Electroweak Unification and the Standard Model

Lecture 5

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Inclusion of fermions

If fermions are to interact with the W^+, W^- and W^0 bosons, they must transform as doublets under SU(2)$_W$, just like the scalar doublet $\Phi(x)$

Consider a fermion doublet (we could do a similar thing for SU(N) ...)

$$\Psi = \begin{pmatrix} \psi_\mathrm{A} \\ \psi_\mathrm{B} \end{pmatrix}$$

where the ψ_A and ψ_B are two mass-degenerate Dirac fermions.

Construct the 'free' Lagrangian density

$$\mathcal{L} = i\overline{\Psi}\gamma^\mu \partial_\mu \Psi - m\,\overline{\Psi}\Psi$$

where $\overline{\Psi} = (\bar{\psi}_A \quad \bar{\psi}_B)$.

Sum of two free Dirac fermion Lagrangian densities, with equal masses.

Now, under a global $SU(2)_W$ gauge transformation, if

$$\Psi(x) \to \Psi'(x) = \mathbb{U}\Psi(x)$$

then

$$\overline{\Psi}(x) \to \overline{\Psi}'(x) = \overline{\Psi}(x)\mathbb{U}^\dagger$$

It follows that the Lagrangian density

$$\mathcal{L} = i\overline{\Psi}\gamma^\mu \partial_\mu \Psi - m\,\overline{\Psi}\Psi$$

must be invariant under global $SU(2)_W$ gauge transformations.

As before, we try to upgrade this to a local $SU(2)_W$ gauge invariance, by writing

$$\mathcal{L} = i\overline{\Psi}\gamma^\mu \mathbb{D}_\mu \Psi - m\,\overline{\Psi}\Psi - \tfrac{1}{2}\mathrm{Tr}\left[\mathbb{F}_{\mu\nu}\,\mathbb{F}^{\mu\nu}\right]$$

where $\mathbb{D}_\mu = \mathbb{1}\partial_\mu + ig\mathbb{A}_\mu(x)$ as before. Invariance is now guaranteed.

Expand the covariant derivate and get the full Lagrangian density

$$\mathcal{L} = \underbrace{i\bar{\Psi}\partial_\mu\Psi - m\bar{\Psi}\Psi}_{\text{free fermion}} \underbrace{-\tfrac{1}{2}\mathrm{Tr}\left[\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu}\right]}_{\text{'free' gauge}} - \underbrace{g\bar{\Psi}\gamma^\mu\mathbb{A}_\mu\Psi}_{\text{interaction term}}$$

Expand the interaction term...

$$\begin{aligned}\mathcal{L}_{\text{int}} &= -g\bar{\Psi}\gamma^\mu\mathbb{A}_\mu\Psi \\ &= -g\bar{\Psi}\gamma^\mu\left(W_\mu^+\mathbb{T}_+ + W_\mu^-\mathbb{T}_- + W_\mu^0\,\mathbb{T}_3\right)\Psi \\ &= -g\bar{\Psi}\gamma^\mu\mathbb{T}_+\Psi\, W_\mu^+ - g\bar{\Psi}\gamma^\mu\mathbb{T}_-\Psi\, W_\mu^- - g\bar{\Psi}\gamma^\mu\mathbb{T}_3\Psi\, W_\mu^0 \\ &\equiv -gj_+^\mu\, W_\mu^+ - gj_-^\mu\, W_\mu^- - gj_0^\mu\, W_\mu^0\end{aligned}$$

$j_\pm^\mu = \bar{\Psi}\gamma^\mu\mathbb{T}_\pm\Psi$ are 'charged' currents

$j_0^\mu = \bar{\Psi}\gamma^\mu\mathbb{T}_3\Psi$ is a 'neutral' current

Write the currents explicitly:

- $j_{+}^{\mu} = \bar{\Psi}\gamma^{\mu}\mathbb{T}_{+}\Psi = \bar{\Psi}\gamma^{\mu}\frac{1}{\sqrt{2}}(\mathbb{T}_1 + i\mathbb{T}_2)\Psi$

$$= \frac{1}{\sqrt{2}}(\bar{\psi}_A \quad \bar{\psi}_B)\gamma^{\mu}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \psi_{\mathrm{A}} \\ \psi_{\mathrm{B}} \end{pmatrix} = \frac{1}{\sqrt{2}}\bar{\psi}_A\gamma^{\mu}\psi_{\mathrm{B}}$$

