Parton Shower with Quantum Interferences

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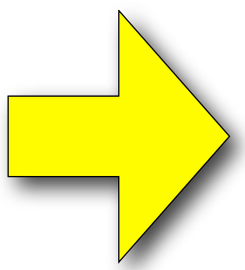
Introduction

- The parton shower relies on the universal soft and collinear factorization of the QCD matrix elements. It is universal property and true at all order. This should be the only approximation ...

... but we have some further approximations:

- ✗ Interference diagrams are treated approximately with the angular ordering
- ✗ Color treatment is valid in the $N_c \to \infty$ limit (correct only in e^+e^- annihilation)
- ✗ Spin treatment is usually approximated.
- ✗ Usually very crude approximation in the phase space
- ✗ "Hidden tricks"



Parton shower as classical statistical mechanics

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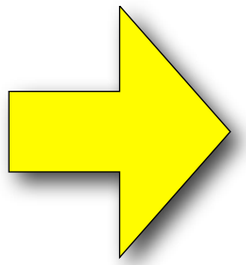

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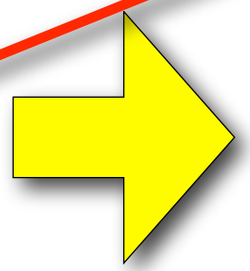

Parton shower as classical statistical mechanics

Density Matrix

The physical cross section is

$$\sigma[F] = \sum_m \int [d\{p,f\}_m] \overbrace{f_{a/A}(\eta_\mathrm{a},\mu_F^2)\, f_{b/B}(\eta_\mathrm{b},\mu_F^2)}^{\text{parton distributions}} \frac{1}{2\eta_\mathrm{a}\eta_\mathrm{b} p_A\cdot p_B}$$
$$\times \langle \mathcal{M}(\{p,f\}_m)| \underbrace{F(\{p,f\}_m)}_{\text{observable}} \underbrace{|\mathcal{M}(\{p,f\}_m)\rangle}_{\text{matrix element}}$$

It is useful to write this as trace in the color and spin space

$$\sigma[F] = \sum_m \int [d\{p,f\}_m] \,\mathrm{Tr}\{\underbrace{\rho(\{p,f\}_m)}_{\text{density operator in color} \otimes \text{ spin space}} F(\{p,f\}_m)\}$$

Density Matrix

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$$\int [d\{p,f\}_m] \equiv \frac{1}{m!}\prod_{i=1}^{m}\left\{\sum_{f_i}\int \frac{d^4p_i}{(2\pi)^4}\,2\pi\delta_+(p_i^2)\right\}\sum_a\int_0^1 d\eta_\mathrm{a}\sum_b\int_0^1 d\eta_\mathrm{b}$$
$$\times (2\pi)^4\delta\left(\eta_\mathrm{a}p_A+\eta_\mathrm{b}p_B-\sum_{i=1}^{m}p_i\right)$$

It

and spin space

$$\sigma[F] = \sum_m \int [d\{p,f\}_m]\,\mathrm{Tr}\{\underbrace{\rho(\{p,f\}_m)}F(\{p,f\}_m)\}$$

density operator in color $\otimes$ spin space

Density Matrix

The density operator is

$$\rho(\{p,f\}_m) = \left|\mathcal{M}(\{p,f\}_m)\right\rangle \frac{f_{a/A}(\eta_\mathrm{a},\mu_F^2) f_{b/B}(\eta_\mathrm{b},\mu_F^2)}{2\eta_\mathrm{a}\eta_\mathrm{b} p_A\cdot p_B} \left\langle\mathcal{M}(\{p,f\}_m)\right|$$

or expanding it on a color and spin basis

$$\rho(\{p,f\}_m) = \sum_{s,c}\sum_{s',c'} \left|\{s,c\}_m\right\rangle \rho(\{p,f,s',c',s,c\}_m) \left\langle\{s',c'\}_m\right|$$

