E}{c},\ p_x,\ p_y,\ p_z\right)}$

One can show that:

$$\boxed{p^\mu p_\mu = m^2} \Rightarrow \boxed{E^2 = \vec{p}^2 + m^2}$$

(in natural units)

$p^2 = m^2 > 0$: Ordinary massive particle

$p^2 = m^2 = 0$: Massless particles like photons, gravitons, etc.

$p^2 < 0$: Tachyon or virtual particles

$p^\mu = 0$: Vacuum

Energy-Momentum Four Vector

Note: the relativistic equations $\vec{p} = \gamma m \vec{v}$ and $E = \gamma m$ do not hold good for massless particles and m=0 is only allowed, if the particle travels with the speed of light.

Problems: 1.1

A pion at rest decays into a muon plus a neutrino. What is the speed of the muon? Use the conventional method and 4-vector method, which one do you like?

Problems: 1.2 $p + p \rightarrow p + p + p + \bar{p}$

What is the threshold energy of the reaction.
Use 4-vector method.

Note the production of antibaryons need more threshold energy !

Why Rapidity?

Successive Lorentz boosts in the same direction is represented by a single boost, where the transformation velocity is given by:

$$\beta'' = |v/c|'' = \frac{\beta + \beta'}{1 + \beta\beta'}$$

Proof:

Assume that velocity *v´* in frame *L* is observed as *v˝* in frame *L˝*, where the frame *L´* is travelling in the *x*-direction with *−v* in frame *L*.

The coordinates *(t´, x¹´)* are expressed in terms of *(t, x¹)* using the usual Lorentz transformation equations.

Omitting coordinates which are not suffered by LT,

$$x^{0'} = \gamma(x^0 + \beta x^1)$$

$$x^{1'} = \gamma(\beta x^0 + x^1)$$

$$\beta' = \frac{v'}{c} = \frac{dx^1}{dx^0}$$

then

Why Rapidity?

$$\beta'' = \frac{dx^{1'}}{dx^{0'}} = \frac{\gamma(\beta dx^0 + dx^1)}{\gamma(dx^0 + \beta dx^1)}$$

$$\Rightarrow \beta'' = \frac{\beta + \beta'}{1 + \beta\beta'}$$

- The velocity is not an additive quantity.
- It is non-linear in successive transformations.

Why Rapidity?

Here comes the need of "rapidity" to circumvent this drawback, by defining

$$\beta = tanh\ y \quad \text{or} \quad y = \frac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right)$$

We can show that rapidity is an additive quantity in successive LT, *i.e.*

$$\boxed{y'' = y + y'}$$

Using rapidity, a Lorentz transformation with finite *y*, can be decomposed into N successive transformations with rapidity:

$$\Delta y = y/N$$

Solving β, γ in terms of *y*, we have

$$\beta = \tanh y, \quad \gamma = \cosh y, \quad \beta\gamma = \sinh y$$

Lorentz boost can now be written as:

$$x^{0'} = (\cosh y)x^0 + (\sinh y)x^1$$

$$x^{1'} = (\sinh y)x^0 + (\cosh y)x^1$$

Why Rapidity?

Comparing this with rotation in *x-y* plane:

$$x' = x\ cos\ \theta - y\ sin\ \theta,$$

$$y' = x\ sin\ \theta + y\ cos\ \theta$$

We can obtain the former equations from the later by substituting:

$$\theta \rightarrow -iy$$

$$x \rightarrow ix^0$$

$$y \rightarrow x^1$$

Lorentz boost (in *x*-direction) is formally a rotation by an angle *(-iy)* in the *x* and imaginary time *(ix^0)* plane.

Rapidity

$$y = \frac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right)$$

Experimental Considerations:

In high-energy collider experiments, the secondary particles produced from the interaction, are boosted in z-direction (along the beam axis). The boosted angular distribution is better expressed as rapidity distribution.

