Structural Relations between Harmonic Sums up to w=6

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DESY

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1. Introduction

- Single scale processes in massless Quantum Field Theories, or being considered in the limit $m^2/Q^2 \to 0$, exhibit significant simplifications when calculated in Mellin space.
- This is, to some extent, due to structure of Feynman parameter integrals which possess a Mellin symmetry.
- Harmonic sums form the appropriate language to derive compact expressions in the respective calculations.
- We will line out the relations of the harmonic sums, resp. their continuations to $N \in \mathbf{Q}, \mathbf{R}, \mathbf{C}$.

x-space results :

Nielsen-type integrals, resp. harmonic polylogarithms (E. Remiddi and J. Vermaseren (1999))

$$S_{n,p,q}(x) = \frac{(-1)^{n+p+q-1}}{\Gamma(n)p!q!} \int_0^1 \frac{dz}{z} \ln^{(n-1)}(z) \ln^p(1-zx) \ln^q(1+zx)$$

2 Loop Wilson Coefficients: x space

Order α_s^2 contributions to the deep inelastic Wilson coefficient

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$$
\begin{aligned}
C_2^{(2),(+)}(x,1) = C_F^2 \Big[& \frac{1+x^2}{1-x} \{4\ln^3(1-x) - (14\ln x + 9)\ln^2(1-x) \\
& - [4\,\mathrm{Li}_2(1-x) - 12\ln^2 x - 12\ln x + 16\zeta(2) + \tfrac{27}{2}]\ln(1-x) - \tfrac{4}{3}\ln^3 x - \tfrac{3}{2}\ln^2 x \\
& + [-24\,\mathrm{Li}_2(-x) + 24\zeta(2) + \tfrac{61}{2}]\ln x + 12\,\mathrm{Li}_3(1-x) - 12 S_{1,2}(1-x) \\
& + 48\,\mathrm{Li}_3(-x) - 6\,\mathrm{Li}_2(1-x) + 32\zeta(3) + 18\zeta(2) + \tfrac{51}{4}\} \\
& + (1+x)\{2\ln x\ln^2(1-x) + 4[\mathrm{Li}_2(1-x) - \ln^2 x]\ln(1-x) \\
& - 4[\mathrm{Li}_2(1-x) + \zeta(2)]\ln x + \tfrac{5}{3}\ln^3 x - 4\,\mathrm{Li}_3(1-x)\} \\
& + \left(40 + 8x - 48x^2 - \tfrac{72}{5}x^3 + \frac{8}{5x^2}\right)[\mathrm{Li}_2(-x) + \ln x\ln(1+x)] \\
& + (-8+40x)[\ln x\,\mathrm{Li}_2(-x) + S_{1,2}(1-x) - 2\,\mathrm{Li}_3(-x) - \zeta(2)\ln(1-x)] + (5+9x)\ln^2(1-x) \\
