Electroweak Unification and the Standard Model

Lecture 4

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Nonabelian Gauge Theory

Consider a scalar multiplet $\Phi(x)$ of length n, i.e.

$$\Phi(x) = \begin{pmatrix} \varphi_1(x) \\ \vdots \\ \vdots \\ \varphi_n(x) \end{pmatrix}$$

where each $\varphi_i(x)$ $(i = 1, \dots, n)$ is a complex scalar field.

Construct the 'free' Lagrangian density

$$\mathcal{L} = (\partial^\mu \Phi)^\dagger \partial_\mu \Phi - M^2 \Phi^\dagger \Phi$$

This is just a shorthand for n mass-degenerate free scalar fields, i.e.

$$\mathcal{L} = \sum_{i=1}^{n} \left(\partial^\mu \varphi_i^* \, \partial_\mu \varphi_i - M^2 \varphi_i^* \varphi_i \right)$$

Now consider a global SU(N) gauge transformation

$$\Phi(x) \rightarrow \Phi'(x) = \mathbb{U}\Phi(x)$$

where $\mathbb{U}$ is a SU(N) matrix, i.e. $\mathbb{U}^{\dagger}\,\mathbb{U} = \mathbb{1}$ and $\det \mathbb{U} = +1$, where

$$\mathbb{U} = \begin{pmatrix} U_{11} & \cdots & U_{1n} \\ \vdots & & \vdots \\ U_{n1} & \cdots & U_{nn} \end{pmatrix}$$

n and N are different (in general)

$n \geq N$

If equal it is the fundamental representation

The number of free (real) parameters in this SU(N) matrix is

$$p = 2N^2 - N - 2\,{}^{N}C_2 - 1 = N^2 - 1$$

We can write this SU(N) transformation in the form $\mathbb{U} = e^{-ig\vec{\theta}.\vec{\mathbb{T}}}$

where the $\vec{\theta} = (\theta_1, \cdots, \theta_p)$ are free (real) parameters

and the $\vec{\mathbb{T}} = (\mathbb{T}_1, \cdots, \mathbb{T}_p)$ are the generators of SU(N)

$$\vec{\theta}.\vec{\mathbb{T}} = \sum_{a=1}^{p} \theta_a \mathbb{T}_a$$

Under this gauge transformation

$$\Phi(x) \;\rightarrow \Phi'(x) \;= \mathbb{U}\,\Phi(x)$$
$$\Phi^\dagger(x) \rightarrow \Phi'^\dagger(x) = \Phi^\dagger(x)\,\mathbb{U}^\dagger$$

The Lagrangian density transforms to

$$\mathcal{L} \rightarrow \; \mathcal{L}' = (\partial^\mu \Phi')^\dagger \partial_\mu \Phi' - M^2 \Phi'^\dagger \Phi'$$

$$= (\partial^\mu \mathbb{U}\Phi)^\dagger \partial_\mu \mathbb{U}\Phi - M^2 \Phi^\dagger \mathbb{U}^\dagger \mathbb{U}\Phi \qquad \text{global}$$

$$= (\partial^\mu \Phi)^\dagger \mathbb{U}^\dagger \mathbb{U} \partial_\mu \Phi - M^2 \Phi^\dagger \mathbb{U}^\dagger \mathbb{U}\Phi \qquad \text{unitary}$$

$$= (\partial^\mu \Phi)^\dagger \partial_\mu \Phi - M^2 \Phi^\dagger \Phi$$

$$= \mathcal{L}$$

Thus, this system of n mass-degenerate free scalar fields possesses a SU(N) global gauge symmetry — with p conserved currents/charges.

The next step is to convert this to a SU(N) local gauge symmetry, i.e.

$$\Phi(x) \;\; \to \Phi'(x) \;\; = \mathbb{U}(x)\,\Phi(x)$$
$$\Phi^\dagger(x) \to \Phi'^\dagger(x) = \Phi^\dagger(x)\,\mathbb{U}^\dagger(x)$$

As in the nonAbelian case, the Lagrangian density will no longer remain gauge invariant...