We use conventional treatment of spin, thus we have orthogonal basis:

$$\left\langle\{s'\}_m\middle|\{s\}_m\right\rangle = \prod_{i=\mathrm{a},\mathrm{b},1,\ldots,m} \delta_{s_i}^{s'_i}$$

Density Matrix

In the color space we use a basis which is normalized but not orthogonal:

$$\langle \{c\}_m | \{c\}_m \rangle = 1 \quad \text{but} \quad \langle \{c'\}_m | \{c\}_m \rangle = \mathcal{O}(1/N_c^2) \quad \text{for } \{c'\}_m \neq \{c\}_m$$

It is useful to introduce a dual basis $|\{c\}_m\rangle_D$ that is defined by

$${}_D\langle \{c'\}_m | \{c\}_m \rangle = \delta^{\{c'\}_m}_{\{c\}_m}$$

and the completeness relations are

$$1 = \sum_{\{c\}_m} |\{c\}_m\rangle \; {}_D\langle \{c\}_m| \qquad \text{and} \qquad 1 = \sum_{\{c\}_m} |\{c\}_m\rangle_D \; \langle \{c\}_m|$$

Classical States

The set of functions $\rho(\{p, f, s', c', s, c\}_m)$ forms a vector space.

Basis: $\left|\{p, f, s', c', s, c\}_m\right)$

Completeness relation :

$$1 = \sum_m \int \left[d\{p, f, s', c', s, c\}_m\right] \left|\{p, f, s', c', s, c\}_m\right)\left(\{p, f, s', c', s, c\}_m\right|$$

where

$$\int \left[d\{p, f, s', c', s, c\}_m\right] \equiv \int \left[d\{p, f\}_m\right] \sum_{s_\mathrm{a}, s'_\mathrm{a}, c_\mathrm{a}, c'_\mathrm{a}} \sum_{s_\mathrm{b}, s'_\mathrm{b}, c_\mathrm{b}, c'_\mathrm{b}} \prod_{i=1}^{m} \left\{ \sum_{s_i, s'_i, c_i, c'_i} \right\}$$

Inner product of the basis states:

$$\left(\{p, f, s', c', s, c\}_m \middle| \{\tilde{p}, \tilde{f}, \tilde{s}', \tilde{c}', \tilde{s}, \tilde{c}\}_{\tilde{m}}\right) = \delta_{m,\tilde{m}}\ \delta(\{p, f, s', c', s, c\}_m; \{\tilde{p}, \tilde{f}, \tilde{s}', \tilde{c}', \tilde{s}, \tilde{c}\}_{\tilde{m}})$$

Classical State

A physical state which is related to the density matrix:

$$|\rho) = \int \left[d\{p, f, s', c', s, c\}_m\right] \rho(\{p, f, s', c', s, c\}_m) \left|\{p, f, s', c', s, c\}_m\right)$$

or the vector corresponding to the measurement function

$$\left(F\middle|\{p, f, s', c', s, c\}_m\right) = \left\langle \{s', c'\}_m \middle| F(\{p, f\}_m) \middle| \{s, c\}_m \right\rangle$$

Then the cross section that corresponding to this measurement function is

$$\sigma[F] = \left(F\middle|\rho\right)$$

Parton Shower Evolution

We use an evolution variable e.g.:

$$\log \frac{Q^2}{q^2} = t \in [0, \infty]$$

The evolution is given by
a linear operator

$$|\rho(t)) = U(t, t_0)|\rho(t_0))$$

Group decomposation

$$U(t_3, t_2)\, U(t_2, t_1) = U(t_3, t_1)$$

Preserves the normalization

$$\big(1\big|A(t_0)\big) = 1 \quad \Rightarrow \quad \big(1\big|U(t, t_0)\big|\rho(t_0)\big) = 1$$

Parton Shower Evolution

Using the factorization properties of the QCD the approximated order by order calculation can be organized according to