At high-energies, each particle has $E \sim pc$, $p_{//} = p\cos\theta$, and its rapidity is approximated by so-called pseudo-rapidity:

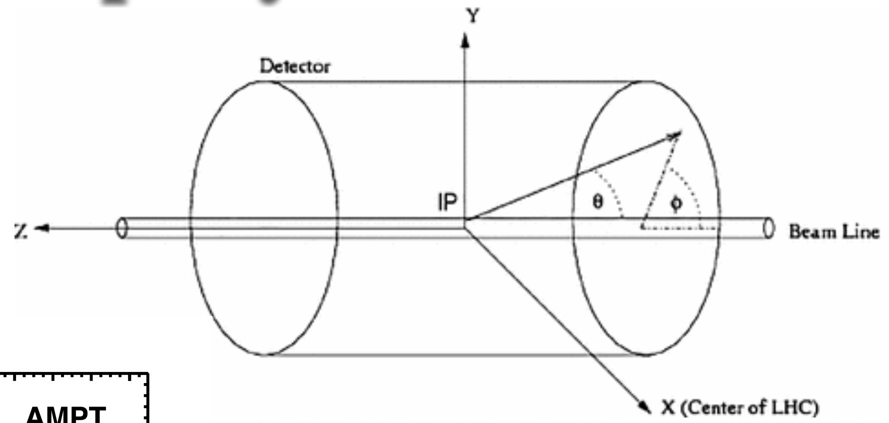
$$\eta' = \frac{1}{2}\, ln\left(\frac{1+\beta_{||}}{1-\beta_{||}}\right) = \frac{1}{2}\, ln\left(\frac{E+p_{||}c}{E-p_{||}c}\right) \sim -ln\; tan(\theta/2)$$

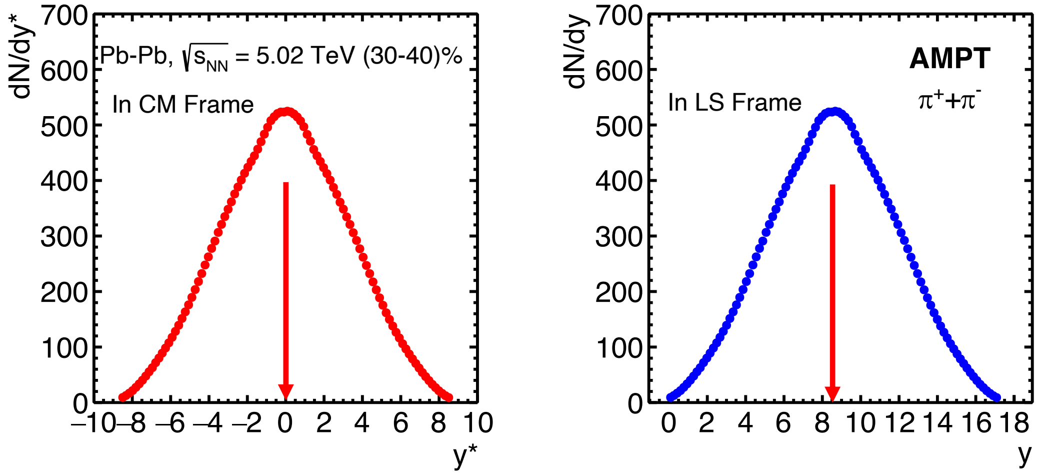
This fact is taken into account in designing detectors, which are divided into modules that span the same solid angle in the η-Φ (azimuthal angle) plane.