$$\mathcal{L} \to \;\mathcal{L}' = (\partial^\mu \Phi')^\dagger \partial_\mu \Phi' - M^2 \Phi'^\dagger \Phi'$$

$$= (\partial^\mu \mathbb{U}\,\Phi)^\dagger \partial_\mu \mathbb{U}\,\Phi - M^2 \Phi^\dagger \mathbb{U}^\dagger \mathbb{U}\,\Phi \quad \text{local}$$

$$= \left(\mathbb{U}\partial_\mu \Phi + \partial_\mu \mathbb{U}\Phi\right)^\dagger \left(\mathbb{U}\partial_\mu \Phi + \partial_\mu \mathbb{U}\Phi\right) - M^2 \Phi^\dagger \mathbb{U}^\dagger \mathbb{U}\Phi$$

$$= \left[\left(\mathbb{1}\partial_\mu + \mathbb{U}^\dagger \partial_\mu \mathbb{U}\right)\Phi\right]^\dagger \mathbb{U}^\dagger \mathbb{U}\left(\mathbb{1}\partial_\mu + \mathbb{U}^\dagger \partial_\mu \mathbb{U}\right)\Phi - M^2 \Phi^\dagger \mathbb{U}^\dagger \mathbb{U}\Phi \quad \text{unitary}$$

$$= \left[\left(\mathbb{1}\partial_\mu + \mathbb{U}^\dagger \partial_\mu \mathbb{U}\right)\Phi\right]^\dagger \left(\mathbb{1}\partial_\mu + \mathbb{U}^\dagger \partial_\mu \mathbb{U}\right)\Phi - M^2 \Phi^\dagger \Phi \qquad \neq \mathcal{L}$$

<u>Solution:</u> define a covariant derivative $\mathbb{D}_\mu = \mathbb{1}\partial_\mu + ig\mathbb{A}_\mu(x)$

where the $\mathbb{A}_\mu(x)$ is a $n \times n$ matrix of gauge fields, i.e.

$$\mathbb{A}^\mu = \begin{pmatrix} a_{11}^\mu & \cdots & a_{1n}^\mu \\ \vdots & & \vdots \\ a_{n1}^\mu & \cdots & a_{nn}^\mu \end{pmatrix}$$

Not all of these need to be independent... ($\mathbb{A}^\mu$ is Hermitian...)

We require the covariant derivative $\mathbb{D}_\mu \Phi$ to transform exactly like Φ, i.e.

$$\mathbb{D}_\mu \Phi \to \mathbb{D}'_\mu \Phi' = \mathbb{U}\, \mathbb{D}_\mu \Phi$$

for then, if we rewrite the Lagrangian density as

$$\mathcal{L} = (\mathbb{D}^\mu \Phi)^\dagger \mathbb{D}_\mu \Phi - M^2 \Phi^\dagger \Phi$$

it will be trivially gauge invariant.

How do we ensure that $\mathbb{D}_\mu \Phi \to \mathbb{D}'_\mu \Phi' = \mathbb{U}\,\mathbb{D}_\mu \Phi$?

By adjusting the transformation of the gauge field matrix $\mathbb{A}^\mu$ ...

$$\mathbb{D}_\mu \Phi \to \mathbb{D}'_\mu \Phi' = \left(\mathbb{1}\partial_\mu + ig\mathbb{A}'_\mu\right)\mathbb{U}\Phi$$

$$= \partial_\mu(\mathbb{U}\Phi) + ig\mathbb{A}'_\mu \mathbb{U}\Phi$$

$$= \mathbb{U}\left(\partial_\mu \Phi\right) + \left(\partial_\mu \mathbb{U}\right)\Phi + ig\mathbb{A}'_\mu \mathbb{U}\Phi$$

$$= \mathbb{U}\left(\partial_\mu \Phi\right) + \mathbb{U}\mathbb{U}^\dagger\left(\partial_\mu \mathbb{U}\right)\Phi + ig\mathbb{U}\mathbb{U}^\dagger\mathbb{A}'_\mu \mathbb{U}\Phi$$

$$= \mathbb{U}\left[\mathbb{1}\partial_\mu + \mathbb{U}^\dagger\partial_\mu \mathbb{U} + ig\mathbb{U}^\dagger\mathbb{A}'_\mu \mathbb{U}\right]\Phi$$

If this is to be the same as

$$\mathbb{D}_\mu \Phi = \left(\mathbb{1}\partial_\mu + ig\mathbb{A}_\mu\right)\Phi$$

we must have $ig\mathbb{A}_\mu = ig\mathbb{U}^\dagger\mathbb{A}'_\mu \mathbb{U} + \mathbb{U}^\dagger\partial_\mu \mathbb{U}$

Rewrite

$$ig\mathbb{A}_\mu = ig\mathbb{U}^\dagger \mathbb{A}'_\mu \mathbb{U} + \mathbb{U}^\dagger \partial_\mu \mathbb{U}$$

as

$$ig\mathbb{U}^\dagger \mathbb{A}'_\mu \mathbb{U} = ig\mathbb{A}_\mu - \mathbb{U}^\dagger \partial_\mu \mathbb{U}$$

or,

$$ig\mathbb{A}'_\mu = ig\mathbb{U}\mathbb{A}_\mu \mathbb{U}^\dagger - (\partial_\mu \mathbb{U})\mathbb{U}^\dagger$$

