Statistical methods and error analysis

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Classical confidence intervals

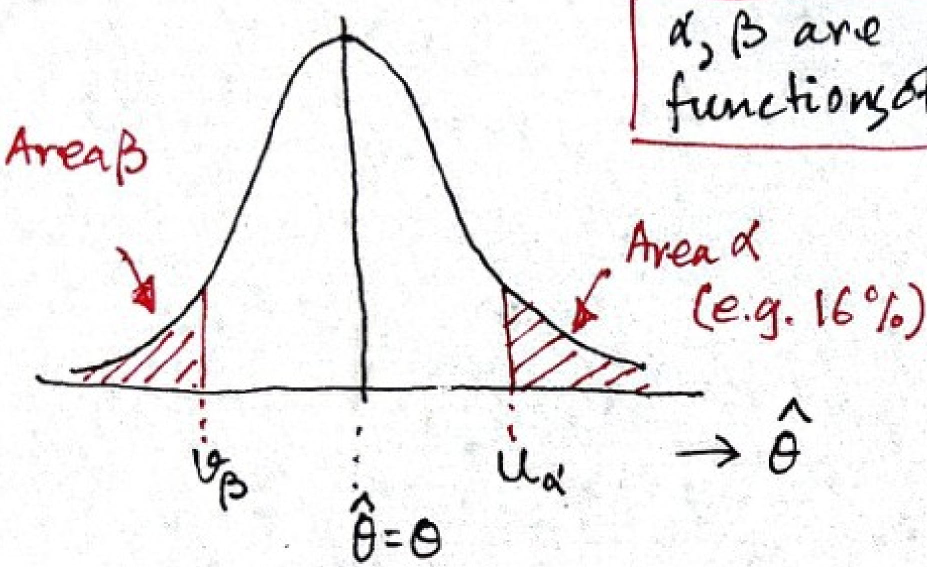
Source: G. Cowan

Due to Neyman, known as Neyman construction

Consider a data set $\{x_1, x_2 \cdots, x_n\}$

From this set an estimator $\hat{\theta}(x_1, x_2, \cdots, x_n)$ is constructed for parameter θ

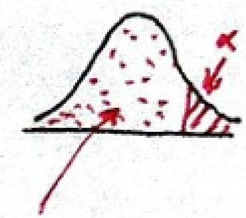
Assume: $g(\hat{\theta};\theta)$, the p.d.f. of θ is known



α, β are functions of θ

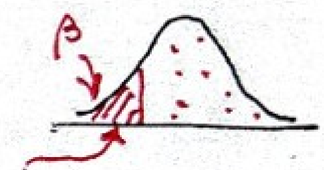
$$\alpha = P(\hat{\theta} \geq u_\alpha(\theta))$$

$$= \int_{u_\alpha}^{\infty} g(\hat{\theta};\theta)\, d\hat{\theta} = 1 - G(u_\alpha;\theta)$$