& + \tfrac{1}{2}(-91+141x)\ln(1-x) - (28+44x)\ln x\ln(1-x) - (14+30x)\,\mathrm{Li}_2(1-x) \\
& + (\tfrac{29}{2} + \tfrac{25}{2}x + 24x^2 + \tfrac{36}{5}x^3)\ln^2 x + \frac{1}{10}\left(13 - 407x + 144x^2 - \frac{16}{x}\right)\ln x + (-10 + 6x - 48x^2 - \tfrac{72}{5}x^3)\zeta(2) \\
& + \tfrac{407}{20} - \tfrac{1917}{20}x + \tfrac{72}{5}x^2 + \frac{8}{5x} + [6\zeta(2)^2 - 78\zeta(3) + 69\zeta(2) + \tfrac{331}{8}]\delta(1-x) \Big] \\
+ C_A C_F \Big[& \frac{1+x^2}{1-x}\{-\tfrac{11}{3}\ln^2(1-x) + [4\,\mathrm{Li}_2(1-x) + 2\ln^2 x + \tfrac{44}{3}\ln x - 4\zeta(2) + \tfrac{367}{18}]\ln(1-x) \\
& - \ln^3 x - \tfrac{55}{6}\ln^2 x + [4\,\mathrm{Li}_2(1-x) + 12\,\mathrm{Li}_2(-x) - \tfrac{239}{6}]\ln x - 12\,\mathrm{Li}_3(1-x) + 12 S_{1,2}(1-x) - 24\,\mathrm{Li}_3(-x) \\
& + \tfrac{22}{3}\mathrm{Li}_2(1-x) + 2\zeta(3) + \tfrac{22}{3}\zeta(2) - \tfrac{3155}{108}\} \\
& + 4(1+x)[\mathrm{Li}_2(1-x) + \ln x\ln(1-x)] + \left(-20 - 4x + 24x^2 + \tfrac{36}{5}x^3 - \frac{4}{5x^2}\right)[\mathrm{Li}_2(-x) + \ln x\ln(1+x)] \\
& + (4-20x)[\ln x\,\mathrm{Li}_2(-x) + S_{1,2}(1-x) - 2\,\mathrm{Li}_3(-x) - \zeta(2)\ln(1-x)] + (\tfrac{133}{6} - \tfrac{1113}{18}x)\ln(1-x) \\
& + (-2 + 2x - 12x^2 - \tfrac{18}{5}x^3)\ln^2 x + \frac{1}{30}\left(13 + 1753x - 216x^2 + \frac{24}{x}\right)\ln x + (-2 - 10x + 24x^2 + \tfrac{36}{5}x^3)\zeta(2) \\
& - \tfrac{9687}{540} + \tfrac{59157}{540}x - \tfrac{36}{5}x^2 - \frac{4}{5x} + [\tfrac{71}{5}\zeta(2)^2 + \tfrac{140}{3}\zeta(3) - \tfrac{251}{3}\zeta(2) - \tfrac{5465}{72}]\delta(1-x) \Big] \\
+ n_f C_F \Big(& \frac{1+x^2}{1-x}[\tfrac{2}{3}\ln^2(1-x) - (\tfrac{8}{3}\ln x + \tfrac{29}{9})\ln(1-x) - \tfrac{4}{3}\mathrm{Li}_2(1-x) + \tfrac{5}{3}\ln^2 x + \tfrac{19}{3}\ln x - \tfrac{4}{3}\zeta(2) + \tfrac{247}{54}] \\
& + \tfrac{1}{3}(1+13x)\ln(1-x) - \tfrac{1}{3}(7+19x)\ln x - \tfrac{23}{18} - \tfrac{27}{2}x + [\tfrac{4}{3}\zeta(3) + \tfrac{38}{3}\zeta(2) + \tfrac{457}{36}]\delta(1-x) \Big), \qquad (9)
\end{aligned}
$$