Note that $\mathbb{U}\mathbb{U}^\dagger = \mathbb{1}$ leads to $(\partial_\mu \mathbb{U})\mathbb{U}^\dagger + \mathbb{U}(\partial_\mu \mathbb{U}^\dagger) = 0$

i.e.

$$ig\mathbb{A}'_\mu = ig\mathbb{U}\mathbb{A}_\mu \mathbb{U}^\dagger + \mathbb{U}(\partial_\mu \mathbb{U}^\dagger) = ig\mathbb{U}\mathbb{A}_\mu \mathbb{U}^\dagger + \mathbb{U}(\partial_\mu \mathbb{U}^\dagger)\mathbb{U}\mathbb{U}^\dagger$$

or, finally,

$$\mathbb{A}'_\mu = \mathbb{U}\left[\mathbb{A}_\mu - \frac{i}{g}(\partial_\mu \mathbb{U}^\dagger)\mathbb{U}\right]\mathbb{U}^\dagger$$

<u>Quick check</u>: suppose $N = 1$ and $n = 1$, i.e. U(1) gauge symmetry

Then $\mathbb{U} = e^{-ig\theta}$ and $\mathbb{A}_\mu = A_\mu$.

Now,

$$\mathbb{A}'_\mu = \mathbb{U}\left[\mathbb{A}_\mu - \frac{i}{g}(\partial_\mu \mathbb{U}^\dagger)\mathbb{U}\right]\mathbb{U}^\dagger$$

assumes the form

$$\begin{aligned} A'_\mu &= e^{-ig\theta}\left[A_\mu - \frac{i}{g}(\partial_\mu e^{+ig\theta})e^{-ig\theta}\right]e^{+ig\theta} \\ &= e^{-ig\theta}\left[A_\mu - \frac{i}{g}(ig\partial_\mu\theta\, e^{+ig\theta})e^{-ig\theta}\right]e^{+ig\theta} \\ &= A_\mu + \partial_\mu\theta \end{aligned}$$

which is what we had derived for the U(1) case.

How many independent fields do we require in the $\mathbb{A}_\mu$ matrix?

$$\mathbb{A}'_\mu = \mathbb{U}\left[\mathbb{A}_\mu - \frac{i}{g}(\partial_\mu \mathbb{U}^\dagger)\mathbb{U}\right]\mathbb{U}^\dagger$$

Since $\mathbb{U} = e^{-ig\vec{\theta}.\vec{\mathbb{T}}}$ i.e. $\mathbb{U}$ has p free parameters, $\mathbb{A}_\mu$ should have p independent fields. This encourages us to expand

$$\mathbb{A}^\mu(x) = \sum_{a=1}^{p} A_a^\mu(x)\,\mathbb{T}_a = \overrightarrow{A^\mu}\,.\,\vec{\mathbb{T}}$$

One can now work out the transformation properties of the $A_a^\mu(x)$ fields in terms of the parameters $\vec{\theta} = (\theta_1, \cdots, \theta_p)$.

(Will do this for specific cases...)

We can also use this expression

$$\mathbb{A}^\mu(x) = \sum_{a=1}^{p} A_a^\mu(x)\,\mathbb{T}_a = \overrightarrow{A^\mu}\,.\,\vec{\mathbb{T}}$$

to write out the interaction terms in the Lagrangian density...

$$\mathcal{L} = (\mathbb{D}^\mu\Phi)^\dagger\mathbb{D}_\mu\Phi - M^2\Phi^\dagger\Phi$$

$$= [(\mathbb{1}\partial^\mu + ig\mathbb{A}^\mu)\Phi]^\dagger\big(\mathbb{1}\partial_\mu + ig\mathbb{A}_\mu\big)\Phi - M^2\Phi^\dagger\Phi$$

$$= (\partial^\mu\Phi)^\dagger\partial_\mu\Phi - M^2\Phi^\dagger\Phi$$ free scalar

$$+ ig\big[(\partial^\mu\Phi)^\dagger\mathbb{A}_\mu\Phi - \Phi^\dagger\mathbb{A}^\mu\partial_\mu\Phi\big]$$ gauge-scalar interaction

$$+ \; g^2\Phi^\dagger\mathbb{A}^\mu\mathbb{A}_\mu\Phi$$ seagull terms

We should complete the Lagrangian density by adding a kinetic term for the gauge fields...

$$\mathbb{F}_{\mu\nu} = -\frac{i}{g}[\mathbb{D}_\mu, \mathbb{D}_\nu] = \partial_\mu \mathbb{A}_\nu - \partial_\nu \mathbb{A}_\mu + ig[\mathbb{A}_\mu, \mathbb{A}_\nu]$$