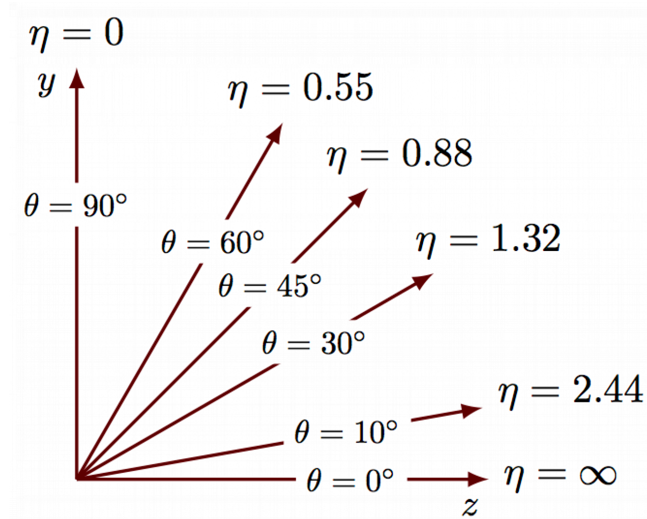
Pseudorapidity

At very high energies, when $p >> m$

$$\boxed{y \approx \eta = -ln\left[tan(\theta/2)\right]}$$







For a symmetric collider, $y_{\rm cm} = \frac{1}{2}(y_a + y_b)$

$$\boxed{y_{\rm beam} = \pm \ln(\sqrt{s_{\rm NN}}/m_p) = \pm y_{\rm max}}$$

$y_{cm} = y_b = 8.52$

Important:
shape invariance of the spectra

Facts and Mysteries

$$\beta = v/c, \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad E = \gamma m$$

- Let's say accelerating a charged particle from *0.1c - 0.2 c* takes ΔE energy, what about accelerating the same particle from *0.8c-0.9*c? Will it take the same energy, ΔE ?

- Why to dig million tons of earth to collide tiny particle?

- Why is LHC(LEP) tunnel 27 Km long?

- CERN is planning to go for 100 Km circumference (FCC) ring to increase the energy to 7-10 times the present one. Will the beam particles reach velocity of light?

Before we move to CM and LS frames, try to decay the pion which was earlier at rest and now having some finite momentum ....solve the problem and realize that working on a frame with net momentum = 0, has many advantages i.e. the CM frame.

The Center of Momentum (CM) Energy and Velocity

- Consider a Lorentz system- call it *Laboratory system*, with two particles m_1, m_2 and 4-momenta p_1 and p_2, respectively.

- Each of the two momenta p_1 and p_2 may again stay (contribute) to the total momentum of the system of particles.

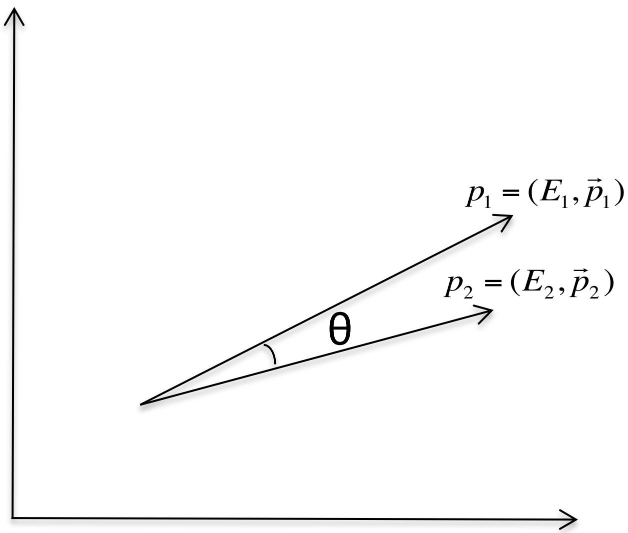
What is the center of mass(momentum) energy, E?

- it is independent of the Lorentz system in which p_1 and p_2 are given.
- it must be possible to give this answer in terms of the three invariants:

$$p_1^2 = m_1^2 \;\; and \;\; p_2^2 = m_2^2$$

$$and$$

$$\left[p_1 p_2 \;\; or \;\; (p_1 + p_2)^2 \;\; or \;\; (p_1 - p_2)^2\right]$$



The answer could be obvious in the CM system itself (we use * in CM frame).