where C_A, C_F denote the colour factors and n_f stands for the number of flavours. Here we have put $\mu^2 = Q^2$. The more general case ($\mu^2 \neq Q^2$) can be easily derived using renormalization group methods (see ref. [14]). In the above expression the terms of the type $\ln^i(1-x)/(1-x)$ have to be understood in the distributional sense [12]. The latter and the coefficient of the delta function can be derived from eq. (16) in ref. [13]. The second part in (8) is given by

$$
\begin{aligned}
C_2^{(2),G}(x,1) = n_f C_F \Big[& 8(1+x)^2 \\
& \times [-4S_{1,2}(-x) - 4\ln(1+x)\,\mathrm{Li}_2(-x) - 2\zeta(2)\ln(1+x) - 2\ln x\ln^2(1+x) + \ln^2 x\ln(1+x)] \\
& + 4(1-x)^2\{\tfrac{5}{6}\ln^3(1-x) - (2\ln x + \tfrac{13}{4})\ln^2(1-x) + [2\,\mathrm{Li}_2(1-x) + 2\ln^2 x + 4\ln x + \tfrac{7}{2}]\ln(1-x) - \tfrac{5}{12}\ln^3 x \\
& + [\mathrm{Li}_2(1-x) - 4\,\mathrm{Li}_2(-x) + 3\zeta(2)]\ln x - 4\,\mathrm{Li}_3(1-x) - S_{1,2}(1-x) + 12\,\mathrm{Li}_3(-x) + 13\zeta(3) + \tfrac{13}{2}\zeta(2)\} \\
& + x^2\{\tfrac{10}{3}\ln^3(1-x) - 12\ln x\ln^2(1-x) + [16\ln^2 x - 16\zeta(2)]\ln(1-x) - 5\ln^3 x \\
& + [12\,\mathrm{Li}_2(1-x) + 20\zeta(2)]\ln x - 8\,\mathrm{Li}_3(1-x) + 12 S_{1,2}(1-x)\} \\
& + \left(48 + \tfrac{64}{3}x + \tfrac{96}{5}x^3 + \frac{8}{15x^2}\right)[\mathrm{Li}_2(-x) + \ln x\ln(1+x)] + (14x - 23x^2)\ln^2(1-x) \\
& + (-12x + 10x^2)\ln(1-x) + (-24x + 56x^2)\ln x\ln(1-x) + 64x\,\mathrm{Li}_3(-x) + (-10+24x)\mathrm{Li}_2(1-x) \\
& + (-\tfrac{3}{2} + \tfrac{22}{3}x - 36x^2 - \tfrac{48}{5}x^3)\ln^2 x + \frac{1}{15}\left(-236 + 339x - 648x^2 - \frac{8}{x}\right)\ln x + (64x + 36x^2)\zeta(3) \\
& + (-\tfrac{20}{3}x + 46x^2 + \tfrac{96}{5}x^3)\zeta(2) - \tfrac{647}{15} + \tfrac{239}{5}x - \tfrac{36}{5}x^2 + \frac{8}{15x} \Big] \\
+ n_f C_A \Big\{ & 4(1+x)^2[S_{1,2}(1-x) - 2\,\mathrm{Li}_3(-x) + 4S_{1,2}(-x) - 2\ln x\,\mathrm{Li}_2(1-x) + 4\ln(1+x)\,\mathrm{Li}_2(-x) \\
& + 2\ln x\,\mathrm{Li}_2(-x) + 2\zeta(2)\ln(1+x) + 2\ln x\ln^2(1+x) + \ln^2 x\ln(1+x)] \\
& + 8(1+2x+2x^2)\left[\mathrm{Li}_3\left(\frac{1-x}{1+x}\right) - \mathrm{Li}_3\left(-\frac{1-x}{1+x}\right) - \ln(1-x)\,\mathrm{Li}_2(-x) - \ln x\ln(1-x)\ln(1+x)\right] \\
& + \left(-24 + \tfrac{80}{3}x^2 - \frac{16}{3x}\right)[\mathrm{Li}_2(-x) + \ln x\ln(1+x)] + x^2[-4S_{1,2}(1-x) + 16\,\mathrm{Li}_3(-x) + 8\ln x\,\mathrm{Li}_2(1-x) \\
& + 8\ln^2 x\ln(1+x)] + \tfrac{2}{3}(1 - 2x + 2x^2)\ln^3(1-x) + (24x - 8x^2)\ln x\ln^2(1-x) \\
& + \left(-2 + 36x - \tfrac{122}{3}x^2 + \frac{8}{3x}\right)\ln^2(1-x) + (-4 - 32x + 8x^2)\ln^2 x\ln(1-x) \\
& + (8 - 144x + 148x^2)\ln x\ln(1-x) + (4 + 40x - 8x^2)\ln(1-x)\,\mathrm{Li}_2(1-x) \\
& + (-20 + 24x - 32x^2)\zeta(2)\ln(1-x) + \frac{1}{9}\left(-186 - 1362x + 1570x^2 + \frac{104}{x}\right)\ln(1-x) \\
& + (-4 - 72x + 8x^2)\,\mathrm{Li}_3(1-x) + \frac{1}{3}\left(12 - 192x + 176x^2 + \frac{16}{x}\right)\mathrm{Li}_2(1-x) + \tfrac{1}{3}(10 + 28x)\ln^3 x \\
& + (-1 + 88x - \tfrac{194}{3}x^2)\ln^2 x + (-48x + 16x^2)\zeta(2)\ln x + (58 + \tfrac{584}{3}x - \tfrac{2090}{9}x^2)\ln x - (10 + 12x + 12x^2)\zeta(3) \\
& + \frac{1}{3}\left(12 - 240x + 268x^2 - \frac{32}{x}\right)\zeta(2) + \tfrac{239}{9} + \tfrac{1072}{9}x - \tfrac{4493}{27}x^2 + \frac{344}{27x} \Big\}, \qquad (5)
\end{aligned}
$$