Now, we have

$$\begin{aligned} \mathbb{D}_\mu \Phi \to \mathbb{D}'_\mu \Phi' &= \mathbb{U}\ \mathbb{D}_\mu \Phi \\ &= \mathbb{U}\ \mathbb{D}_\mu \mathbb{U}^\dagger\ \mathbb{U}\Phi \\ &= \mathbb{U}\ \mathbb{D}_\mu \mathbb{U}^\dagger\ \Phi' \qquad \Rightarrow\ \ \mathbb{D}'_\mu = \mathbb{U}\ \mathbb{D}_\mu \mathbb{U}^\dagger \end{aligned}$$

Thus,

$$\mathbb{F}_{\mu\nu} \to \mathbb{F}'_{\mu\nu} = -\frac{i}{g}[\mathbb{D}'_\mu, \mathbb{D}'_\nu] = -\frac{i}{g}[\mathbb{U}\ \mathbb{D}_\mu \mathbb{U}^\dagger, \mathbb{U}\ \mathbb{D}_\nu \mathbb{U}^\dagger] = \mathbb{U}\ \mathbb{F}_{\mu\nu} \mathbb{U}^\dagger$$

To get gauge invariance, we have to take the trace...

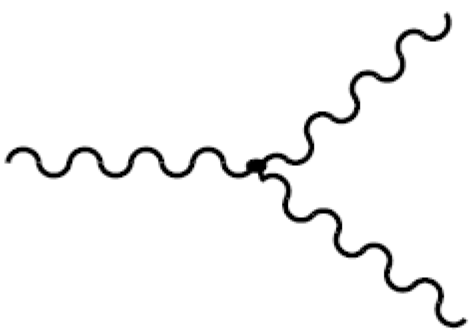
The full Lagrangian density is now

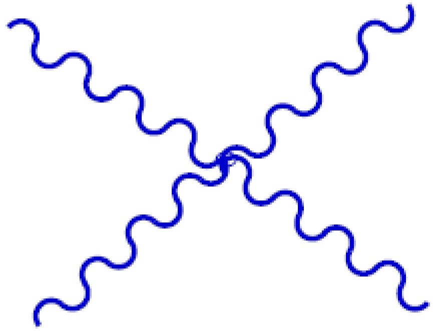
$$\mathcal{L} = (\mathbb{D}^{\mu}\Phi)^{\dagger}\mathbb{D}_{\mu}\Phi - M^{2}\Phi^{\dagger}\Phi - \frac{1}{n}\mathrm{Tr}\left[\mathbb{F}_{\mu\nu}\,\mathbb{F}^{\mu\nu}\right]$$

Since $\mathbb{F}_{\mu\nu} = \left(\partial_{\mu}\mathbb{A}_{\nu} - \partial_{\nu}\mathbb{A}_{\mu}\right) + ig\left[\mathbb{A}_{\mu},\mathbb{A}_{\nu}\right]$

$$\mathbb{F}^{\mu\nu} = (\partial^{\mu}\mathbb{A}^{\nu} - \partial^{\nu}\mathbb{A}^{\mu}) + ig[\mathbb{A}^{\mu},\mathbb{A}^{\nu}]$$

Leads to triple gauge vertices and quadruple gauge vertices





absent in an Abelian gauge theory, e.g. QED

SU(2) Gauge Theory

Recall that for weak interactions we needed three gauge bosons, the

$$W_\mu^+, W_\mu^-, W_\mu^0$$

This seems to indicate a gauge theory with three generators

and the obvious one to take is an SU(2) gauge theory.

All of the above formalism will work, except that now we must take the generators as

$$\mathbb{T}_1 = \tfrac{1}{2}\sigma_1 \qquad \mathbb{T}_2 = \tfrac{1}{2}\sigma_2 \qquad \mathbb{T}_3 = \tfrac{1}{2}\sigma_3$$

obeying the Lie algebra

$$[\mathbb{T}_a, \mathbb{T}_b] = i\varepsilon_{abc}\,\mathbb{T}_c$$

The full Lagrangian for this is

$$\mathcal{L} = (\partial^\mu \Phi)^\dagger \partial_\mu \Phi - M^2 \Phi^\dagger \Phi + ig\left[(\partial^\mu \Phi)^\dagger \mathbb{A}_\mu \Phi - \Phi^\dagger \mathbb{A}^\mu \partial_\mu \Phi\right]$$

$$+ g^2 \Phi^\dagger \mathbb{A}^\mu \mathbb{A}_\mu \Phi - \tfrac{1}{2}\mathrm{Tr}\left[\mathbb{F}_{\mu\nu} \mathbb{F}^{\mu\nu}\right]$$

$$\Phi = \begin{pmatrix} \varphi_\mathrm{A} \\ \varphi_\mathrm{B} \end{pmatrix}$$

where

$$\mathbb{A}^\mu = A_1^\mu \mathbb{T}_1 + A_2^\mu \mathbb{T}_2 + A_3^\mu \mathbb{T}_3$$