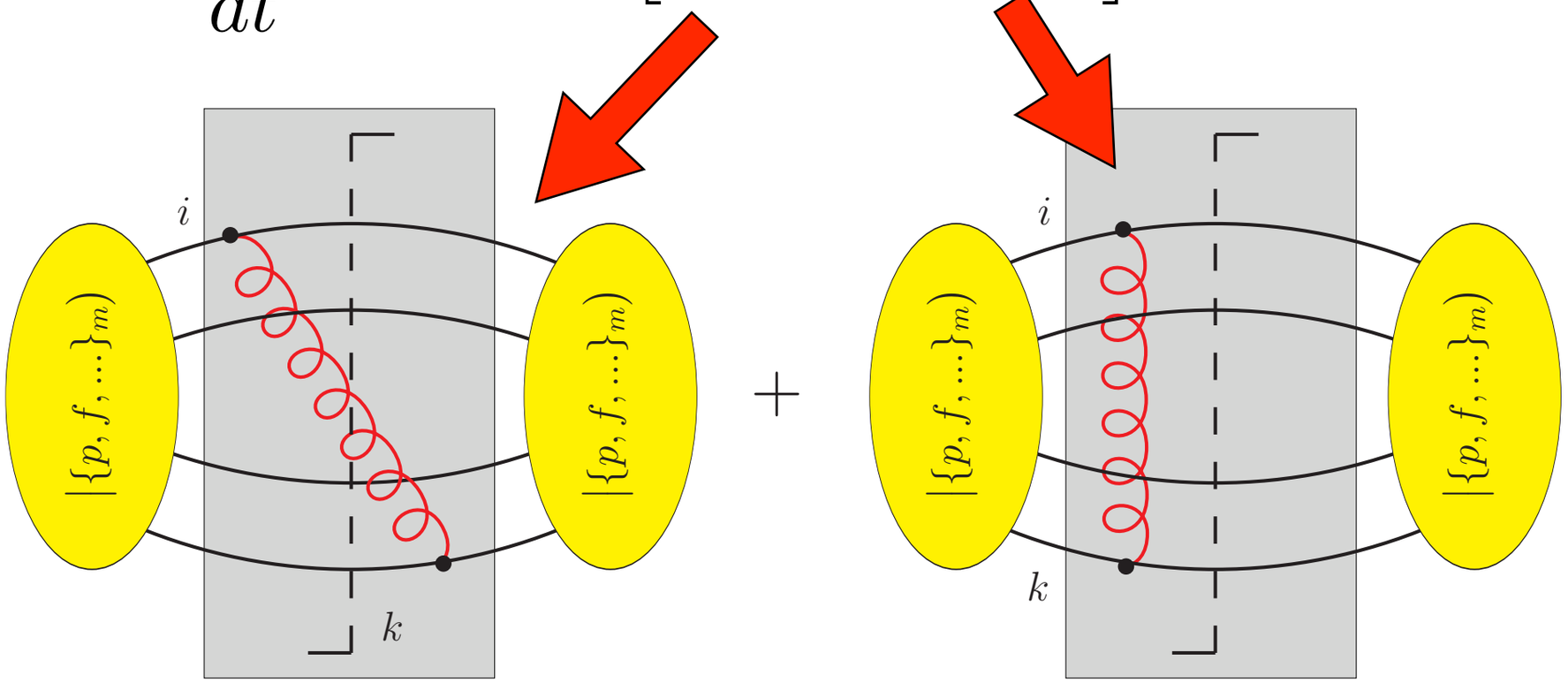
$$\mathcal{U}(t,t') = 1 + \int_{t'}^{t} d\tau\, \mathcal{U}(t,\tau)\left[\mathcal{H}_I(\tau) - \mathcal{V}(\tau)\right]$$

unresolved radiation

resolved radiations

or the differential equation

$$\frac{d}{dt}\mathcal{U}(t,t') = \left[\mathcal{H}_I(t) - \mathcal{V}(t)\right]\mathcal{U}(t,t')$$