- $j_{-}^{\mu} = \bar{\Psi}\gamma^{\mu}\mathbb{T}_{-}\Psi = \bar{\Psi}\gamma^{\mu}\frac{1}{\sqrt{2}}(\mathbb{T}_1 - i\mathbb{T}_2)\Psi$

$$= \frac{1}{\sqrt{2}}(\bar{\psi}_A \quad \bar{\psi}_B)\gamma^{\mu}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \psi_{\mathrm{A}} \\ \psi_{\mathrm{B}} \end{pmatrix} = \frac{1}{\sqrt{2}}\bar{\psi}_B\gamma^{\mu}\psi_{\mathrm{A}}$$

- $j_{0}^{\mu} = \bar{\Psi}\gamma^{\mu}\mathbb{T}_3\Psi$

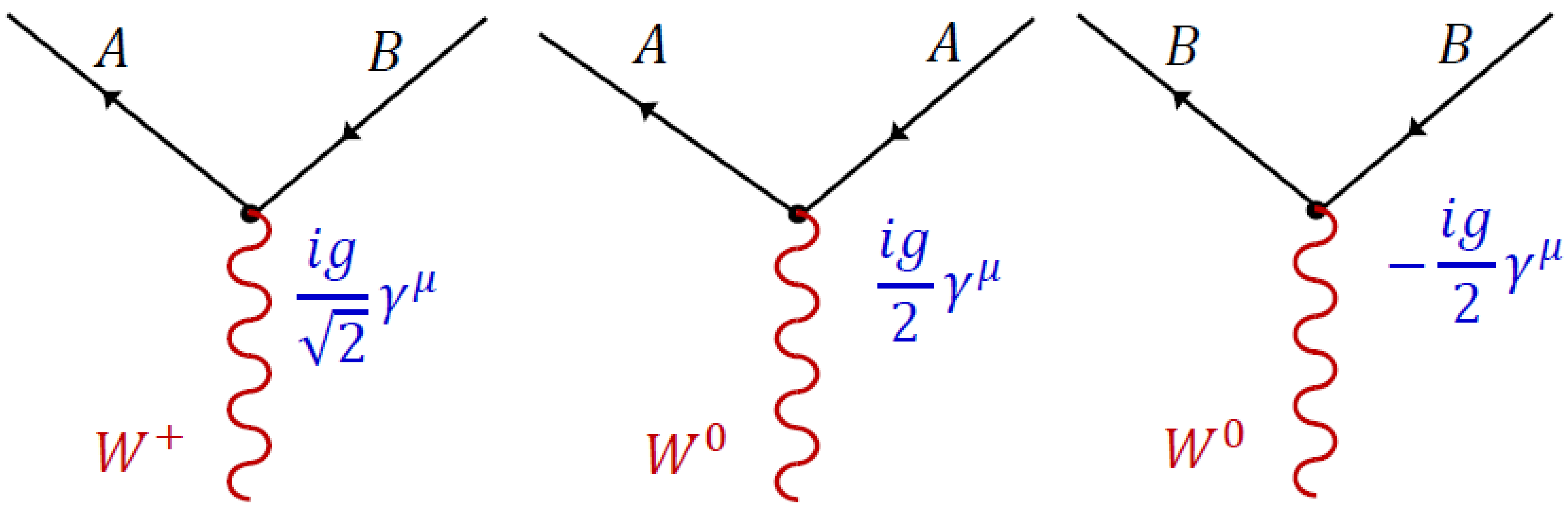
$$= \frac{1}{2}(\bar{\psi}_A \quad \bar{\psi}_B)\gamma^{\mu}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \psi_{\mathrm{A}} \\ \psi_{\mathrm{B}} \end{pmatrix} = \frac{1}{2}(\bar{\psi}_A\gamma^{\mu}\psi_{\mathrm{A}} - \bar{\psi}_B\gamma^{\mu}\psi_{\mathrm{B}})$$

$$\mathcal{L}_{\text{int}} = -gj_+^\mu W_\mu^+ - gj_-^\mu W_\mu^- - gj_0^\mu W_\mu^0$$

$$= -\frac{g}{\sqrt{2}}\bar{\psi}_A\gamma^\mu\psi_{\mathrm{B}}\, W_\mu^+ - \frac{g}{\sqrt{2}}\bar{\psi}_B\gamma^\mu\psi_{\mathrm{A}}\, W_\mu^- \quad \text{c.c. interactions}$$

$$-\frac{g}{2}(\bar{\psi}_A\gamma^\mu\psi_{\mathrm{A}} - \bar{\psi}_B\gamma^\mu\psi_{\mathrm{B}})\, W_\mu^0 \quad \text{n.c. interactions}$$