The Center of Momentum (CM) Energy and Velocity

$$\vec{p_1}^* + \vec{p_2}^* = 0$$

$$\Rightarrow \; p_1 + p_2 = (E_1^* + E_2^*, 0)$$

$$and \;\; E^* = E_1^* + E_2^*$$

Hence, $E^{*2} = (E_1^* + E_2^*)^2 = (p_1^* + p_2^*)^2$

$$= (p_1 + p_2)^2 \;\; since \;\; (p_1 + p_2)^2 \;\; is \; L.I.$$

Let $M \equiv$ total mass of the system.

$$\Rightarrow M^2 = (p_1 + p_2)^2 = P^2 = E^{*2}$$

$$= (E_1 + E_2)^2 - (\vec{p_1} + \vec{p_2})^2 = \;\; Invariant \longrightarrow (1)$$

i.e. kinematically our two particles p_1 and p_2 are equivalent to one single particle with 4-momentum P and mass $M = E_{cm}$

The Center of Momentum (CM) Energy and Velocity

Furthermore,

$$\vec{p} = \gamma m \vec{v} \qquad \gamma = \frac{1}{\sqrt{1-v^2}} \longrightarrow (2)$$

$$E = \gamma m$$

Hence the total 4-momentum of the two-particle system $P = M\beta\gamma$

$$E = M\gamma$$

$$\beta_{cm} = \frac{P}{E} = \frac{(p_1 + p_2)}{(E_1 + E_2)}$$ *is the velocity of the CM seen from the lab.*

$$\gamma_{cm} = \frac{1}{\sqrt{1-\beta^2}} = \frac{E}{M}$$

$$= \frac{E_1 + E_2}{\sqrt{(E_1+E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2}} = \frac{E_1+E_2}{E_{cm}} \longrightarrow (3)$$

is the Lorentz boost of the CM.

$$\gamma_{cm} = \frac{Sum\ of\ the\ energies\ of\ the\ particles\ in\ LS}{Energy\ of\ the\ CM}$$

The Center of Momentum (CM) Energy and Velocity

For practical calculations one has to express everything either in

$$E_1,\ E_2\ and\ cos\theta$$

$$or\ in\ |\vec{p_1}|, |\vec{p_2}|\ and\ cos\theta$$

$$using\ E = \sqrt{m^2 + \vec{p}^{\,2}}$$

It could be shown that

$$\boxed{E^{*2} = (m_1^2 + m_2^2) \ + 2E_1E_2 \ \mp 2\sqrt{(E_1^2 - m_1^2)\,(E_2^2 - m_2^2)}}$$

- sign for parallel motion and + sign for antiparallel motion.

Problems: 3.1 (Hagedorn)

In nuclei the K.E. of bound nucleons goes up to the order of 20 MeV. Illustrate formulas (1) and (2) by calculating the effect of this motion when it is parallel or antiparallel to an incoming beam of 25 GeV (K.E.) protons (put m= 1 GeV).

(a) What is the difference in the CM energy?

(b) What energies must the incoming protons have to produce the same CM energies on nucleon at rest?

(a) What is the difference in β and γ of the CM system?

The Center of Momentum (CM) Energy and Velocity

Problems: 3.2 (Hagedorn)

Suppose that a group of *A* nucleons (at rest) as a whole would interact with an incoming proton of 25 GeV kinetic energy.

(a) What energy would be available for the production of particles and for kinetic energy? (put m = 1 GeV)

(a) How do β_{cm} and γ_{cm} depend on *A*?

The E, P and v of one particle as seen from the rest system of another particle

- Suppose, we sit on particle 1, moving with it, what would be the energy of particle 2, for us?
- The answer to this question must always be the same, no matter in which Lorentz system we start.
- It must be therefore expressible by invariants and the only invariants are again.

$$p_1^2 = m_1^2 \quad and \quad p_2^2 = m_2^2$$

$$and$$

$$\left[p_1 p_2 \;\; or \;\; (p_1 + p_2)^2 \;\; or \;\; (p_1 - p_2)^2\right]$$

Let E_{21} : energy of particle *2* if we look at it in the rest system of *1*.

$E_{21} = E_2$ in the system where $\vec{p}_1 = 0$.

Let's write E_{21} in an invariant form.