W.L. van Neerven et al.: 79 functions 80 objects would be maximal.

2. Algebraic Relations

cf. J.Blümlein, Comput. Phys. Commun. **159** (2004) 19

Number of harmonic sums up to weight w : 3^{w-1}.

Harmonic sums form a quasi-shuffle algebra through $\sqcup\!\sqcup$. (M. Hoffman)

$$\begin{aligned} S_{a_1,a_2} \sqcup\!\sqcup S_{a_3,a_4} &= S_{a_1,a_2,a_3,a_4} + S_{a_1,a_3,a_2,a_4} + S_{a_1,a_2,a_4,a_2} \\ &+ S_{a_3,a_4,a_1,a_2} + S_{a_3,a_1,a_4,a_2} + S_{a_3,a_1,a_2,a_4} \quad etc. \end{aligned}$$

Solve all the linear equations possible for the harmonic sums $\Longrightarrow$ algebraic basis.

Let $\{a, a, a, ..., b, b, ..., ..., z, z\}$ a set of n_1 a's, n_2 b's etc. The number of basis elements corresponding to all words formed by ALL the above letters is:

$$l_n(n_1, ..., n_q) = \frac{1}{n} \sum_{d|n_i} \mu(d) \frac{(n/d)!}{(n_1/d)!...(n_d/d)!}, \quad \sum_i n_i = n$$

(E. Witt, 1937) $\Longrightarrow$ # Lyndon words

w	1	2	3	4	5	6
$\#_c$	2	8	26	80	242	728
$\#_r$	0	1	7	23	69	183

Algebraic Relations

Observation in Quantum Field Theory :

At least up to $O(\alpha_s^3)$ the contributing harmonic sums never exhibit any index $a_k = -1$ applying a compact representation.

The number of sums of this type is

$$N_{\neg\{-1\}}(w) = \frac{1}{2}\left[\left(1-\sqrt{2}\right)^w + \left(1+\sqrt{2}\right)\right]$$

$$N^{\rm basic}_{\neg\{-1\}}(\mathrm{w}) \;=\; \frac{2}{\mathrm{w}} \sum_{d|\mathrm{w}} \mu\left(\frac{\mathrm{w}}{d}\right) N^{\rm basic}_{\neg\{-1\}}(d)\;.$$

w	1	2	3	4	5	6
$\#_c$	1	4	11	28	69	168
$\#_r$	1	3	7	14	30	60

Algebraic Relations

Side Remark:

Harmonic, Generalized Harmonic Polylogarithms and Multiple Polylogarithms also form shuffle algebras. As shuffle algebras are sub-sets of the quasi-shuffle algebra studied above, the respective algebraic relations can be derived directly.

- Form the index alphabet.
- Solve the shuffle-relations $\Longrightarrow$ Basis

As the relations in J.Blümlein, Comput. Phys. Commun. **159** (2004) 19 are of arbitrary weight (general alphabet) and depth $d \leq 6$ the corresponding relations can be read off there.

Algorithms to extend this scenario are available and can be run.

