

DGLAP and BFKL equations in $N = 4$ SUSY

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1 BFKL equation (1975)

Production amplitude in the MR kinematics

$$M_{2 \rightarrow 1+n} \sim \frac{s_1^{\omega_1}}{|q_1|^2} g T_{c_2 c_1}^{d_1} C(q_2, q_1) \frac{s_2^{\omega_2}}{|q_2|^2} \dots C(q_n, q_{n-1}) \frac{s_n^{\omega_n}}{|q_n|^2}$$

Gluon Regge trajectory and production vertex

$$\omega(-|q|^2) = -\frac{g^2 N_c}{16 \pi^3} \int \frac{|q|^2 d^2 k}{|k|^2 |q - k|^2}, \quad C(q_2, q_1) = \frac{q_2 q_1^*}{q_2 - q_1}$$

Impact parameter coordinates and momenta

$$\rho_k = x_k + i y_k, \quad \rho_k^* = x_k - i y_k, \quad p_k = i \frac{\partial}{\partial \rho_k}, \quad p_k^* = i \frac{\partial}{\partial \rho_k^*}$$

Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation

$$E \Psi(\vec{\rho}_1, \vec{\rho}_2) = H_{12} \Psi(\vec{\rho}_1, \vec{\rho}_2), \quad \Delta = -\frac{\alpha_s N_c}{2\pi} E$$

BFKL Hamiltonian

$$H_{12} = \ln |p_1 p_2|^2 + \frac{1}{p_1 p_2^*} (\ln |\rho_{12}|^2) p_1 p_2^* \\ + \frac{1}{p_1^* p_2} (\ln |\rho_{12}|^2) p_1^* p_2 - 4\psi(1), \quad \rho_{12} = \rho_1 - \rho_2$$

2 Properties of BKP equations

Möbius invariance (L. (1986))

$$\rho_k \rightarrow \frac{a\rho_k + b}{c\rho_k + d}$$

Eigenvalue equations for Casimir operators

$$M^2 f_{m,\tilde{m}} = m(m-1) f_{m,\tilde{m}}, \quad M^{*2} f_{m,\tilde{m}} = \tilde{m}(\tilde{m}-1) f_{m,\tilde{m}}$$

Principal series of unitary representations

$$m = 1/2 + i\nu + n/2, \quad \tilde{m} = 1/2 + i\nu - n/2$$

Bartels-Kwiecinski-Praszalowicz equation (1980)

$$E \Psi(\vec{\rho}_1, \dots, \vec{\rho}_n) = H \Psi(\vec{\rho}_1, \dots, \vec{\rho}_n), \quad H = \sum_{k < l} \frac{\vec{T}_k \vec{T}_l}{-N_c} H_{kl}$$

Holomorphic separability for $N_c \rightarrow \infty$ (L. (1988))

$$H = \frac{h + h^*}{2}, \quad h = \sum_{k=1}^n h_{k,k+1},$$

$$\Psi(\vec{\rho}_1, \vec{\rho}_2, \dots, \vec{\rho}_n) = \sum_{r,s} a_{r,s} \Psi_r(\rho_1, \dots, \rho_n) \Psi_s(\rho_1^*, \dots, \rho_n^*)$$

3 Duality and integrability

Cyclic and duality symmetry (L. (1998))

$$\rho_i \rightarrow \rho_{i+1}, \ p_i \rightarrow p_{i+1},$$

$$\rho_{r,r+1} \rightarrow p_r \rightarrow \rho_{r-1,r}$$

First integral of motion

$$A = q_n = \rho_{12}\rho_{23}...\rho_{n1} p_1p_2...p_n, \ [h, A] = 0$$

Transfer and monodromy matrices (L. (1993))

$$T(u) = \text{tr } t(u), \ t(u) = L_1 L_2 ... L_n = \sum_{r=0}^n u^{n-r} q_r,$$

$$L_k = \begin{pmatrix} u + \rho_k p_k & p_k \\ -\rho_k^2 p_k & u - \rho_k p_k \end{pmatrix}, \ \hat{l} = u \hat{1} + i \hat{P}$$

Yang-Baxter equation (L. (1993))

$$t_{r'_1}^{s_1}(u) t_{r'_2}^{s_2}(v) l_{r'_1 r'_2}^{r'_1 r'_2}(v-u) = l_{s'_1 s'_2}^{s_1 s_2}(v-u) t_{r'_2}^{s'_2}(v) t_{r'_1}^{s'_1}(u)$$