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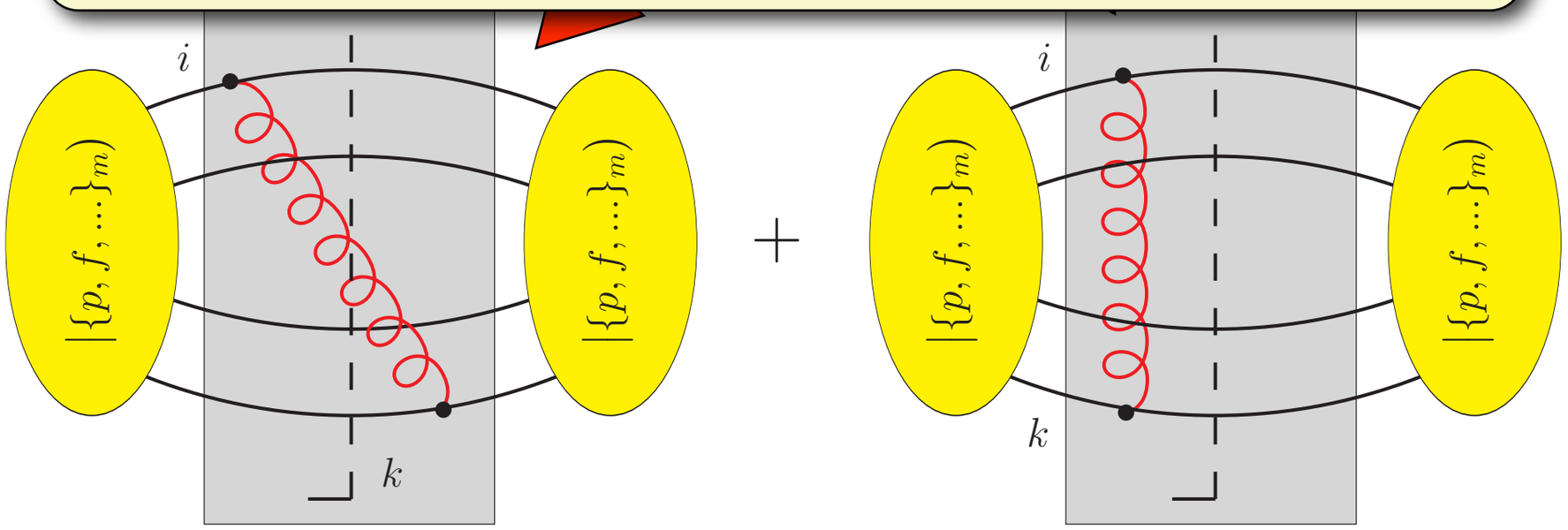
$$\mathcal{U}(t,t') = \mathbb{T}\exp\left(\int_{t'}^{t} d\tau\left[\mathcal{H}_I(\tau) - \mathcal{V}(\tau)\right]\right)$$



Parton Shower Evolution

The unitary condition fixes the unresolved part of the splitting operator

$$\big(\{\hat{c}',\hat{c}\}_m\big|\mathcal{V}(t,\{p,f\}_m)\big|\{c',c\}_m\big)$$
$$= {}_D\big\langle\{\hat{c}\}_m\big|h(t,\{p,f\}_m)\big|\{c\}_m\big\rangle\,\delta^{\{\hat{c}'\}_m}_{\{c'\}_m} + \delta^{\{\hat{c}\}_m}_{\{c\}_m}\,\big\langle\{c'\}_m\big|h(t,\{p,f\}_m)\big|\{\hat{c}'\}_m\big\rangle_D$$

where

$$\big(1\big|\mathcal{H}_I(t)\big|\{p,f,s',c',s,c\}_m\big) = 2\,\big\langle\{s'\}_m\big|\{s\}_m\big\rangle\,\underbrace{\big\langle\{c'\}_m\big|h(t,\{p,f\}_m)\big|\{c\}_m\big\rangle}_{\text{non-trivial color structure}}$$