3. Structural Relations

w = 1:

$$\frac{1}{1-x} \quad \& \quad \frac{1}{1+x}$$

$$\frac{1}{1-x^2} = \frac{1}{2}\left[\frac{1}{1-x} + \frac{1}{1+x}\right]$$

$$\mathbf{M}\left[\left(\frac{1}{1-x}\right)_+\right]\left(\frac{N}{2}\right) = \mathbf{M}\left[\left(\frac{1}{1-x}\right)_+\right](N) + \mathbf{M}\left[\frac{1}{1+x}\right](N) + \ln(2)$$

$$\begin{aligned} -\psi\left(\frac{N}{2}\right) - \gamma_E &= -\psi(N) - \gamma_E + \beta(N) + \ln(2) \\ \beta(N) &= \frac{1}{2}\left[\psi\left(\frac{N+1}{2}\right) - \psi\left(\frac{N}{2}\right)\right] \end{aligned}$$

- $S_{-1}(N)$ depends on $S_1(N)$ for $N \epsilon \mathbf{Q}$

Structural Relations

$N \,\epsilon\, \mathbf{R}$:

$$S_2(N) = -\frac{d}{dN} S_1(N) + \zeta_2 \quad \text{(etc.)}$$

For $N \,\epsilon\, \mathbf{R}$: only one independent single sum occurs.

$$S_1(N) = \sum_{k=1}^{N} \frac{1}{k} = \psi(N+1) + \gamma_E$$

w = 2:

$$\mathbf{M}\left[\frac{\ln(1-x)}{1+x}\right](N) = -\mathbf{M}\left[\frac{\ln(1+x)}{1+x}\right](N) - [\psi(N) + \gamma_E + \ln(2)]\beta)N) + \beta'(N)$$

See also relations for Nielsen´s ξ, η, ξ_1 and ξ_2 functions.

$$F_1(N) := \mathbf{M}\left[\frac{\ln(1+x)}{1+x}\right](N) \rightarrow S_{1,-1}(N)$$

Structural Relations

The Reduction for $\mathrm{Li}_k(-x)/(x \pm 1)$:

$$\frac{1}{2^{k-2}} \frac{\mathrm{Li}_k(x^2)}{1-x^2} = \frac{\mathrm{Li}_k(x)}{1-x} + \frac{\mathrm{Li}_k(x)}{1+x} + \frac{\mathrm{Li}_k(-x)}{1-x} + \frac{\mathrm{Li}_k(-x)}{1+x} \rightarrow \frac{\mathrm{Li}_k(-x)}{1-x}$$

- There always exists another IBP relation to express also $\mathrm{Li}_k(-x)/(1+x)$
- At even k there exists an algebraic relation which yields an additional relation for $\mathrm{Li}_k(x)/(1+x)$.
- Applying differential operators one may show :

For $N \epsilon \mathbf{R}$ double harmonic sums can always be represented by one basic function for even weight and two basic functions for odd weight.

w = 3:

$$\rightarrow \frac{\mathrm{Li}_2(x)}{x \pm 1}, \qquad \frac{\ln^2(1+x)}{x \pm 1}$$

Structural Relations

w = 4; $i \neq -1, \rightarrow$

$$\frac{\mathrm{Li}_3(x)}{x+1}, \qquad \frac{S_{1,2}(x)}{x \pm 1}$$

w = 5; $i \neq -1, \rightarrow$

$$\frac{\mathrm{Li}_4(x)}{x \pm 1} \qquad \frac{S_{1,3}(x)}{x+1} \qquad \frac{S_{2,2}(x)}{x \pm 1} \qquad \frac{\mathrm{Li}_2^2(x)}{x+1} \qquad \frac{S_{2,2}(x) - \mathrm{Li}_2^2(x)/2}{x \pm 1}$$

w = 6; $i \neq -1, \rightarrow$

$$\frac{\mathrm{Li}_5(x)}{x+1} \qquad \frac{S_{3,2}(x)}{x \pm 1} \qquad \frac{S_{2,3}(x)}{x \pm 1} \qquad \frac{S_{1,4}(x)}{x \pm 1} \qquad \frac{\mathrm{Li}_2(x)\mathrm{Li}_3(x)}{x \pm 1}$$

$$\frac{\mathrm{Li}_2(-x)\mathrm{Li}_3(-x)}{x-1} \qquad \frac{S_{3,2}(-x)}{x-1} \qquad \frac{A_1(x)}{x+1} \qquad \frac{A_1(-x)}{x-1} \qquad \frac{A_2(x)}{x+1}, + \ldots$$