This leads to vertices



Comparing with the IVB hypothesis for the $W_\mu^{\pm}$, we should be able to identify

$$\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix} = \begin{pmatrix}p\\ n\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix} = \begin{pmatrix}\nu_e\\ e\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix} = \begin{pmatrix}\nu_\mu\\ \mu\end{pmatrix}$$

Q. Can we identify the W_μ^0 with the photon (forgetting the mass)?

If the $W_\mu^{\pm}$ are charged, we will have, under $U(1)_{em}$

$$W_\mu^+ \to W_\mu^{\prime +} = e^{-ie\theta}\, W_\mu^+ \qquad\qquad W_\mu^- \to W_\mu^{\prime -} = e^{+ie\theta}\, W_\mu^-$$

Now, if the term $\bar{\psi}_A \gamma^\mu \psi_B\, W_\mu^+$ is to remain invariant, we must assign charges $q_A e$ and $q_B e$ to the A and B, s.t. the term transforms as

$$\bar{\psi}_A \gamma^\mu \psi_B\, W_\mu^+ \to e^{-ie\theta + iq_A e\theta - iq_B e\theta}\, \bar{\psi}_A \gamma^\mu \psi_B\, W_\mu^+$$

To keep the Lagrangian neutral, we require $q_A - q_B = 1$

But if we look at the W_μ^0 vertices, and consider them to be QED vertices, we must identify

$$\frac{g}{2} = -q_A e \qquad \text{and} \qquad -\frac{g}{2} = -q_B e$$

i.e. $q_A = -q_B$.

Now solve the equations: $q_A - q_B = 1$ and $q_A = -q_B$ ...

result is

$$q_A = -q_B = \frac{1}{2}$$

Two alternatives:

- A and B cannot be the Fermi-IVB particles (defeats whole effort...)
- W_μ^0 cannot be the photon... (already hinted by the mass)

Electroweak unification

Why not just include the U(1)$_{em}$ group as a direct product with the SU(2)$_W$ group?

The transformation matrix on a fermion of charge qe will then look like

$$\mathbb{U} = e^{-ig\vec{\theta}.\vec{\mathbb{T}} - iqe\,\theta'\,\mathbb{T}'}$$

where $\mathbb{T}'$ is the generator of U(1)$_{em}$ and the direct product means that

$$[\mathbb{T}', \mathbb{T}_a] = 0 \quad \forall a$$

The gauge field matrix should expand to

$$g\mathbb{A}_\mu = gW_\mu^+\mathbb{T}_+ + gW_\mu^-\mathbb{T}_- + gW_\mu^0\,\mathbb{T}_3 + qeA_\mu\mathbb{T}'$$

and give us interaction terms as before...

i.e., to the interaction terms with the W boson we must now add interaction terms with the photon:

$$\mathcal{L}_{\text{int}} = -\frac{g}{\sqrt{2}}\bar{\psi}_A\gamma^\mu\psi_{\text{B}}\,W_\mu^+ - \frac{g}{\sqrt{2}}\bar{\psi}_B\gamma^\mu\psi_{\text{A}}\,W_\mu^-$$
$$-\frac{g}{2}\bar{\psi}_A\gamma^\mu\psi_{\text{A}}\,W_\mu^0 + \frac{g}{2}\bar{\psi}_B\gamma^\mu\psi_{\text{B}}\,W_\mu^0$$
$$-q_A e\,\bar{\psi}_A\gamma^\mu\psi_{\text{A}}A_\mu - q_B e\,\bar{\psi}_B\gamma^\mu\psi_{\text{B}}A_\mu$$