4 Pomeron in $N = 4$ SUSY

BFKL kernel in two loops (F., L. (1998))

$$\omega = 4 \hat{a} \chi(n, \gamma) + 4 \hat{a}^2 \Delta(n, \gamma), \quad \hat{a} = g^2 N_c / (16\pi^2)$$

Non-analytic terms in QCD (K.,L. (2000))

$$\Delta_{QCD}(n, \gamma) = c_0 \delta_{n,0} + c_2 \delta_{n,2} + \dots$$

Hermitian separability in $N = 4$ SUSY (K.,L. (2000))

$$\Delta(n, \gamma) = \phi(M) + \phi(M^*) - \frac{\rho(M) + \rho(M^*)}{2\hat{a}/\omega}, \quad M = \gamma + \frac{|n|}{2},$$

$$\rho(M) = \beta'(M) + \frac{1}{2} \zeta(2), \quad \beta'(z) = \frac{1}{4} \left[\Psi' \left(\frac{z+1}{2} \right) - \Psi' \left(\frac{z}{2} \right) \right]$$

Maximal transcedentality (K.,L. (2002))

$$\phi(M) = 3\zeta(3) + \Psi''(M) - 2\Phi(M) + 2\beta'(M) \left(\Psi(1) - \Psi(M) \right)$$

$$\Phi(M) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+M} \left(\Psi'(k+1) - \frac{\Psi(k+1) - \Psi(1)}{k+M} \right)$$

5 $N = 4$ DGLAP equation

Anomalous dimension matrix (L. (1997))

$$\begin{aligned}\gamma_{gg} &= -\frac{8}{j-1} + \frac{8}{j} - \frac{8}{j+1} + \frac{8}{j+2} + 8S_1(j), \quad \gamma_{q\varphi} = -\frac{16}{j}, \\ \gamma_{gq} &= -\frac{8}{j-1} + \frac{8}{j} - \frac{4}{j+1}, \quad \gamma_{g\varphi} = -\frac{8}{j-1} + \frac{8}{j}, \\ \gamma_{qg} &= -\frac{16}{j} + \frac{32}{j+1} - \frac{32}{j+2}, \quad \gamma_{qq} = -\frac{16}{j} + \frac{16}{j+1} + 8S_1(j), \\ \gamma_{\varphi g} &= -\frac{24}{j+1} + \frac{24}{j+2}, \quad \gamma_{\varphi q} = -\frac{12}{j+1}, \quad \gamma_{\varphi\varphi} = 8S_1(j), \\ \tilde{\gamma}_{gg} &= -\frac{16}{j} + \frac{16}{j+1} + 8S_1(j), \quad \tilde{\gamma}_{gq} = -\frac{8}{j} + \frac{4}{j+1}, \\ \tilde{\gamma}_{qg} &= \frac{16}{j} - \frac{32}{j+1}, \quad \tilde{\gamma}_{qq} = \frac{8}{j} - \frac{8}{j+1} + 8S_1(j)\end{aligned}$$

Diagonalization in the Born Approximation

$$\left| \begin{array}{ccc} S_1(j-2) & 0 & 0 \\ 0 & S_1(j) & 0 \\ 0 & 0 & S_1(j+2) \end{array} \right|, \left| \begin{array}{ccc} S_1(j-1) & 0 & 0 \\ 0 & 0 & S_1(j+1) \end{array} \right|$$

Integrable Heisenberg spin model (L. (1997))

6 Maximal transcendentality

Most transcendental functions (K.,L. (2002))

$$\gamma(j) = \hat{a}\gamma_1(j) + \hat{a}^2\gamma_2(j) + \hat{a}^3\gamma_3(j) + \dots, \quad \gamma_1(j+2) = -4S_1(j)$$

