QCD Parton Dynamics, 30 years later

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plan

1. Three loops (scary movie)
2. Parton Dynamics made simple(r)
   - Innovative Bookkeeping
   - Divide and Conquer
3. $\mathcal{N} = 4$ SYM serving QCD
4. Conclusions

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4. Conclusions

1-loop drill,
2-loop thrill,
3-loop chill ...

3rd loop non-singlet a.d.

$$P_{\rm ns}^{(2)+}(x) = 16 C_A C_F n_f \Bigg(\frac{1}{6} p_{\rm qq}(x) \Bigg[\frac{10}{3}\zeta_2 - \frac{209}{36} - 9\zeta_3 - \frac{167}{18}\mathrm{H}_0 + 2\mathrm{H}_0\zeta_2 - 7\mathrm{H}_0$$

$$+3\mathrm{H}_{1,0,0} - \mathrm{H}_3 \Bigg] + \frac{1}{3} p_{\rm qq}(-x) \Bigg[\frac{3}{2}\zeta_3 - \frac{5}{3}\zeta_2 - \mathrm{H}_{-2,0} - 2\mathrm{H}_{-1}\zeta_2 - \frac{10}{3}\mathrm{H}_{-1,0} - \mathrm{H}_-$$

$$+2\mathrm{H}_{-1,2} + \frac{1}{2}\mathrm{H}_0\zeta_2 + \frac{5}{3}\mathrm{H}_{0,0} + \mathrm{H}_{0,0,0} - \mathrm{H}_3 \Bigg] + (1-x)\Bigg[\frac{1}{6}\zeta_2 - \frac{257}{54} - \frac{43}{18}\mathrm{H}_0 -$$

$$-(1+x)\Bigg[\frac{2}{3}\mathrm{H}_{-1,0} + \frac{1}{2}\mathrm{H}_2 \Bigg] + \frac{1}{3}\zeta_2 + \mathrm{H}_0 + \frac{1}{6}\mathrm{H}_{0,0} + \delta(1-x)\Bigg[\frac{5}{4} - \frac{167}{54}\zeta_2 + \frac{1}{20}\zeta_2$$

$$+16\, C_A {C_F}^2 \Bigg(p_{\rm qq}(x) \Bigg[\frac{5}{6}\zeta_3 - \frac{69}{20}{\zeta_2}^2 - \mathrm{H}_{-3,0} - 3\mathrm{H}_{-2}\zeta_2 - 14\mathrm{H}_{-2,-1,0} + 3\mathrm{H}_{-2,0}$$

$$-4\mathrm{H}_{-2,2} - \frac{151}{48}\mathrm{H}_0 + \frac{41}{12}\mathrm{H}_0\zeta_2 - \frac{17}{2}\mathrm{H}_0\zeta_3 - \frac{13}{4}\mathrm{H}_{0,0} - 4\mathrm{H}_{0,0}\zeta_2 - \frac{23}{12}\mathrm{H}_{0,0,0} + 5\mathrm{H}$$

$$-24\mathrm{H}_1\zeta_3 - 16\mathrm{H}_{1,-2,0} + \frac{67}{9}\mathrm{H}_{1,0} - 2\mathrm{H}_{1,0}\zeta_2 + \frac{31}{3}\mathrm{H}_{1,0,0} + 11\mathrm{H}_{1,0,0,0} + 8\mathrm{H}_{1,1,0,0}$$

3rd loop, more

$$+\frac{67}{9}\mathrm{H}_2 - 2\mathrm{H}_2\zeta_2 + \frac{11}{3}\mathrm{H}_{2,0} + 5\mathrm{H}_{2,0,0} + \mathrm{H}_{3,0}\Big] + p_{\mathrm{qq}}(-x)\Big[\frac{1}{4}\zeta_2{}^2 - \frac{67}{9}\zeta_2 + \frac{31}{4}\zeta_3$$

$$-32\mathrm{H}_{-2}\zeta_2 - 4\mathrm{H}_{-2,-1,0} - \frac{31}{6}\mathrm{H}_{-2,0} + 21\mathrm{H}_{-2,0,0} + 30\mathrm{H}_{-2,2} - \frac{31}{3}\mathrm{H}_{-1}\zeta_2 - 42\mathrm{H}$$

$$-4\mathrm{H}_{-1,-2,0} + 56\mathrm{H}_{-1,-1}\zeta_2 - 36\mathrm{H}_{-1,-1,0,0} - 56\mathrm{H}_{-1,-1,2} - \frac{134}{9}\mathrm{H}_{-1,0} - 42\mathrm{H}_{-1}$$