Working back, we can write this as

$$\mathcal{L}_{\text{int}} = -(\bar{\psi}_A \quad \bar{\psi}_B)\gamma^\mu \begin{pmatrix} \frac{g}{2}\,W_\mu^0 + q_A e A_\mu & \frac{g}{\sqrt{2}}\,W_\mu^+ \\ \frac{g}{\sqrt{2}}\,W_\mu^- & -\frac{g}{2}\,W_\mu^0 + q_B e A_\mu \end{pmatrix}\begin{pmatrix}\psi_{\text{A}} \\ \psi_{\text{B}}\end{pmatrix}$$

$$= -\overline{\Psi}(g\overrightarrow{A^\mu}.\vec{\mathbb{T}} + eA_\mu\mathbb{T}')\Psi$$ where $$\mathbb{T}' = \begin{pmatrix} q_A & 0 \\ 0 & q_B \end{pmatrix}$$

This generator of $U(1)_{em}$ can be rewritten

$$\mathbb{T}' = \begin{pmatrix} q_A & 0 \\ 0 & q_B \end{pmatrix} = \frac{q_A + q_B}{2}\mathbb{1} + \frac{q_A - q_B}{2}\mathbb{T}_3$$

If we remember that $q_A - q_B = 1$, then

$$\mathbb{T}' = (2q_A + 1)\mathbb{1} + \frac{1}{2}\mathbb{T}_3$$

Paradox!

$$[\mathbb{T}', \mathbb{T}_a] \neq 0 \quad \text{for} \quad a = 1{,}2$$

Glashow (1961) :

We cannot treat weak interactions and electromagnetism as separate (direct product) gauge theories $\Rightarrow$ electroweak unification

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$SU(2)_W x U(1)_Y$ model

Introduce a new $U(1)_y$ which is different from $U(1)_{em}$ and exists as a direct product with the $SU(2)_W$...

The gauge transformation matrix will become

$$\mathbb{U} = e^{-ig\vec{\theta}.\vec{\mathbb{T}}+ig'\theta'\mathbb{T}'}$$

where $\mathbb{T}' = \frac{y}{2}\mathbb{1}$, which, by construction, will commute with all the $\vec{\mathbb{T}}$

We now expand the gauge field matrix as

$$g\mathbb{A}_\mu = gW_\mu^+\mathbb{T}_+ + gW_\mu^-\mathbb{T}_- + gW_\mu^0\,\mathbb{T}_3 - g'B_\mu\mathbb{T}'$$

B_μ is a new gauge field and y is a new quantum number which is clearly same for both the A and B component of the fermion doublet.

We now construct the gauge-fermion interaction term as before

$$\mathcal{L}_{\text{int}} = -g\bar{\Psi}\gamma^{\mu}\mathbb{A}_{\mu}\Psi$$

$$= -\bar{\Psi}\gamma^{\mu}\left(gW_{\mu}^{+}\mathbb{T}_{+} + gW_{\mu}^{-}\mathbb{T}_{-} + gW_{\mu}^{0}\,\mathbb{T}_{3} - g'B_{\mu}\mathbb{T}'\right)\Psi$$

Expanding as before

$$\mathcal{L}_{\text{int}} = -(\bar{\psi}_A \quad \bar{\psi}_B)\gamma^{\mu}\begin{pmatrix} \frac{g}{2}\,W_{\mu}^{0} - \frac{g'y}{2}B_{\mu} & \frac{g}{\sqrt{2}}\,W_{\mu}^{+} \\ \frac{g}{\sqrt{2}}\,W_{\mu}^{-} & -\frac{g}{2}\,W_{\mu}^{0} - \frac{g'y}{2}B_{\mu} \end{pmatrix}\begin{pmatrix}\psi_{\mathrm{A}} \\ \psi_{\mathrm{B}}\end{pmatrix}$$

$$= -\frac{g}{\sqrt{2}}\bar{\psi}_A\gamma^{\mu}\psi_{\mathrm{B}}\,W_{\mu}^{+} - \frac{g}{\sqrt{2}}\bar{\psi}_B\gamma^{\mu}\psi_{\mathrm{A}}\,W_{\mu}^{-}$$

$$-\bar{\psi}_A\gamma^{\mu}\psi_{\mathrm{A}}\left(\frac{g}{2}\,W_{\mu}^{0} - \frac{g'y}{2}B_{\mu}\right) + \bar{\psi}_B\gamma^{\mu}\psi_{\mathrm{B}}\left(\frac{g}{2}\,W_{\mu}^{0} + \frac{g'y}{2}B_{\mu}\right)$$

Glashow (1961): *for some reason*, the W_μ^0 and B_μ mix, i.e. the physical states are orthonormal combinations (demanded by gauge kinetic terms) of the W_μ^0 and B_μ ...