The E, P and v of one particle as seen from the rest system of another particle

Now

$$p_1 p_2 = E_1 E_2 - \vec{p}_1 . \vec{p}_2$$
$$= m_1 E_2 \qquad (as \quad \vec{p}_1 = 0)$$

Hence

$$E_{21} = E_2 = \frac{p_1 p_2}{m_1}$$ (the RHS is already in an invariant form)

$$|\vec{p}_{21}| = E_{21}^2 - m_2^2 = \frac{(p_1 p_2)^2 - m_1^2 m_2^2}{m_1^2}$$

If p_1 and p_2 are the momentum 4-vectors of any two particles in any Lorentz system, then

The E, P and v of one particle as seen from the rest system of another particle

$$E_{21} = \frac{p_1 p_2}{m_1}$$

$$|\vec{p}_{21}|^2 = \frac{(p_1 p_2)^2 - m_1^2 m_2^2}{m_1^2} \qquad \left(as \quad p_1 p_2 = E_1 E_2 - \vec{p}_1 . \vec{p}_2\right)$$

$$v_{21}^2 = \frac{|\vec{p}_{21}|^2}{E_{21}^2} = \frac{(p_1 p_2)^2 - m_1^2 m_2^2}{(p_1 p_2)^2}$$

$\longrightarrow$ (4)

are the energy E_{21} and momentum $|\vec{p}_{21}|$ of particle *2* seen from particle *1* and v_{21} is the relative velocity (symmetric in *1* and *2*).

All these expressions are invariants and can be evaluated in any Lorentz system.

The E, P and v of a particle as seen from the CM system

This problem is like all the above quantities are as seen from a fictitious particle *M*, called the *"center-of-momentum-particle"*, whose 4-momentum is:

$$P = p_1 + p_2$$

We need to apply formulas (4) with p_1 replaced by P and p_2 by the 4-momentum of that particle whose energy, momentum and velocity we wish to know.

With CM quantities as *, from eqn. (4),

$$\boxed{E_1^* = \frac{Pp_1}{M} \qquad v_1^{*2} = \frac{(Pp_1)^2 - M^2 m_1^2}{(Pp_1)^2} \qquad |\vec{p}_1^{\,*}|^2 = \frac{(Pp_1)^2 - M^2 m_1^2}{M^2} \qquad \text{with } P = p_1 + p_2} \longrightarrow (5)$$

The E, P and v of a particle as seen from the CM system

$$and \quad p_1 p_2 = \frac{1}{2}\left[(p_1+p_2)^2 - p_1^2 - p_2^2\right]$$
$$= \frac{1}{2}\left(M^2 - m_1^2 - m_2^2\right)$$

Using the above equations, one obtains:

$$E_1^* = \frac{M^2 + \left(m_1^2 - m_2^2\right)}{2M} \qquad E_2^* = \frac{M^2 - \left(m_1^2 - m_2^2\right)}{2M} \qquad E_1^* + E_2^* = M$$

$$|\vec{p}^{\,*}|^2 = |\vec{p}_1^{\,*}|^2 = |\vec{p}_2^{\,*}|^2$$

$$= \frac{M^4 - 2M^2\left(m_1^2 + m_2^2\right) + \left(m_1^2 - m_2^2\right)^2}{4M^2} \qquad v_1^{*2} = \left(\frac{|\vec{p}^{\,*}|}{E_1^*}\right)^2$$

$$= \frac{\left[M^2 - \left(m_1 + m_2\right)^2\right].\left[M^2 - \left(m_1 - m_2\right)^2\right]}{4M^2} \longrightarrow (6)$$

The E, P and v of a particle as seen from the CM system

where E_1^*, v_1^* are energy and velocity of particle *1*, as seen from either common *CM system* and $M^2 = P^2 = (p_1 + p_2)^2$ is the total mass squared.

Eqn. (6) gives the energies, the momenta and the velocities of two particles m_1 and m_2 for which $M = m_1 + m_2$

Problems: 3.3 (Hagedorn)

Given $M = m_1 + m_2$

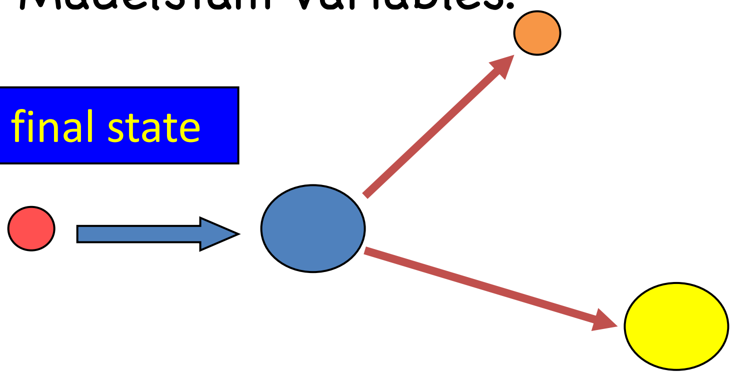
What are the energy and momentum of particle *2* seen from particle *1* ?

