Harry Desmond
astro-ph/1607.01800.

Superfluidity = spontaneous breaking of global $U(1)$, at finite charge density.

Simplest example: $\mathcal{L} = -\partial_\mu \psi \, \partial^\mu \psi^* - m^2 |\psi|^2 - \frac{g}{2} |\psi|^4 .$

$g > 0$.

Invariant under: $\psi \to e^{i\alpha} \psi \, , \quad \alpha \in \mathbb{R} .$

Conserved (Noether) current: $j^\mu = i(\psi^* \partial^\mu \psi - \psi \, \partial^\mu \psi^*)$

Charge density:

$$n = j^0 = -i(\psi^* \partial_t \psi - \psi \, \partial_t \psi^*)$$

↑ number density.

Condensate: Work in mean-field approx'n.

Classical eq'n of motion: $\partial^2 \psi_0 = m^2 \psi_0 + g |\psi_0|^2 \psi_0 .$

Ansatz: $\boxed{\psi_0 = v \, e^{i \mu_R t}}$ where μ_R = (relativistic) chemical potential.

SUBSTITUTE, $\boxed{\mu_R^2 = m^2 + g v^2}$

TO RELATE μ_R TO NON-REL. CHEM. POT. μ,
TAKE NON-REL. LIMIT:

$$\mu_R = \sqrt{m^2 + g v^2} \simeq \underset{\substack{\uparrow \\ \text{REST MASS} \\ \text{ENERGY}}}{m} + \underbrace{\frac{g v^2}{2m}}_{\substack{\text{INTERACTION} \\ \text{ENERGY}}} \quad (1)$$

$\frac{g v^2}{2m} \equiv \mu$

(NOTE: FREE BOSE GAS, $-\infty < \mu < 0$.
FOR $T \to 0$, $\mu \to 0$.)

$$\partial_t n + \nabla \cdot \vec{j} = 0 .$$

$$n \sim \frac{1}{a^3} .$$

SUBSTITUTE THIS ANSATZ INTO n :

$$\boxed{n = 2 \mu_R v^2 \simeq 2 m v^2} \quad (2)$$

$$\therefore \quad \boxed{n = \frac{4 m^2 \mu}{g}}$$

FROM GOLDSTONE'S THM, EXPECT MASSLESS / GAPLESS
MODE $\longleftrightarrow$ PHONON.

EXCITATION: $\psi = (v + \rho(\vec{x},t)) \cdot e^{i(\mu_R t + \pi(\vec{x},t))}$

SUBSTITUTE INTO $\mathcal{L}$:

$$\mathcal{L} = -(\partial_\mu \rho)^2 + (v+\rho)^2 \left[g v^2 + 2\mu_R \dot{\pi} + \dot{\pi}^2 - (\vec{\nabla}\pi)^2 \right] - \frac{g}{2}(v+\rho)^4$$

$\cdot \quad \rho^2 \dot{\pi} \quad , \quad \rho^2 \dot{\pi}^2$

$$\simeq -(\partial_\mu \rho)^2 - g v^2 \rho^2 + \ldots$$

$$\Rightarrow \quad m_\rho^2 = g v^2 .$$

(HIGGS)

AT LOW ENERGY/MOMENTUM, CAN INTEGRATE OUT ρ. TREAT ITS DERIVATIVES PERTURBATIVELY.

TO ZEROTH ORDER, IGNORE GRADIENTS OF ρ:

$$(v+\rho)\left[g v^2 + 2\mu_R \dot{\pi} + \dot{\pi}^2 - (\vec{\nabla}\pi)^2 \right] - g(v+\rho)^3 = 0 .$$

$$\Longrightarrow \quad g(v+\rho)^2 = \underbrace{g v^2 + 2\mu_R \dot{\pi} + \cancel{\dot{\pi}^2} - (\vec{\nabla}\pi)^2}_{\equiv\, X_R}$$

SUBSTITUTE BACK INTO $\mathcal{L}$:

$$\boxed{\mathcal{L} = \frac{1}{2g} X_R^2}$$

NON-REL. LIMIT : $\mu_R \simeq m$, $\dot{\pi} \ll m$.

$$\underline{\underline{X_R}} \simeq g v^2 + 2m\dot{\pi} - (\vec{\nabla}\pi)^2$$

$$= 2m\left(\underbrace{\frac{g v^2}{2m}}_{\equiv\,\mu} + \dot{\pi} - \frac{(\vec{\nabla}\pi)^2}{2m}\right)$$

$$= \underline{\underline{2m}}\Bigg(\underbrace{\mu + \dot{\pi} - \frac{(\vec{\nabla}\pi)^2}{2m}}_{\equiv\, \underline{\underline{X}}}\Bigg)$$

$$\therefore \quad \boxed{\mathcal{L} = \frac{2m^2}{g} X^2}$$

WHERE $\boxed{X = \mu + \dot{\pi} - \frac{(\vec{\nabla}\pi)^2}{2m}}$

EXPAND TO QUADRATIC ORDER IN π,

$$\mathcal{L} \simeq \overbrace{\frac{2m^2\mu^2}{g}}^{\text{CONSTANT}} + \underbrace{\frac{4m^2\mu}{g}}_{\text{TOTAL DERIVATIVE}}\dot{\pi} + \frac{2m^2}{g}\left(\dot{\pi}^2 - \frac{\mu}{m}(\vec{\nabla}\pi)^2\right)$$

$$\frac{4m^2\mu}{g} \equiv n$$

DISPERSION REL'N : $\omega_k = c_s k$

WHERE $\boxed{c_s^2 = \frac{\mu}{m} = \frac{g\,n}{4m^3}}$

PHONONS.