Two-loop dimension (K.,L.,V. (2003))

$$\frac{\gamma_2(j+2)}{8} = 2S_1(S_2 + S_{-2}) - 2S_{-2,1} + S_3 + S_{-3}$$

Three-loop dimension (K.,L.,O.,V. (2004))

$$\begin{aligned} \gamma_3(j+2)/32 &= -12(S_{-3,1,1} + S_{-2,1,2} + S_{-2,2,1}) \\ &+ 6(S_{-4,1} + S_{-3,2} + S_{-2,3}) - 3S_{-5} - 2S_3S_{-2} - S_5 \\ &- 2S_1^2(3S_{-3} + S_3 - 2S_{-2,1}) - S_2(S_{-3} + S_3 - 2S_{-2,1}) \\ &+ 24S_{-2,1,1,1} - S_1(8S_{-4} + S_{-2}^2 + 4S_2S_{-2} + 2S_2^2) \\ &- S_1(3S_4 - 12S_{-3,1} - 10S_{-2,2} + 16S_{-2,1,1}), \end{aligned}$$

$$S_a(j) = \sum_{m=1}^j \frac{1}{m^a}, \quad S_{a,b,c,\dots}(j) = \sum_{m=1}^j \frac{1}{m^a} S_{b,c,\dots}(m),$$

$$S_{-a}(j) = \sum_{m=1}^j \frac{(-1)^m}{m^a}, \quad S_{-a,b,\dots}(j) = \sum_{m=1}^j \frac{(-1)^m}{m^a} S_{b,\dots}(m),$$

$$\overline{S}_{-a,b,c,\dots}(j) = (-1)^j S_{-a,b,\dots}(j) + S_{-a,b,\dots}(\infty) \left(1 - (-1)^j \right)$$

7 Special cases

Singularities at $j = 1 + \omega \rightarrow 1$

$$\gamma_{uni}^{N=4}(j) = \hat{a} \frac{4}{\omega} + 0\hat{a}^2 + 32\zeta_3 \hat{a}^3 \frac{1}{\omega} - \frac{16\hat{a}^4}{\omega^4} \left(32\zeta_3 + \frac{\pi^4}{9}\omega \right)$$

DL resummation at $j + 2r = \omega \rightarrow 0$

$$\begin{aligned} \omega\gamma_{uni} &= \gamma_{uni}^2 + 16\hat{a}^2(S_2 + \zeta_2 - S_1^2) \\ &+ 4\hat{a} \left(1 - \omega S_1 - \omega^2(S_2 + \zeta_2) + \gamma^2(S_2 + S_{-2}) \right) \end{aligned}$$

Anomalous dimensions at large j

$$\gamma_{uni} = a(z) \ln j, \quad z = \frac{\alpha N_c}{\pi} = 4\hat{a}$$

Perturbative results

$$a = -z + \frac{\pi^2}{12} z^2 - \frac{11}{720} \pi^4 z^3 + \dots$$

Gubser, Klebanov, Polyakov prediction

$$\lim_{z \rightarrow \infty} a = -z^{1/2} + \frac{3 \ln 2}{4\pi} + \dots$$

Resummation

$$\tilde{a} = -z + \frac{\pi^2}{12} \tilde{a}^2 = -z + \frac{\pi^2}{12} z^2 - \frac{1}{72} \pi^4 z^3 + \dots$$

8 Eden-Staudacher approach

Linear set of ES equations for $a(z)$ (K.,L. (2006))

$$a_{n,\epsilon} = \sum_{n'=1}^{\infty} K_{n,n'}(\epsilon) (\delta_{n',1} - a_{n',\epsilon}) , \quad a(z) = \frac{2(1-a_{1,\epsilon})}{\epsilon^2},$$

$$\epsilon = \frac{1}{g\sqrt{2}}, \quad K_{n,n'}(\epsilon) = 2n \sum_{R=0}^{\infty} (-1)^R \frac{2^{-2R-n-n'}}{\epsilon^{2R+n+n'}}$$

$$\zeta(2R+n+n') \frac{(2R+n+n'-1)! (2R+n+n')!}{R! (R+n)! (R+n')! (R+n+n')!}$$

Transcedentality with integer coefficients

$$\gamma(\epsilon) = 8 \sum_{k=1}^{\infty} \left(-\frac{1}{4\epsilon^2} \right)^k \sum_{[s_t]} c_{[s_t]} \prod_r \zeta(s_r), \quad \sum_t s_t = 2k-2$$