We can also expand

$$\mathbb{F}^{\mu\nu} = \partial_\mu \mathbb{A}_\nu - \partial_\nu \mathbb{A}_\mu + ig\left[\mathbb{A}_\mu, \mathbb{A}_\nu\right]$$

$$= F_1^{\mu\nu} \mathbb{T}_1 + F_2^{\mu\nu} \mathbb{T}_2 + F_3^{\mu\nu} \mathbb{T}_3$$

where

$$F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu - g\varepsilon_{abc} A_b^\mu A_c^\nu$$

Mass generation:

To break this symmetry spontaneously, we now replace the scalar mass term by a potential

$-M^2\Phi^\dagger\Phi \to -\mathrm{V}(\Phi)$

$$\mathrm{V}(\Phi) = -M^2\Phi^\dagger\Phi + \lambda\left(\Phi^\dagger\Phi\right)^2$$

i.e. this is a theory with n massless scalars and some self-interactions

As before, if we define a real field

$$\Phi^\dagger(x)\Phi(x) \equiv \eta(x)^2$$

then we can write the potential as

$$\mathrm{V}(\eta) = -M^2\eta^2 + \lambda\eta^4$$

with a local maximum at $\eta = 0$; local minima at $\eta = v/\sqrt{2} = \sqrt{M^2/2\lambda}$

These local minima correspond to

$$\Phi^\dagger\Phi = \eta^2 = \frac{M^2}{2\lambda}$$

Recall that

$$\Phi = \begin{pmatrix}\varphi_{\mathrm{A}} \\ \varphi_{\mathrm{B}}\end{pmatrix} = \begin{pmatrix}\frac{\varphi_1 + i\varphi_2}{\sqrt{2}} \\ \frac{\varphi_3 + i\varphi_4}{\sqrt{2}}\end{pmatrix}$$

so that $\Phi^\dagger\Phi = |\varphi_{\mathrm{A}}|^2 + |\varphi_{\mathrm{B}}|^2 = \frac{1}{2}(\varphi_1^2 + \varphi_2^2 + \varphi_3^2 + \varphi_4^2)$

i.e.

$$\varphi_1^2 + \varphi_2^2 + \varphi_3^2 + \varphi_4^2 = \frac{M^2}{\lambda}$$

Equation of a 4-sphere – only one of these points can be the vacuum

Hidden Symmetry!!

Vacuum manifold in a U(1) gauge theory is a circle

- The scalar field is

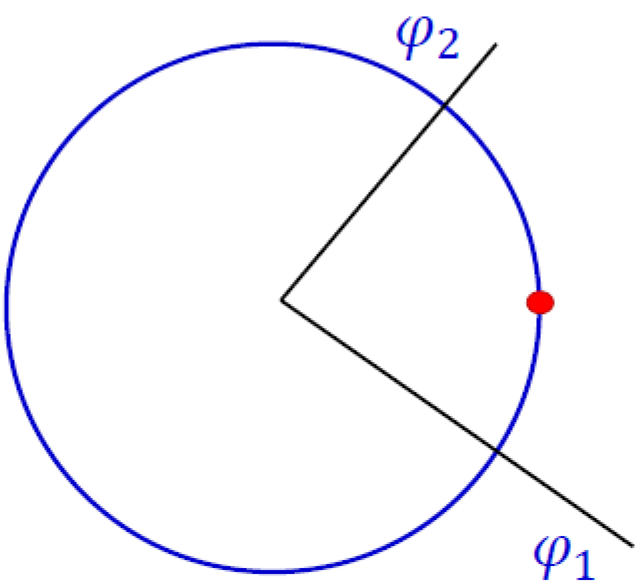
$$\varphi = \frac{\varphi_1 + i\varphi_2}{\sqrt{2}}$$

- Traditional to orient the axes in the φ-space such that only the φ_1 has a vacuum expectation value

$$\varphi_0 \equiv \langle \varphi_1 \rangle = v$$

i.e.