At leading order level:

$$h^{(0)}(t,\{p,f\}_m) = \underbrace{\mathbf{1}P(t,\{p,f\}_m)}_{\text{collinear}} + \sum_{\substack{i,k\\ i\neq k}} \underbrace{\boldsymbol{T}_i\cdot\boldsymbol{T}_k S_{ik}(t;\{p\}_m)}_{\text{pure soft}}$$

One can solve the evolution equation by using small steps

$$\mathcal{U}(t+\Delta t,0)\big|\rho(0)\big) = \underbrace{[1-\mathcal{V}(t)\Delta t]}_{\textit{changes the color states}}\mathcal{U}(t,0)\big|\rho(0)\big) + \underbrace{\mathcal{H}_{\rm I}(t)\Delta t}_{\textit{changes everything}}\mathcal{U}(t,0')\big|\rho(0)\big)$$

Parton Shower Evolution

There is a more tradition way to solve the evolution equation

$$\mathcal{U}(t,t') = \mathcal{N}(t,t') + \int_{t'}^{t} d\tau\, \mathcal{U}(t,\tau)\big[\mathcal{H}_I(\tau) - \mathcal{V}_S(\tau)\big]\mathcal{N}(\tau,t')$$

where

$$\mathcal{V}_E(\tau) + \mathcal{V}_S(\tau) = \mathcal{V}(\tau) \qquad \mathcal{N}(t,t') = \mathbb{T}\exp\left(-\int_{t'}^{t} d\tau\, \mathcal{V}_E(\tau)\right)$$

We exponentiate the color conserving part of the operator $\mathcal{V}(\tau)$ and subtract the remaining. The color conserving part is the leading color contribution. In this case the basis vectors $\big|\{p,f,...\}\big)$ are eigenvectors of the operator $\mathcal{V}_E(\tau)$, thus

$$N(t',t)|\{p,f,s'c',s,c\}_m) = \underbrace{\Delta(\{p,f,c',c\}_m;t',t)}_{\text{Sudakov factor}}|\{p,f,s'c',s,c\}_m)$$

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$$\Delta(\{p, f, c', c\}_m; t_2, t_1) = \exp\left(-\int_{t_1}^{t_2} d\tau\, \underbrace{\omega(\tau; \{p, f, c', c\}_m)}\right)$$