Another representation for $K_{n,n'}(\epsilon)$

$$\frac{\Gamma^2(\frac{n+n'+1}{2})\Gamma(\frac{n+n'}{2}+1)\Gamma(\frac{n+n'}{2})}{\pi\Gamma(n)\Gamma(n'+1)\Gamma(n+n'+1)\epsilon^{n+n'}} \sum_{k=1}^{\infty} \frac{2^{n+n'}}{k^{n+n'}} F_n^{n'}\left(-\frac{4}{k^2\epsilon^2}\right)$$

Generalized hypergeometric function ($\bar{n} = \frac{n+n'}{2}$)

$$F_n^{n'}(x) =_4 F_3\left(\bar{n}+\frac{1}{2}, \bar{n}, \bar{n}+1, \bar{n}+\frac{1}{2}; n+1, n'+1, 2\bar{n}+1; x\right)$$

9 AdS/CFT correspondence

Beisert-Eden-Staudacher equation

$$\left(\epsilon + K_0 + K_1 + \frac{2}{\epsilon} K_1 K_0 \right) F(x) = \left(1 + \frac{2}{\epsilon} K_1 \right) K_0 \delta(x),$$

$$K_s F(x) = \int_0^\infty dx' \frac{t'}{e^{t'} - 1} K_s(x, x') F(x'),$$

$$K_s(x, y) = \frac{2}{xy} \sum_{r=1}^{\infty} (2r - 1 + s) J_{2r-1+s}(x) J_{2r-1+s}(y)$$

Essential singularity at $z = 0$ and $g \rightarrow \infty$

$$\chi_{sing}^{BES}(z) = \sum_{k=1}^{\infty} \frac{d_k}{z^k} = - \frac{i}{I_0(2\epsilon^{-1})} \int_L \frac{dz'}{2\pi i} \frac{\exp \frac{z'^2 - 1}{i\epsilon z'}}{z - z'}$$

Recurrent relations

$$\epsilon d_1 = 1 - id_2, \quad n\epsilon d_n = -id_{n-1} - id_{n+1}$$

Agreement with the AdS/CFT prediction (K.,L.)

$$\gamma_{sing} = \frac{2}{\epsilon} \frac{I_1(2\epsilon^{-1})}{I_0(2\epsilon^{-1})} \approx 2\sqrt{2} g - \frac{1}{2}$$

10 Pomeron and graviton

BFKL Pomeron in a diffusion approximation

$$j = 2 - \Delta - D \nu^2$$

Anomalous dimension of twist-2 operators

$$\gamma = 1 + \frac{j - 2}{2} + i\nu$$

Constraint from the conservation of $T_{\mu\nu}$

$$\gamma = (j - 2) \left(\frac{1}{2} - \frac{1/\Delta}{1 + \sqrt{1 + (j - 2)/\Delta}} \right)$$

AdS/CFT for the graviton Regge trajectory

$$j = 2 + \frac{\alpha'}{2} t, \quad t = E^2/R^2, \quad \alpha' = \frac{R^2}{2} \Delta$$

Gubser, Klebanov and Polyakov prediction

$$\gamma|_{z,j \rightarrow \infty} = -\sqrt{j-2} \Delta_{|j \rightarrow \infty}^{-1/2} = \sqrt{\pi j} z^{1/4}$$

Pomeron intercept at large α (K.,L.,O.,V.)

$$j = 2 - \Delta, \quad \Delta = \frac{1}{\pi} z^{-1/2} \approx \frac{\sqrt{3}}{2\pi} z^{-1/2},$$

$$\frac{\pi^2}{6} z = -\tilde{b} + \frac{1}{2} \tilde{b}^2, \quad b = \gamma'(2) = -\frac{\pi^2}{6} z + \frac{\pi^4}{72} z^2 - \frac{\pi^6}{540} z^3$$

11 Discussion

1. High energy limit and reggeized gluon interactions.
2. Möbius and duality symmetry.
3. Holomorphic factorization and integrability.
4. Analyticity of the BFKL kernel in $|n|$ for N=4.
8. Highest transcedentality of anomalous dimensions.
9. Analytic properties of anomalous dimensions.
10. AdS/CFT correspondence and resummation.
- 11.ES equation for γ at $j \rightarrow \infty$.
12. Transcedentality in all loops.
13. Singularities of $\gamma(j)$.
14. Beisert-Eden-Staudacher equation.
15. Agreement with the string prediction at $g \rightarrow \infty$.
16. Pomeron and Reggeized graviton.