$$\langle \varphi \rangle = \frac{v}{\sqrt{2}}$$



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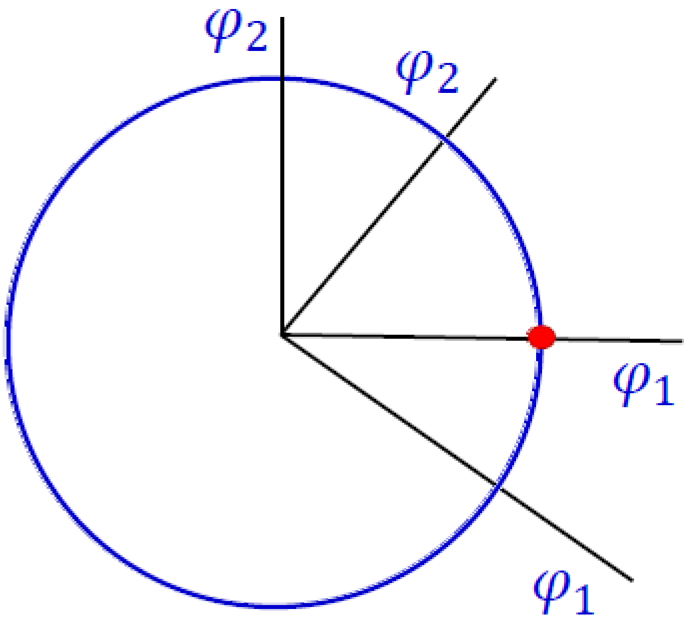


Vacuum manifold in a SU(2) gauge theory is a four-sphere

- The scalar field is

$$\Phi = \begin{pmatrix} \frac{\varphi_1 + i\varphi_2}{\sqrt{2}} \\ \frac{\varphi_3 + i\varphi_4}{\sqrt{2}} \end{pmatrix}$$

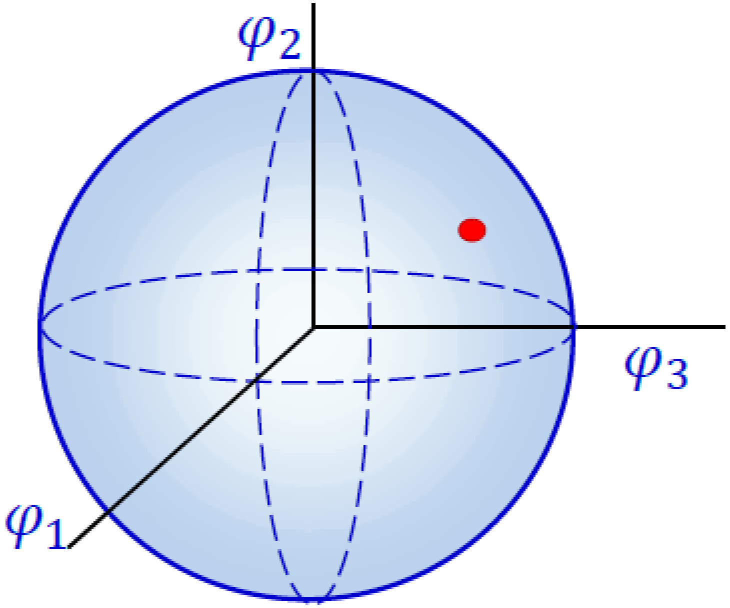
- Traditional to orient the axes in the φ-space such that only the φ_3 has a vacuum expectation value

$$\langle \varphi_3 \rangle = v$$

i.e.

$$\langle \Phi \rangle = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}$$

- Now shift $\Phi = \langle \Phi \rangle + \Phi'$



(The φ_4 axis is not shown...)

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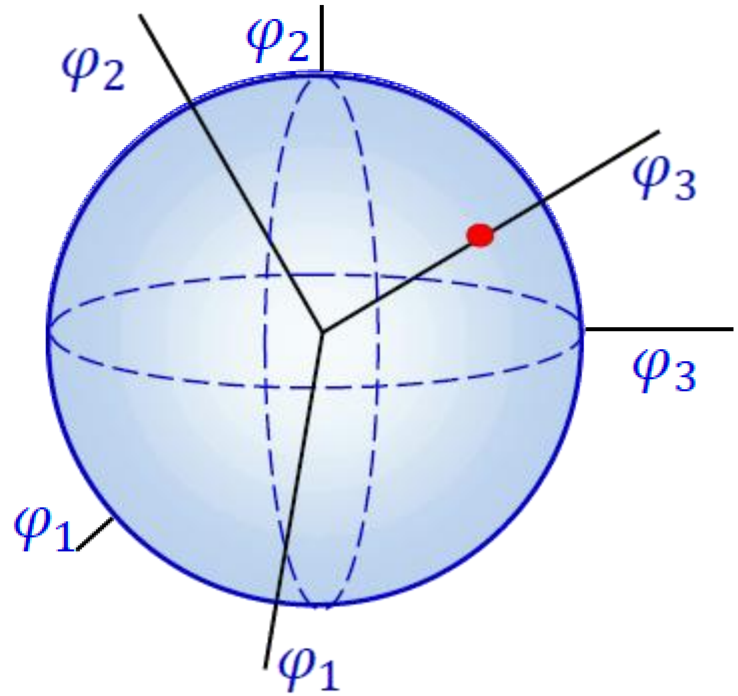

(The φ_4 axis is not shown...)