$$\begin{pmatrix} W_\mu^0 \\ B_\mu \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} \qquad c = \cos\omega,\ \ s = \sin\omega$$

In terms of this, the neutral current terms come out to be

$$\mathcal{L}_{\rm nc} = -\bar{\psi}_A \gamma^\mu \psi_{\rm A} \left(\frac{g}{2} W_\mu^0 - \frac{g' y}{2} B_\mu \right) + \bar{\psi}_B \gamma^\mu \psi_{\rm B} \left(\frac{g}{2} W_\mu^0 + \frac{g' y}{2} B_\mu \right)$$

$$= -\frac{1}{2} \bar{\psi}_A \gamma^\mu \psi_{\rm A} \left[(gc - g'ys) Z_\mu - (gs + g'yc) A_\mu \right]$$

$$-\frac{1}{2} \bar{\psi}_B \gamma^\mu \psi_{\rm B} \left[(gc - g'ys) Z_\mu + (gs - g'yc) A_\mu \right]$$

If we now wish to identify A_μ with the photon, we require to set

$$-\frac{1}{2}(gs + g'yc) = q_A e \qquad \frac{1}{2}(gs - g'yc) = q_B e$$

Solving for g and g' we get

$$-gs = (q_A - q_B)e \qquad -g'yc = (q_A + q_B)e$$

Recall that $q_A - q_B = 1$. It follows that

$$e = -gs \qquad e = -g'c\frac{y}{q_A + q_B}$$

Choose $-y = q_A + q_B$. Then

$$e = -g\sin\omega \qquad g' = g\tan\omega$$

Note that ω is some arbitrary angle... it must be nonzero, else $e = 0$

We can also obtain

$$q_A = \frac{1}{2} + \frac{y}{2} \qquad\qquad q_B = -\frac{1}{2} + \frac{y}{2}$$

Now, these $\pm\frac{1}{2}$ are precisely the eigenvalues of the $\mathbb{T}_3$ operator

i.e. we can write a general relation

Sheldon L. Glashow

$$q = t_3 + \frac{y}{2}$$

Looks exactly like the Gell-Mann-Nishijima relation...

Call t_3 the weak isospin and y the weak hypercharge

This gauge theory works pretty well and can give the correct couplings of all the gauge bosons... up to the angle ω, which is not determined by the fermion sector...

Determination of ω : (Salam 1966-67, Weinberg 1967)

Back to the gauge boson mass term...

$$\mathcal{L}_{\text{mass}} = g^2 (\mathbb{A}^\mu \langle\Phi\rangle)^\dagger \left(\mathbb{A}_\mu \langle\Phi\rangle\right) = (g\mathbb{A}^\mu \langle\Phi\rangle)^\dagger \left(g\mathbb{A}_\mu \langle\Phi\rangle\right)$$

For the Glashow theory, we must include the $U(1)_Y$ field in the gauge field matrix, i.e.

$$g\mathbb{A}_\mu = gW_\mu^+ \mathbb{T}_+ + gW_\mu^- \mathbb{T}_- + gW_\mu^0\, \mathbb{T}_3 - g' B_\mu \mathbb{T}'$$

$$= \begin{pmatrix} \frac{g}{2} W_\mu^0 - \frac{g' Y}{2} B_\mu & \frac{g}{\sqrt{2}} W_\mu^+ \\ \frac{g}{\sqrt{2}} W_\mu^- & -\frac{g}{2} W_\mu^0 - \frac{g' Y}{2} B_\mu \end{pmatrix}$$

where Y is the hypercharge of the Φ field.

Thus,

$$g\mathbb{A}_\mu \langle \Phi \rangle = \begin{pmatrix} \frac{g}{2}\, W_\mu^0 - \frac{g' Y}{2} B_\mu & \frac{g}{\sqrt{2}}\, W_\mu^+ \\ \frac{g}{\sqrt{2}}\, W_\mu^- & -\frac{g}{2}\, W_\mu^0 - \frac{g' Y}{2} B_\mu \end{pmatrix} \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{gv}{2}\, W_\mu^+ \\ -\frac{gv}{2\sqrt{2}} (g\, W_\mu^0 + g' Y B_\mu) \end{pmatrix}$$

and

$$(g\mathbb{A}^\mu \langle \Phi \rangle)^\dagger = \begin{pmatrix} \frac{gv}{2} W^{\mu -} & -\frac{gv}{2\sqrt{2}} (g\, W^{\mu 0} + g' Y B^\mu) \end{pmatrix}$$