$$+32\mathrm{H}_{-1,3} - \frac{31}{6}\mathrm{H}_{-1,0,0} + 17\mathrm{H}_{-1,0,0,0} + \frac{31}{3}\mathrm{H}_{-1,2} + 2\mathrm{H}_{-1,2,0} + \frac{13}{12}\mathrm{H}_0\zeta_2 + \frac{29}{2}\mathrm{H}$$

$$+13\mathrm{H}_{0,0}\zeta_2 + \frac{89}{12}\mathrm{H}_{0,0,0} - 5\mathrm{H}_{0,0,0,0} - 7\mathrm{H}_2\zeta_2 - \frac{31}{6}\mathrm{H}_3 - 10\mathrm{H}_4\Big] + (1-x)\Big[\frac{133}{36} +$$

$$-\frac{167}{4}\zeta_3 - 2\mathrm{H}_0\zeta_3 - 2\mathrm{H}_{-3,0} + \mathrm{H}_{-2}\zeta_2 + 2\mathrm{H}_{-2,-1,0} - 3\mathrm{H}_{-2,0,0} + \frac{77}{4}\mathrm{H}_{0,0,0} - \frac{20}{6}$$

$$+4\mathrm{H}_{1,0,0} + \frac{14}{3}\mathrm{H}_{1,0}\Big] + (1+x)\Big[\frac{43}{2}\zeta_2 - 3\zeta_2{}^2 + \frac{25}{2}\mathrm{H}_{-2,0} - 31\mathrm{H}_{-1}\zeta_2 - 14\mathrm{H}_{-1,-}$$

$$+24\mathrm{H}_{-1,2} + 23\mathrm{H}_{-1,0,0} + \frac{55}{2}\mathrm{H}_0\zeta_2 + 5\mathrm{H}_{0,0}\zeta_2 + \frac{1457}{48}\mathrm{H}_0 - \frac{1025}{36}\mathrm{H}_{0,0} - \frac{155}{6}\mathrm{H}_2$$

3rd loop, and more

$$+2\mathrm{H}_{2,0,0} - 3\mathrm{H}_4\Big] - 5\zeta_2 - \frac{1}{2}\zeta_2{}^2 + 50\zeta_3 - 2\mathrm{H}_{-3,0} - 7\mathrm{H}_{-2,0} - \mathrm{H}_0\zeta_3 - \frac{37}{2}\mathrm{H}_0\zeta_2$$

$$-2\mathrm{H}_{0,0}\zeta_2 + \frac{185}{6}\mathrm{H}_{0,0} - 22\mathrm{H}_{0,0,0} - 4\mathrm{H}_{0,0,0,0} + \frac{28}{3}\mathrm{H}_2 + 6\mathrm{H}_3 + \delta(1-x)\Big[\frac{151}{64} +$$

$$-\frac{247}{60}\zeta_2{}^2 + \frac{211}{12}\zeta_3 + \frac{15}{2}\zeta_5\Big]\Big) + 16\,C_A{}^2C_F\Big(p_{\mathrm{qq}}(x)\Big[\frac{245}{48} - \frac{67}{18}\zeta_2 + \frac{12}{5}\zeta_2{}^2 + \frac{1}{2}$$

$$+\mathrm{H}_{-3,0} + 4\mathrm{H}_{-2,-1,0} - \frac{3}{2}\mathrm{H}_{-2,0} - \mathrm{H}_{-2,0,0} + 2\mathrm{H}_{-2,2} - \frac{31}{12}\mathrm{H}_0\zeta_2 + 4\mathrm{H}_0\zeta_3 + \frac{389}{72}$$

$$-\mathrm{H}_{0,0,0,0} + 9\mathrm{H}_1\zeta_3 + 6\mathrm{H}_{1,-2,0} - \mathrm{H}_{1,0}\zeta_2 - \frac{11}{4}\mathrm{H}_{1,0,0} - 3\mathrm{H}_{1,0,0,0} - 4\mathrm{H}_{1,1,0,0} + 4$$

$$+\frac{11}{12}\mathrm{H}_3 + \mathrm{H}_4\Big] + p_{\mathrm{qq}}(-x)\Big[\frac{67}{18}\zeta_2 - \zeta_2{}^2 - \frac{11}{4}\zeta_3 - \mathrm{H}_{-3,0} + 8\mathrm{H}_{-2}\zeta_2 + \frac{11}{6}\mathrm{H}_{-2,0}$$