related to the parton splitting

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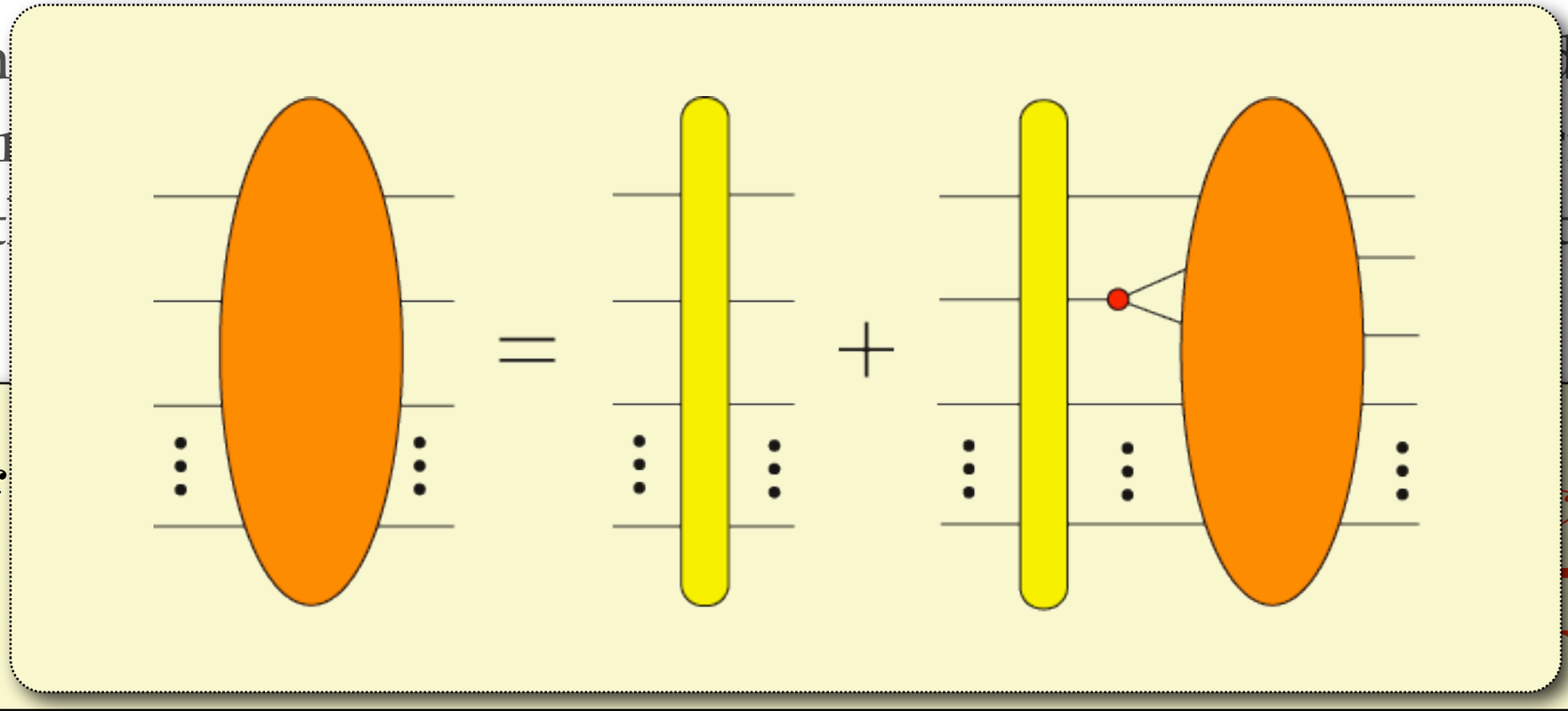


Splitting Operator

The splitting operator is based on the soft and collinear factorization formulas. At LO level we have

$$\mathcal{H}_I(t) = \mathcal{D}_{\rm a}(t) + \mathcal{D}_{\rm b}(t) + \sum_{i=1}^{m} \mathcal{D}_i(t) + \mathcal{O}(\alpha_s^2)$$

- The splitting kernel is organized according to the collinear splittings (even the soft radiations)
- This decomposition can be also done at higher order level.
- We need to define phase space mapping to get *m+1* parton configurations from *m* parton configurations. This can be also done systematically even for higher order.
- It is important that the phase space mapping must be exact since the classical states are defined on phase space surface in the momentum space.
- The mapping must be consistent with the higher order; *e.g.* we cannot use the Catani-Seymour dipole factorization and phase space mapping.

Splitting Operator

The physical states are given by the density matrix which is the direct product of the matrix element vector:

$$\rho(\{p,f\}_{m+1}) \sim \left|\mathcal{M}(\{p,f\}_{m+1})\right\rangle\left\langle\mathcal{M}(\{p,f\}_{m+1})\right|$$