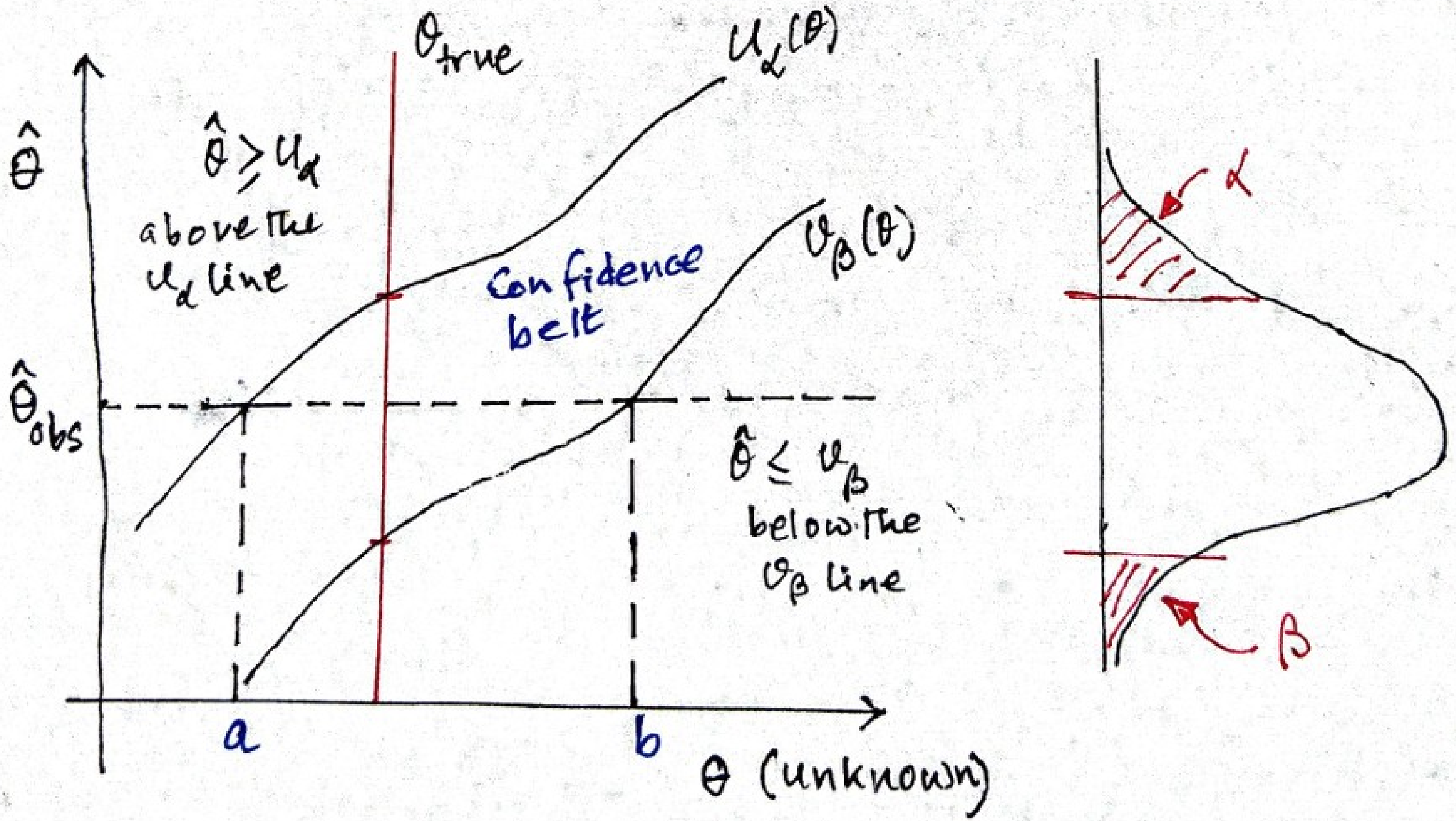


$$\beta = P(\hat{\theta} \leq v_\beta(\theta))$$

$$= \int_{-\infty}^{v_\beta} g(\hat{\theta};\theta)\, d\hat{\theta} = G(v_\beta;\theta)$$



Neyman construction

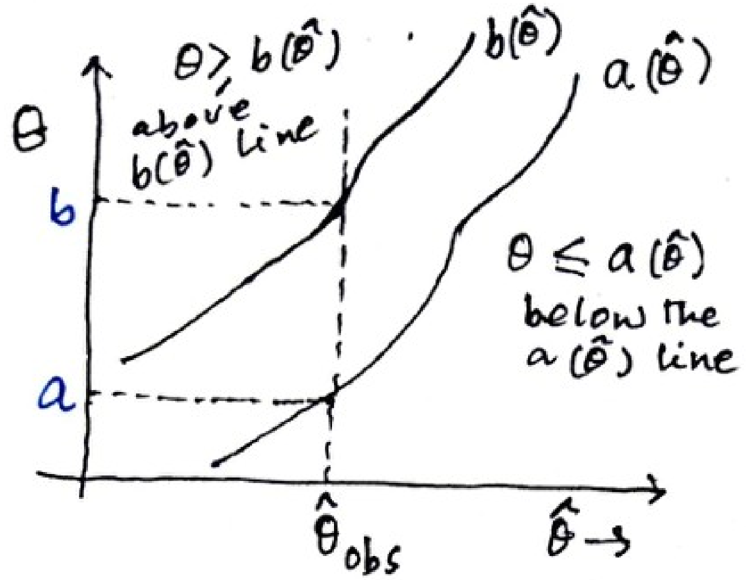


Probability of $\hat{\theta}$ falling inside the interval

$$P\left(v_\beta \leq \hat{\theta} \leq u_\alpha\right) = 1 - \alpha - \beta$$

Neyman construction - 2

If $u_\alpha(\theta)$, $v_\beta(\theta)$ are monotonically increasing functions of θ, u_α, v_β can be inverted



$$a(\hat\theta) = u_\alpha^{-1}(\hat\theta)$$

$$b(\hat\theta) = v_\beta^{-1}(\hat\theta)$$

$$\hat\theta \geq u_\alpha(\theta) \Rightarrow a(\hat\theta) \geq \theta$$

$$\hat\theta \leq v_\beta(\theta) \Rightarrow b(\hat\theta) \leq \theta$$

Therefore: $P(a(\hat\theta) \leq \theta \leq b(\hat\theta)) = 1 - \alpha - \beta$

$[a, b]$ is the classical confidence interval around the observed $\hat\theta = \hat\theta_{obs}$.