Multiplying these

$$\mathcal{L}_{\text{mass}} = \left(\frac{gv}{2}\right)^2 W_\mu^+ W^{\mu -} + \left(\frac{v}{2\sqrt{2}}\right)^2 (g\, W^{\mu 0} + g' Y B^\mu)(g\, W_\mu^0 + g' Y B_\mu)$$

Consider only the neutral bosons:

$$(g\, W^{\mu 0} + g' Y B^{\mu})(g\, W_{\mu}^{0} + g' Y B_{\mu})$$
$$= g^2\, W^{\mu 0}\, W_{\mu}^{0} + g g' Y\, W^{\mu 0} B_{\mu} + g g' Y B^{\mu}\, W_{\mu}^{0} + (g' Y)^2 B^{\mu} B_{\mu}$$

One cannot have mass terms of the form $W^{\mu 0} B_{\mu}$ and $B^{\mu}\, W_{\mu}^{0}$ in a viable field theory, since our starting point is always a theory with free fields.

Thus, it is essential to transform to orthogonal states

$$\begin{pmatrix} W_{\mu}^{0} \\ B_{\mu} \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} Z_{\mu} \\ A_{\mu} \end{pmatrix} \qquad c = \cos\omega,\ s = \sin\omega$$

and choose ω to cancel out cross terms...

Rewrite the neutral boson mass terms as

$$(g\, W^{\mu 0} + g^{'} Y B^{\mu})(g\, W_{\mu}^{0} + g^{'} Y B_{\mu})$$

$$= g^2\, W^{\mu 0}\, W_{\mu}^{0} + g g^{'} Y\, W^{\mu 0} B_{\mu} + g g^{'} Y B^{\mu}\, W_{\mu}^{0} + (g^{'} Y)^2 B^{\mu} B_{\mu}$$

$$= (\, W^{\mu 0} \quad B^{\mu}) \begin{pmatrix} g^2 & g g' Y \\ g g' Y & (g^{'} Y)^2 \end{pmatrix} \begin{pmatrix} W_{\mu}^{0} \\ B_{\mu} \end{pmatrix}$$

The diagonalising matrix will be

$$\begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix}$$

where

$$\tan\omega = -\frac{g' Y}{g}$$

How to determine Y ?

Write out the interaction terms for the gauge bosons with the scalar doublet. One finds that once again, to match the couplings to the charges of the W bosons, we get the Gell-Mann-Nishijima relation, i.e.

$$q = t_3 + \frac{Y}{2}$$

Now, the lower component φ_B develops a vacuum expectation value, so it must be neutral, i.e.

$$0 = -\frac{1}{2} + \frac{Y}{2} \quad \Rightarrow Y = 1$$

It follows that

Weinberg angle

$$-\tan\omega = \frac{g'}{g} = \tan\theta_W$$

Eigenvalues of the mass matrix:


Steven Weinberg

$$\begin{pmatrix} g^2 & gg' \\ gg' & g'^2 \end{pmatrix}$$

Determinant = 0 ; trace = $g^2 + g'^2$, i.e.

$$M_A = 0$$

and

$$M_Z^2 = 2\left(\frac{v}{2\sqrt{2}}\right)^2 (g^2 + g'^2) = \left(\frac{gv}{2}\right)^2 \left(1 + \frac{g'^2}{g^2}\right) = M_W^2 (1 + \tan^2\theta_W)$$

$$= M_W^2 \sec^2\theta_W$$

$$\Rightarrow \qquad M_Z = \frac{M_W}{\cos\theta_W}$$

Determination of parameters:

$$\frac{e^2}{4\pi} = \alpha \approx \frac{1}{137} \qquad M_Z = \frac{M_W}{\cos\theta_W}$$

$$e = g\sin\theta_W \qquad g' = g\tan\theta_W$$

Carlo Rubbia

Experimental measurements show that

$M_W \approx 80.4$ GeV and $M_Z \approx 91.2$ GeV

It follows that $\cos\theta_W = M_W/M_Z \approx 0.8816 \Rightarrow \theta_W \approx 28^o.17$

We can now calculate: $e = \sqrt{4\pi\alpha} \approx 0.303$

$$g = e/\sin\theta_W \approx 0.642$$

$$g' = g\tan\theta_W \approx 0.344$$