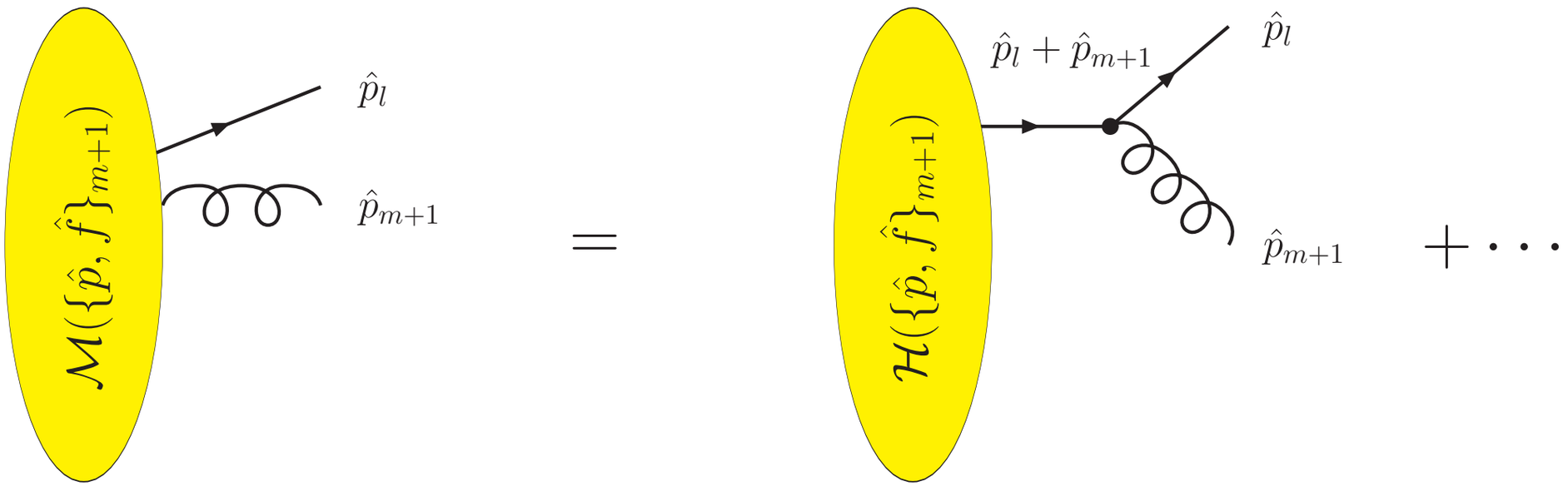
The splitting operator is based on the soft collinear factorization of the matrix element. Thus in the limit when parton l and $m+1$ become collinear we have

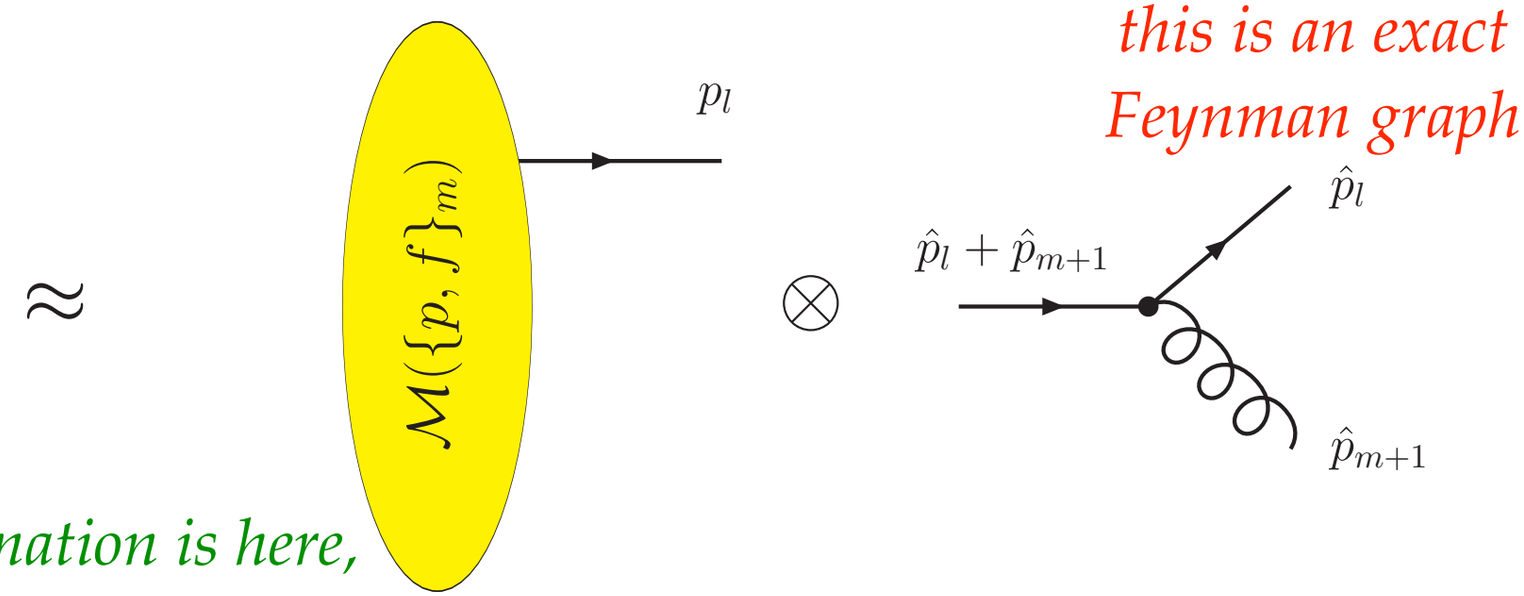
$$\left|M(\{\hat{p},\hat{f}\}_{m+1})\right\rangle \sim \underbrace{t_l^\dagger(f_l \to \hat{f}_l + \hat{f}_{m+1})}_{\textit{Color operator}}\ \underbrace{V_l^\dagger(\{\hat{p},\hat{f}\}_{m+1})}_{\textit{Spin operator}}\ \left|M(\{p,f\}_m)\right\rangle$$