Intervals and limits

$[a,b]$ is the classical confidence interval around the observed $\hat{\theta} = \hat{\theta}_{obs}$.

- a, b are random numbers, they change with $\hat{\theta}_{obs}$
- However we just showed that they have a coverage probability $(1-\alpha-\beta)$

i.e. if we construct a 68% interval in this way, $(1-\alpha-\beta) = 0.68$ then in 68% of repeat. experiments $[a,b]$ will contain the true but unknown value of θ.

Intervals and limits

If $\beta = 0$, $\quad P(a \leq \theta) = 1 - \alpha$

lower limit on θ at
$(1-\alpha) \times 100\%$ confidence level
for $\alpha = 0.05 \rightarrow 95\%$ C.L.

If $\alpha = 0$, $\quad P(\theta \leq b) = 1 - \beta$

Upper limit on θ at
$(1-\beta) \times 100\%$ confidence
level

If $\alpha = \beta = \frac{\gamma}{2}$, central interval with C.L. $(1-\gamma) \times 100\%$

e.g. $\gamma = 0.32 \Rightarrow \underset{\times 100}{(1-.32)}\% = 68\%$ C.L.

Interval on Poisson mean

- Single observation n_{obs} of a Poisson process: $\hat{\nu} = n_{\mathrm{obs}}$

$$\alpha = P(\hat{\nu} \geq \hat{\nu}_{\mathrm{obs}}; a),$$

$$\beta = P(\hat{\nu} \leq \hat{\nu}_{\mathrm{obs}}; b),$$

$$\alpha = \sum_{n=n_{\mathrm{obs}}}^{\infty} f(n;a) = 1 - \sum_{n=0}^{n_{\mathrm{obs}}-1} f(n;a) = 1 - \sum_{n=0}^{n_{\mathrm{obs}}-1} \frac{a^n}{n!} e^{-a},$$

$$\beta = \sum_{n=0}^{n_{\mathrm{obs}}} f(n;b) = \sum_{n=0}^{n_{\mathrm{obs}}} \frac{b^n}{n!} e^{-b}.$$

Poisson interval - 2

$$\sum_{n=0}^{n_{\rm obs}} \frac{\nu^n}{n!} e^{-\nu} = \int_{2\nu}^{\infty} f_{\chi^2}(z; n_{\rm d} = 2(n_{\rm obs}+1))\, dz$$

$$= 1 - F_{\chi^2}(2\nu; n_{\rm d} = 2(n_{\rm obs}+1)),$$

- Using the above ralation between chi-sqared and Possion, we get

$$a = \tfrac{1}{2} F^{-1}_{\chi^2}(\alpha; n_{\rm d} = 2n_{\rm obs}),$$

$$b = \tfrac{1}{2} F^{-1}_{\chi^2}(1-\beta; n_{\rm d} = 2(n_{\rm obs}+1))$$

Poisson Interval -3

- Poisson intervals are conservatively large

$$
\begin{aligned}
P(\nu \geq a) &\geq 1-\alpha, \\
P(\nu \leq b) &\geq 1-\beta, \\
P(a \leq \nu \leq b) &\geq 1-\alpha-\beta.
\end{aligned}
$$

- Special case is zero observation --> upper limit ~ 3

$$\beta = \sum_{n=0}^{0} \frac{b^n \, e^{-b}}{n!} = e^{-b}$$

Error on efficiency- CP interval

- Error on efficiency with binomial error does not work if efficiency is close to 1 or 0
- Clopper Pearson is a classical interval, known also as an exact interval

The Clopper–Pearson interval can be written as

$$S_{\leq} \cap S_{\geq}$$

or equivalently,

$$(\inf S_{\geq}\ ,\ \sup S_{\leq})$$

with

$$S_{\leq} := \left\{ \theta \;\middle|\; P\left[\mathrm{Bin}(n;\theta) \leq x\right] > \frac{\alpha}{2} \right\} \text{ and } S_{\geq} := \left\{ \theta \;\middle|\; P\left[\mathrm{Bin}(n;\theta) \geq x\right] > \frac{\alpha}{2} \right\}$$

Source: Wikipedia

- There are other intervals, Agresti-Coull implemented in ROOT