Structural Relations

New numerator functions :

$$
\begin{aligned}
A_1(x) &= \int_0^x \frac{dy}{y} \mathrm{Li}_2^2(y) \\
A_2(x) &= \int_0^x \frac{dy}{y} [\mathrm{Li}_4(1-y) - \zeta_4], \ldots
\end{aligned}
$$

Representation of some Observables

- Unpolarized and Polarized Drell-Yan an Higgs-Boson Production Cross Section $O(\alpha_s^2)$ w = 4 J.B. and V. Ravindran, Nucl. Phys. **B716** (2005) 128.
- Unpolarized and Polarized Time-like Anomalous Dimensions and Wilson Coefficients $O(\alpha_s^2)$ w = 4 J.B. and V. Ravindran, Nucl. Phys. **B749** (2006) 1.
- Anomalous Dimensions and Wilson Coefficients $O(\alpha_s^3)$ w = 5, 6 from: S. Moch, J. Vermaseren, A. Vogt, Nucl. Phys. **B688** (2004) 101; **691** (2004) 129; **B724** (2005) 3 $\longrightarrow$ J.B., DESY 07-042
- Polarized and Unpolarized Wilson Coefficients $O(\alpha_s^2)$ w = 4 J.B. and S. Moch, to appear
- Polarized and Unpolarized asymptotic Heavy Flavor Wilson Coefficients $O(\alpha_s^{2(3)})$ w = 4 J.B., A. de Freitas, W. van Neerven, S. Klein, Nucl. Phys. **B755** (2006) 272; I. Bierenbaum, J.B., S. Klein, DESY 07-026, J.B. and S. Klein, DESY 07-027
- Virtual and soft corrections to Bhabha Scattering $O(\alpha^2)$ w = 4 J.B. and S. Klein

Example: Bhabha s+v

$$
\begin{aligned}
T_0 \;=\;& \frac{248+15\,N^2+N^4}{2(N-2)(N-1)N(N+1)(N+2)}S_{1,1,1,1}(N)+\frac{-2}{(N-1)(N+1)}\mathbf{S_{2,1,1}}(N)\\
&+\frac{-340+120\,N+17\,N^2+18\,N^3-31\,N^4}{2(N-2)(N-1)N(N+1)(N+2)}S_{3,1}(N)+\frac{1344-502\,N-69\,N^2-2\,N^3+57\,N^4}{8(N-2)(N-1)N(N+1)(N+2)}S_4(N)\\
&+\frac{304-328\,N-500\,N^2+330\,N^3-6\,N^4+6\,N^5-2\,N^6+4\,N^7}{(N-2)^2(N-1)^2N^2(N+1)(N+2)}\mathbf{S_{2,1}}(N)\\
&+\frac{-112-4\,N^2-4\,N^4}{(N-2)(N-1)N(N+1)(N+2)}S_{2,1}(N)\mathbf{S_1}(N)+\frac{-48+8\,N+6\,N^2+7\,N^3}{(N-1)N(N+1)(N+2)}S_3(N)S_1(N)\\
&+\frac{-1840+292\,N+5532\,N^2+827\,N^3-1978\,N^4-274\,N^5+36\,N^6+19\,N^7-22\,N^8}{4(N-2)^2(N-1)^2N^2(N+1)^2(N+2)}S_{1,1,1}(N)\\
&+\frac{128-56\,N-252\,N^2+54\,N^3+177\,N^4-91\,N^5+19\,N^6+9\,N^7}{2(N-2)(N-1)^2N^2(N+1)^2(N+2)}S_3(N)\\
&+\frac{4032-2048\,N-14200\,N^2+5036\,N^3+23610\,N^4+2521\,N^5-12342\,N^6}{4(N-2)^3(N-1)^3N^3(N+1)^3(N+2)}S_{1,1}(N)\\
&+\frac{-3365\,N^7+2148\,N^8+903\,N^9+14\,N^{10}-167\,N^{11}+50\,N^{12}}{4(N-2)^3(N-1)^3N^3(N+1)^3(N+2)}S_{1,1}(N)\\
&+\frac{-124+16\,N+24\,N^2-4\,N^3-14\,N^4}{(N-2)(N-1)N(N+1)(N+2)}S_{1,1}(N)\zeta(2)+\frac{424-118\,N+9\,N^2-2\,N^3+23\,N^4}{4(N-2)(N-1)N(N+1)(N+2)}S_2(N)S_{1,1}(N)\\
&+\frac{224+144\,N-1216\,N^2-56\,N^3+1786\,N^4+641\,N^5-406\,N^6}{4(N-2)^2(N-1)^3N^3(N+1)^3(N+2)}S_2(N)\\
&+\frac{+17\,N^7-308\,N^8+141\,N^9-56\,N^{10}+N^{11}}{4(N-2)^2(N-1)^3N^3(N+1)^3(N+2)}S_2(N)+\frac{58+21\,N+N^2+15\,N^3+10\,N^4}{(N-2)(N-1)N(N+1)(N+2)}S_2(N)\zeta(2)
\end{aligned}
$$