Seagull term:

$$\mathcal{L}_{\text{sg}} = g^2 \Phi^\dagger \mathbb{A}^\mu \mathbb{A}_\mu \Phi \to g^2 (\langle\Phi\rangle + \Phi')^\dagger \mathbb{A}^\mu \mathbb{A}_\mu (\langle\Phi\rangle + \Phi')$$
$$= g^2 \langle\Phi\rangle^\dagger \mathbb{A}^\mu \mathbb{A}_\mu \langle\Phi\rangle + \cdots$$

We thus get a mass term for the gauge bosons, viz.

$$\mathcal{L}_{\text{mass}} = g^2 \langle\Phi\rangle^\dagger \mathbb{A}^\mu \mathbb{A}_\mu \langle\Phi\rangle = g^2 (\mathbb{A}^\mu \langle\Phi\rangle)^\dagger \big(\mathbb{A}_\mu \langle\Phi\rangle\big)$$

Expand this...

$$\mathbb{A}_\mu = A_{\mu 1}\mathbb{T}_1 + A_{\mu 2}\mathbb{T}_2 + A_{\mu 3}\mathbb{T}_3 = \tfrac{1}{2}\big(A_{\mu 1}\sigma_1 + A_{\mu 2}\sigma_2 + A_{\mu 3}\sigma_3\big)$$
$$= \begin{pmatrix} \dfrac{A_{\mu 3}}{2} & \dfrac{A_{\mu 1} - iA_{\mu 2}}{2} \\ \dfrac{A_{\mu 1} + iA_{\mu 2}}{2} & -\dfrac{A_{\mu 3}}{2} \end{pmatrix} \equiv \begin{pmatrix} \dfrac{W_\mu^0}{2} & \dfrac{W_\mu^+}{\sqrt{2}} \\ \dfrac{W_\mu^-}{\sqrt{2}} & -\dfrac{W_\mu^0}{2} \end{pmatrix}$$

$$\mathbb{A}_\mu \langle \Phi \rangle = \begin{pmatrix} \frac{W_\mu^0}{2} & \frac{W_\mu^+}{\sqrt{2}} \\ \frac{W_\mu^-}{\sqrt{2}} & -\frac{W_\mu^0}{2} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{v}{2} W_\mu^+ \\ -\frac{v}{2\sqrt{2}} W_\mu^0 \end{pmatrix}$$

and

$$(\mathbb{A}^\mu \langle \Phi \rangle)^\dagger = \begin{pmatrix} \frac{v}{2} W^{\mu -} & -\frac{v}{2\sqrt{2}} W^{\mu 0} \end{pmatrix}$$

Thus,

$$\mathcal{L}_{\text{mass}} = g^2 (\mathbb{A}^\mu \langle \Phi \rangle)^\dagger \big(\mathbb{A}_\mu \langle \Phi \rangle \big) = \left(\frac{g^2 v^2}{4} W_\mu^+ W^{\mu -} + \frac{g^2 v^2}{4} W_\mu^0 W^{\mu 0} \right)$$

$$= M_W^2 W_\mu^+ W^{\mu -} + \tfrac{1}{2} M_W^2 W_\mu^0 W^{\mu 0}$$

where $M_W = \frac{1}{2} g v$

In a hidden U(1) gauge theory: $\varphi = \langle \varphi \rangle + \varphi'$

$$\frac{\varphi_1 + i\varphi_2}{\sqrt{2}} = \frac{v}{\sqrt{2}} + \frac{\varphi'_1 + i\varphi'_2}{\sqrt{2}} = \frac{(\varphi'_1 + v) + i\varphi'_2}{\sqrt{2}}$$

When substituted into the potential, this leads to a correct-sign mass for φ'_1 (massive scalar) and keeps φ'_2 massless (Goldstone boson)

In a hidden SU(2) gauge theory: $\Phi = \langle \Phi \rangle + \Phi'$

$$\begin{pmatrix} \frac{\varphi_1 + i\varphi_2}{\sqrt{2}} \\ \frac{\varphi_3 + i\varphi_4}{\sqrt{2}} \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} + \begin{pmatrix} \frac{\varphi'_1 + i\varphi'_2}{\sqrt{2}} \\ \frac{\varphi'_3 + i\varphi'_4}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{\varphi'_1 + i\varphi'_2}{\sqrt{2}} \\ \frac{(\varphi'_3 + v) + i\varphi'_4}{\sqrt{2}} \end{pmatrix}$$

When substituted into the potential, this leads to a correct-sign mass for φ'_3 (massive scalar) and keeps $\varphi'_{1,2,4}$ massless (Goldstone bosons)