Mandelstam Variables

In the interaction of two incoming particles 1 and 2 with two particles in the final state, 3 and 4, one can construct the following Lorentz invariants called Madelstam Variables.

$1+2 \rightarrow 3+4$

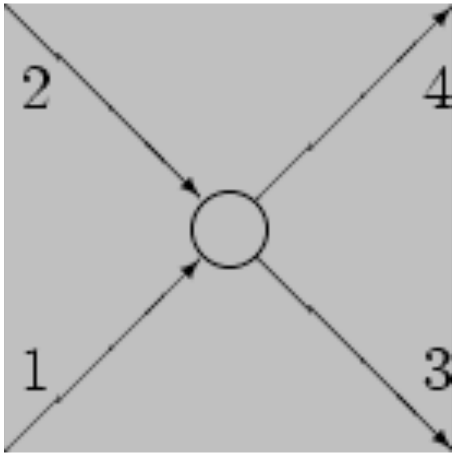
Two-particle final state



$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2$$

$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2$$

$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2$$



- *s* is the square of the center of mass energy
- *t* is 4-momentum transfer square, if 1 and 3 are the same particle.

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$$

$$s = (p_1 + p_2)^2 = m_1^2 + m_2^2 + 2p_1.p_2$$

Mandelstam Variables

- The minimum value *s* can have is: $(m_1 + m_2)^2$.
- As $s = (p_3 + p_4)^2$, the minimum value *s* can have to produce final state particles is $(m_3 + m_4)^2$.

Hence the minimum CM energy required for a reaction to occur is given by:

$$s_0 = \max\left[(m_1 + m_2)^2, (m_3 + m_4)^2\right]$$

It could be shown that *t* is negative for real processes.

Mandelstam Variables

Example 2.4 **Find the center-of-mass energy of A, in terms of the Mandelstam variables and the particles masses.**

Solution

A+B $\rightarrow$ C+D

Ref: Relativistic Kinematics for Beginners:
Raghunath Sahoo & Basanta K. Nandi
(CRC Press- 2020)

$$
\begin{aligned}
s &= (p_A + p_B)^2 \\
&= p_A^2 + p_B^2 + 2p_A.p_B \\
&= p_A^2 + p_B^2 + 2(E_A E_B - \vec{p}_A.\vec{p}_B)
\end{aligned}
$$

In CM frame, $\vec{p}_A.\vec{p}_B = -\vec{p}_A^2$.

$$\therefore\ p_A^2 = E_A^2 - \vec{p}_A^2 \tag{2.44}$$

$$\Longrightarrow \vec{p}_A^2 = E_A^2 - p_A^2 = -\vec{p}_A.\vec{p}_B \tag{2.45}$$

and $p_B^2 = E_B^2 - \vec{p}_B^2 = E_B^2 - \vec{p}_A^2$ $(\because |\vec{p}_A| = |\vec{p}_B|$ in CM system)

$$
\begin{aligned}
\Longrightarrow E_B &= \sqrt{p_B^2 + \vec{p}_A^2} \\
&= \sqrt{p_B^2 + E_A^2 - p_A^2} \quad \text{[by using Eqn. 2.45]}
\end{aligned} \tag{2.46}
$$

Now $s = p_A^2 + p_B^2 + 2[E_A\sqrt{p_B^2 + E_A^2 - p_A^2} + E_A^2 - p_A^2]$ (by using Eqns. 2.45 and 2.46).