Example: Bhabha s+v

$$
\begin{aligned}
&+\frac{232-384\,N^2-17\,N^3+286\,N^4-128\,N^5-14\,N^6+N^7}{4(N-2)(N-1)^2N^2(N+1)^2(N+2)}S_2(N)S_1(N)\\
&+\frac{-560-26\,N-31\,N^2-10\,N^3-33\,N^4}{8(N-2)(N-1)N(N+1)(N+2)}S_2(N)^2\\
&+\frac{576+1088\,N-3280\,N^2-5136\,N^3+11764\,N^4+20392\,N^5-17385\,N^6-30114\,N^7}{4(N-2)^3(N-1)^4N^4(N+1)^4(N+2)}S_1(N)\\
&+\frac{+5984\,N^8+17228\,N^9-1228\,N^{10}-2754\,N^{11}-112\,N^{12}-8\,N^{13}+33\,N^{14}-24\,N^{15}}{4(N-2)^3(N-1)^4N^4(N+1)^4(N+2)}S_1(N)\\
&+\frac{-56+336\,N+522\,N^2+424\,N^3-53\,N^4-500\,N^5+60\,N^6+28\,N^7-5\,N^8}{2(N-2)^2(N-1)^2N^2(N+1)^2(N+2)}S_1(N)\zeta(2)\\
&+\frac{64+6\,N^2+N^3}{(N-2)(N-1)N(N+1)}S_1(N)\zeta(3)+\frac{2112+608\,N+76\,N^2-140\,N^3+107\,N^4}{10(N-2)(N-1)N(N+1)(N+2)}\zeta(2)^2\\
&+\frac{-224-136\,N+1688\,N^2+1290\,N^3-1998\,N^4-1997\,N^5+198\,N^6}{2(N-1)^3N^3(N-2)^2(N+2)(N+1)^3}\zeta(2)\\
&+\frac{+405\,N^7+376\,N^8-119\,N^9+56\,N^{10}+5\,N^{11}}{2(N-1)^3N^3(N-2)^2(N+2)(N+1)^3}\zeta(2)\\
&+\frac{-552+144\,N+1654\,N^2-370\,N^3-361\,N^4+19\,N^5+35\,N^6-25\,N^7}{2(N-2)^2(N-1)^2N^2(N+1)^2}\zeta(3)\\
&+\frac{320-64\,N-1920\,N^2+1600\,N^3+6524\,N^4-14872\,N^5-19036\,N^6+31543\,N^7-43960\,N^8-13935\,N^9}{16(N-1)^5(N+1)^5(N-2)^3N^5(N+2)}\\
&+\frac{+65372\,N^{10}+26822\,N^{11}-44576\,N^{12}-9558\,N^{13}+9840\,N^{14}+339\,N^{15}+428\,N^{16}-371\,N^{17}+128\,N^{18}}{16(N-1)^5(N+1)^5(N-2)^3N^5(N+2)}\\
&+4\frac{N^4-N^2+12}{(N-2)(N-1)N(N+1)(N+2)}f_{0,2}+(-2)\frac{N^4-N^2+12}{(N-2)(N-1)N(N+1)(N+2)}f_{0,1}^2
\end{aligned}
$$

z-space: A. Penin, (2005)

$\Longrightarrow$ 3 basic sums only; no alternating sums.