EXERCISE: WORKING TO SUBLEADING ORDER IN $\vec{\nabla}\rho$, FIND :

$$\boxed{\omega_k^2 = c_s^2 k^2 + \frac{k^4}{4m^2}}$$

- $g \to 0 \Rightarrow c_s \to 0$. (NO SUPERFLUIDITY)

  & $\omega_k \to \frac{k^2}{2m}$ (STD DISPERSION REL'N NON-REL. PARTICLES)

- STABILITY ($c_s^2 > 0$) REQUIRES $g > 0$.

  $\longleftrightarrow$ REPULSIVE INTERACTIONS.

EQU'N OF STATE : SET $\pi = 0$, $X = \mu$

$$\therefore \quad \mathcal{L} = \underline{\frac{2m^2}{g}\mu^2} = \underset{\text{PRESSURE}}{\underset{\uparrow}{P(\mu)}}$$

FROM THERMO,

$$\underset{\uparrow}{\underline{\underline{n}}} = \left(\frac{\partial P}{\partial \mu}\right)_T = \frac{4m^2}{g}\underset{\uparrow}{\underline{\underline{\mu}}}$$

$$\therefore \quad \boxed{P(n) \sim n^2}$$

GENERAL EFT : • AT LOW ENERGY / MOMENTUM, RELEVANT DEGREE OF FREEDOM IS GOLDSTONE θ.

• U(1) ACTS NON-LINEARLY :

$$\theta \to \theta + c .$$

• DEMAND GALILEAN INV., IN PARTICULAR GALILEAN BOOSTS :

$$\theta \to \theta + m\vec{v}\cdot\vec{x} + t\,\vec{v}\cdot\vec{\nabla}\theta . \quad (*)$$

TO LEADING ORDER IN DERIVATIVES,

$$\boxed{\mathcal{L} = P(X)} \qquad \boxed{X = \dot{\theta} - \frac{(\vec{\nabla}\theta)^2}{2m}}$$

GREITER, WILCZEK, WITTEN (1989).

EXERCISE: SHOW THAT $\mathcal{L} = P(X)$ IS INVARIANT UNDER $(*)$.

EXERCISE: i) FINITE CHARGE DENSITY:

$$\theta = \mu t + \pi$$

$$\Longrightarrow \boxed{n = \frac{\partial P}{\partial \mu} = P_{,X}\Big|_{\pi=0}}$$

ii) EXPAND TO QUADRATIC ORDER IN π,

$$c_s^2 = \frac{1}{m}\frac{P_{,\mu}}{P_{,\mu\mu}}$$

$$= \frac{P_{,\mu}}{m}\cdot\frac{\partial \mu}{\partial n} = \frac{1}{m}\frac{\partial P}{\partial \mu}\frac{\partial \mu}{\partial n}$$

$$= \frac{\partial P}{\partial \rho}, \quad \rho = mn.$$

✓

EXAMPLES:

- $P(X) \sim X^2 \Longrightarrow P(n) \sim n^2$

  ($2 \to 2$ SCATTERING, $V(\psi) \sim |\psi|^4$)

- $P(X) \sim X^{3/2} \Longrightarrow P(n) \sim n^3$

  ($3 \to 3$ SCATTERING $V(\psi) \sim |\psi|^6$)

- $P(X) \sim \underline{X}^{5/2} \Longrightarrow P(n) \sim n^{5/3}$

(UNITARY FERMI GAS $a \to \infty$)

BEC ——|—— BCS

$a \to \infty$.

CONSIDER SUPERFLUID IN PRESENCE OF <u>EXTERNAL POTENTIAL</u> $V(\vec{x})$.

↳ FOR COLD ATOMS IN LAB, $V(\vec{x}) = V_{trap}(\vec{x})$.

↳ FOR DM IN GALAXIES,

$$V(\vec{x}) = m\Phi(\vec{x}).$$

IN THIS CASE,

$$X = \mu - \underline{V(\vec{x})} + \dot{\pi} - \frac{(\vec{\nabla}\pi)^2}{2m}$$

IN PARTICULAR,

$$\boxed{X = \mu - m\Phi(\vec{x}) + \dot{\pi} - \frac{(\vec{\nabla}\pi)^2}{2m}}$$

<u>WHY?</u> • $\mathcal{L} = -\partial\psi\partial\psi^* - \underline{m^2|\psi|^2} - \frac{g}{2}|\psi|^4 - \Phi\,\underbrace{m|\psi|^2}_{mn = \rho}$.

• EXERCISE: NEWTONIAN GRAV. + SUPERFLUID.

$$\mathcal{L} = -\frac{1}{8\pi G}(\vec{\nabla}\Phi)^2 + P(X)$$

VARY WRT Φ, & SET $\pi = 0$ (~~PHONON~~),

$$\vec{\nabla}^2 \Phi = 4\pi G\, m \underbrace{P_{,X}\Big|_{\pi=0}}_{=n}$$

$$= 4\pi G \rho .$$