And in the soft limit we have

$$\left|M(\{\hat{p},\hat{f}\}_{m+1})\right\rangle \sim \sum_l t_l^\dagger(f_l \to \hat{f}_l + \hat{f}_{m+1})\, V_l^\dagger(\{\hat{p},\hat{f}\}_{m+1})\, \left|M(\{p,f\}_m)\right\rangle$$

Splitting Operator





this is an exact Feynman graph

approximation is here, the kinematics is an m body configuration

Splitting Operator

What about the momentum mapping and phase space?

$$\hat{p}_l + \hat{p}_{m+1} = \lambda p_l + \frac{1-\lambda+y}{2a_l}\, Q$$

$$\lambda = \sqrt{\frac{(1+y)^2 - 4a_l(y+b_l)}{1-4a_l b_l}} \qquad a_l = \frac{Q^2}{2p_l\cdot Q}\ , \qquad b_l = \frac{m^2(f_l)}{2p_l\cdot Q}$$

For the spectators we have

$$\hat{p}_j^\mu = \Lambda^\mu{}_\nu(Q-\hat{p}_l-\hat{p}_{m+1}, Q-p_l)\; p_j^\nu \qquad\qquad j\notin\{l,m+1\}$$

and the Lorentz transformation is

$$\Lambda(\hat{K},K)^\mu{}_\nu = g^\mu_\nu - \frac{2(\hat{K}+K)^\mu(\hat{K}+K)_\nu}{(\hat{K}+K)^2} + \frac{2\hat{K}^\mu K_\nu}{K^2}$$

Standard Shower

1. The interference diagrams are treated approximately with angular ordering

$$\Big|\{c', c\}_m\Big) \longrightarrow \Big|\{c\}_m\Big) \qquad + \textit{special evolution variable}$$

2. Color treatment is valid in the $N_c \to \infty$ limit

$$\mathcal{V}_S(\tau) = \mathcal{O}(1/N_c^2), \quad \mathcal{V}_E(\tau) \text{ is diagonal}$$

3. Spin treatment is approximated

$$\Big|\{s', s\}_m\Big) \longrightarrow \Big|\{s\}_m\Big) \qquad + \textit{spin averaged splitting functions}$$

4. Usually crude approximation in the phase space

$\Big|\{p\}_m\Big)$ is NOT defined on the phase space surface properly

Shower Cross Section

The evolution starts from the kinematically simplest configuration and the shower cross section is

$$\sigma[F] = \left(F\middle|D(t_\mathrm{f})U(t_\mathrm{f},t_0)\middle|\mathcal{M}_2\right)$$

$D(t_\mathrm{f})$ represents the hadronization. Tuning is allowed only here.