We now have to worry about <u>three</u> Goldstone bosons

The Higgs mechanism works here too…

Exactly as before: parametrise $\Phi(x) = e^{i\vec{\xi}(x).\vec{\mathbb{T}}} \begin{pmatrix} 0 \\ \eta(x) \end{pmatrix}$ (polar form)

Consider the unbroken (i.e. gauge invariant) Lagrangian density

$$\mathcal{L} = -\tfrac{1}{2}\mathrm{Tr}\left[\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu}\right] + \left(\mathbb{D}^{\mu}\Phi\right)^{\dagger}\mathbb{D}_{\mu}\Phi - V(\Phi)$$

where $V(\varphi) = -M^2\Phi^{\dagger}\Phi + \lambda\left(\Phi^{\dagger}\Phi\right)^2$

At this level, we are free to make any gauge choice we wish…

Make a gauge transformation

$$\Phi(x) \to U(x)\Phi(x) = e^{-ig\vec{\theta}(x).\vec{\mathbb{T}}}\Phi(x) = e^{i[g\vec{\theta}(x)-\vec{\xi}(x)].\vec{\mathbb{T}}} \begin{pmatrix} 0 \\ \eta(x) \end{pmatrix}$$

We might as well choose a special gauge, since the gauge symmetry is going to be broken anyway...

Choose the three gauge functions $\vec{\theta}(x)$ such that

$$g\vec{\theta}(x) - \vec{\xi}(x) = \vec{0}$$

This is called the unitary gauge.

In this gauge, $\Phi(x) = \Phi_\eta(x) = \begin{pmatrix} 0 \\ \eta(x) \end{pmatrix}$ and the Lagrangian becomes

$$\mathcal{L} = -\tfrac{1}{2}\mathrm{Tr}[\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu}] + (\mathbb{D}^\mu\Phi_\eta)^\dagger\mathbb{D}_\mu\Phi_\eta - V(\eta)$$

where $V(\eta) = -M^2\eta^2 + \lambda\eta^4$

The ground state is still at $v/\sqrt{2}$ so we must shift

$$\eta = \frac{v}{\sqrt{2}} + \eta'$$

This will lead to

1. $\mathcal{L}_{mass} = M_W^2 W_\mu^+ W^{\mu-} + \frac{1}{2} M_W^2 W_\mu^0 W^{\mu 0}$ with $M_W = \frac{1}{2} g v$
2. $V\left(\frac{v}{\sqrt{2}} + \eta'\right) = +\frac{1}{2} 4M^2\eta^2 + \cdots$ i.e. $M_\eta = 2M$
3. and there are no Goldstone bosons...

if we had kept the $\vec{\xi}(x)$ they would have been the Goldstone bosons

These three degrees of freedom reappear in the longitudinal polarisations of the three W^+, W^- and W^0.



The gauge field matrix expands to

$$\mathbb{A}_\mu = A_{\mu 1}\mathbb{T}_1 + A_{\mu 2}\mathbb{T}_2 + A_{\mu 3}\mathbb{T}_3$$

Now,

$$W_\mu^+ = \frac{A_{\mu 1} - iA_{\mu 2}}{\sqrt{2}} \qquad W_\mu^- = \frac{A_{\mu 1} + iA_{\mu 2}}{\sqrt{2}} \qquad W_\mu^0 = A_{\mu 3}$$

$$\Rightarrow \quad A_{\mu 1} = \frac{1}{\sqrt{2}}\left(W_\mu^+ + W_\mu^-\right) \qquad A_{\mu 2} = \frac{i}{\sqrt{2}}\left(W_\mu^+ - W_\mu^-\right) \qquad A_{\mu 3} = W_\mu^0$$

i.e.

$$\mathbb{A}_\mu = \frac{1}{\sqrt{2}}\left(W_\mu^+ + W_\mu^-\right)\mathbb{T}_1 + \frac{i}{\sqrt{2}}\left(W_\mu^+ - W_\mu^-\right)\mathbb{T}_2 + W_\mu^0\,\mathbb{T}_3$$

$$= \frac{1}{\sqrt{2}}(\mathbb{T}_1 + i\mathbb{T}_2)W_\mu^+ + \frac{1}{\sqrt{2}}(\mathbb{T}_1 - i\mathbb{T}_2)W_\mu^- + W_\mu^0\,\mathbb{T}_3$$

$$\equiv W_\mu^+\mathbb{T}_+ + W_\mu^-\mathbb{T}_- + W_\mu^0\,\mathbb{T}_3 \quad \text{where } \mathbb{T}_\pm = \frac{1}{\sqrt{2}}(\mathbb{T}_1 \pm i\mathbb{T}_2)$$