$$s + p_A^2 - p_B^2 - 2E_A^2 = 2E_A\sqrt{p_B^2 + E_A^2 - p_A^2}$$

Squaring both the sides of the above equation, we get:

Mandelstam Variables

$$
\begin{aligned}
(s + p_A^2 - p_B^2)^2 + 4E_A^4 - 4E_A^2(s + p_A^2 - p_B^2) &= 4E_A^2(p_B^2 + E_A^2 - p_A^2) \\
\Longrightarrow (s + p_A^2 - p_B^2)^2 &= 4E_A^2(p_B^2 - p_A^2 + s + p_A^2 - p_B^2) \\
&= 4E_A^2 s \\
\Longrightarrow E_A^2 &= \frac{(s + p_A^2 - p_B^2)^2}{4s} \\
\Longrightarrow E_A^{CM} &= \frac{(s + m_A^2 - m_B^2)}{2\sqrt{s}}
\end{aligned}
$$

Example 2.7 For elastic scattering of identical particles like the Moller scattering ($e^-e^- \to e^-e^-$), show that the Mandelstam variables become

$$
\begin{aligned}
s &= 4(\vec{p}^2 + m^2) \\
t &= -2\vec{p}^2(1 - \cos\theta) \\
u &= -2\vec{p}^2(1 + \cos\theta)
\end{aligned}
$$

Here $\vec{p}$ is the CM momentum of the incident particle and θ is the scattering angle.

Ref: Relativistic Kinematics for Beginners:
Raghunath Sahoo & Basanta K. Nandi
(CRC Press- 2020)

Mandelstam Variables

Solution

We know $s = (p_A + p_B)^2 = (E_A + E_B)^2 - (\vec{p_A} + \vec{p_B})^2$.
In CM frame, $\vec{p_A} + \vec{p_B} = 0$ and $E_A = E_B = \sqrt{\vec{p}^2 + m^2}$

$$\begin{aligned} s &= (2E_A)^2 \\ &= 4(p^2 + m^2) \end{aligned}$$

Further by definition,

$$\begin{aligned} t &= (p_A - p_C)^2 \\ &= (E_A - E_C)^2 - (\vec{p_A} - \vec{p_C})^2 \end{aligned}$$

But $E_A = E_C$ and

$$\begin{aligned} (\vec{p_A} - \vec{p_C})^2 &= \vec{p_A}^2 + \vec{p_C}^2 - 2\vec{p_A}.\vec{p_C} \\ &= 2\vec{p}^2(1 - \cos\theta) \\ \implies t &= -2\vec{p}^2(1 - \cos\theta) \end{aligned}$$

Again as per the definition of u,

$$\begin{aligned} u &= (p_A - p_D)^2 \\ &= (E_A - E_D)^2 - (\vec{p_A} - \vec{p_D})^2 \\ &= -(\vec{p_A}^2 + \vec{p_D}^2 - 2\vec{p_A}.\vec{p_D}) \;\; (\because E_A = E_D) \\ &= -2\vec{p}^2(1 + \cos\theta) \\ \implies u &= -2\vec{p}^2(1 + \cos\theta) \end{aligned}$$

Ref: Relativistic Kinematics for Beginners:
Raghunath Sahoo & Basanta K. Nandi
(CRC Press- 2020)

Lecture-2

Recap.........what did we learn so far…

$\lambda = \dfrac{h}{p}$ $E = mc^2$ Two essential formulae to go to high energies.

$E \approx pc \gg mc^2$ Particle is said to be relativistic.

$$\beta'' = |v/c|'' = \frac{\beta + \beta'}{1 + \beta\beta'}$$

Velocity is not an additive quantity. It is non-linear in successive Lorentz transformations.

$$y = \frac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right) \qquad \boxed{y'' = y + y'}$$

Rapidity is additive under successive LT and preserves the shape.

$$a^2 \equiv a.a = (a^0)^2 - \vec{a}^2$$

4-vector scalar product is LI.

$$\boxed{p^\mu p_\mu = m^2} \Rightarrow \boxed{E^2 = \vec{p}^2 + m^2}$$

$\boxed{\hbar = c = 1}$ Natural Unit