4. Factorial Series

Consider

$$\Omega(z) = \int_0^1 dt\, t^{z-1}\, \varphi(t); \qquad \varphi(1-t) = \sum_{k=0}^{\infty} a_k t^k$$

$$Re(z) > 0, \qquad \Omega(z) = \sum_{k=0}^{\infty} \frac{a_{k+1} k!}{z(z+1)\ldots(z+k)}$$

- $\Omega(z)$ is meromorphic in $z \,\epsilon\, \mathbf{C}$, obeys a recursion $z \rightarrow z+1$ and has an analytic asymptotic representation.

Example:

$$\begin{aligned}
F_5(z) &= \mathbf{M}\left[\frac{\mathrm{Li}_2(z)}{1+z}\right](z) \\
F_5(z+1) &= -F_5(z) + \frac{1}{z}\left[\zeta_2 - \frac{\psi(z+1)+\gamma_E}{z}\right] \\
\mathrm{Asymp.\ ser.} : \mathrm{Li}_2(z) &\rightarrow \mathrm{Li}_2(1-z) \\
\mathbf{M}\left[\frac{\mathrm{Li}_2(1-z)}{1+z}\right](N) &\propto \frac{1}{2N^2} + \frac{1}{4N^3} - \frac{7}{24}\frac{1}{N^4} - \frac{1}{3}\frac{1}{N^5} + \frac{73}{120}\frac{1}{N^6}\ldots
\end{aligned}$$

5. The Basis

$$
\begin{array}{llll}
w=1 & 1/(x-1)_+ & & \\
w=2 & \ln(1+x)/(x+1) & & \\
w=3 & \mathrm{Li}_2(x)/(x\pm 1) & & \\
w=4 & \mathrm{Li}_3(x)/(x+1) & S_{1,2}(x)/(x\pm 1) & \\
w=5 & \mathrm{Li}_4(x)/(x\pm 1) & S_{1,3}(x)/(x+1) & S_{2,2}(x)/(x\pm 1) \\
 & \mathrm{Li}_2^2(x)/(x+1) & [S_{2,2}(-x)-\mathrm{Li}_2^2(-x)/2]/(x\pm 1) & \\
w=6 & \mathrm{Li}_5(x)/(x+1) & S_{1,4}(x)/(x\pm 1) & S_{2,3}(x)/(x\pm 1) \\
 & S_{3,2}(x)/(x\pm 1) & \ldots &
\end{array}
$$

- $O(\alpha)$ Wilson Coefficients/anom. dim. $\#1$
- $O(\alpha^2)$ Anomalous Dimensions $\#2$
- $O(\alpha^2)$ Wilson Coefficients $\# \leq 5$
- $O(\alpha^3)$ Anomalous Dimensions $\#15$
- $O(\alpha^3)$ Wilson Coefficients $\#29+$

6. Conclusions

- The single-scale quantities in Quantum Field Theories to 3 Loop Order $\Leftrightarrow$ w = 6 can be represented in a polynomial ring spanned by a few Mellin transforms of the above basic functions, which are the same for all known processes. This points to their general nature.
- The basic Mellin transforms are meromorphic functions with single poles at the non-positive integers.
- The total amount of harmonic sums reduces due to algebraic relations (index structure), and structural relations N ϵ $\mathbf{Q}$, N ϵ $\mathbf{R}$.
- They can be represented in terms of factorial series up to simple "soft components". This allows an exact analytic continuation.
- Up to w = 6 physical (pseudo-) observables are free of harmonic sums with index = {-1}. Up to w = 5 all numerator functions are Nielsen integrals.