$$-3\mathrm{H}_{-1,0,0,0} + \frac{11}{3}\mathrm{H}_{-1}\zeta_2 + 12\mathrm{H}_{-1}\zeta_3 - 16\mathrm{H}_{-1,-1}\zeta_2 + 8\mathrm{H}_{-1,-1,0,0} + 16\mathrm{H}_{-1,-1,2}$$

$$-8\mathrm{H}_{-2,2} + 11\mathrm{H}_{-1,0}\zeta_2 + \frac{11}{6}\mathrm{H}_{-1,0,0} - \frac{11}{3}\mathrm{H}_{-1,2} - 8\mathrm{H}_{-1,3} - \frac{3}{4}\mathrm{H}_0 - \frac{1}{6}\mathrm{H}_0\zeta_2 - 4$$

3rd loop, and again

$$-3\mathrm{H}_{0,0}\zeta_2 - \frac{31}{12}\mathrm{H}_{0,0,0} + \mathrm{H}_{0,0,0,0} + 2\mathrm{H}_2\zeta_2 + \frac{11}{6}\mathrm{H}_3 + 2\mathrm{H}_4\Big] + (1-x)\Big[\frac{1883}{108} - \frac{1}{2}$$

$$-\mathrm{H}_{-2,-1,0} + \frac{1}{2}\mathrm{H}_{-3,0} - \frac{1}{2}\mathrm{H}_{-2}\zeta_2 + \frac{1}{2}\mathrm{H}_{-2,0,0} + \frac{523}{36}\mathrm{H}_0 + \mathrm{H}_0\zeta_3 - \frac{13}{3}\mathrm{H}_{0,0} - \frac{5}{2}\mathrm{H}$$

$$-2\mathrm{H}_{1,0,0}\Big] + (1+x)\Big[8\mathrm{H}_{-1}\zeta_2 + 4\mathrm{H}_{-1,-1,0} + \frac{8}{3}\mathrm{H}_{-1,0} - 5\mathrm{H}_{-1,0,0} - 6\mathrm{H}_{-1,2} - \frac{13}{3}$$

$$-\frac{43}{4}\zeta_3 - \frac{5}{2}\mathrm{H}_{-2,0} - \frac{11}{2}\mathrm{H}_0\zeta_2 - \frac{1}{2}\mathrm{H}_2\zeta_2 - \frac{5}{4}\mathrm{H}_{0,0}\zeta_2 + 7\mathrm{H}_2 - \frac{1}{4}\mathrm{H}_{2,0,0} + 3\mathrm{H}_3 + \frac{3}{4}$$

$$+\frac{1}{4}\zeta_2{}^2 - \frac{8}{3}\zeta_2 + \frac{17}{2}\zeta_3 + \mathrm{H}_{-2,0} - \frac{19}{2}\mathrm{H}_0 + \frac{5}{2}\mathrm{H}_0\zeta_2 - \mathrm{H}_0\zeta_3 + \frac{13}{3}\mathrm{H}_{0,0} + \frac{5}{2}\mathrm{H}_{0,0,0}$$

$$-\delta(1-x)\Big[\frac{1657}{576} - \frac{281}{27}\zeta_2 + \frac{1}{8}\zeta_2{}^2 + \frac{97}{9}\zeta_3 - \frac{5}{2}\zeta_5\Big]\Big) + 16\, C_F n_f^2\Big(\frac{1}{18}p_{\mathrm{qq}}(x)\Big[\mathrm{H}_{0,}$$

$$+(1-x)\Big[\frac{13}{54} + \frac{1}{9}\mathrm{H}_0\Big] - \delta(1-x)\Big[\frac{17}{144} - \frac{5}{27}\zeta_2 + \frac{1}{9}\zeta_3\Big]\Big) + 16\, C_F{}^2 n_f\Big(\frac{1}{3}p_{\mathrm{qq}}(x)\Big[$$

3rd loop, and still some more

$$-\frac{55}{16} + \frac{5}{8}\mathrm{H}_0 + \mathrm{H}_0\zeta_2 + \frac{3}{2}\mathrm{H}_{0,0} - \mathrm{H}_{0,0,0} - \frac{10}{3}\mathrm{H}_{1,0} - \frac{10}{3}\mathrm{H}_2 - 2\mathrm{H}_{2,0} - 2\mathrm{H}_3\Big] + \frac{2}{3}$$

$$-\frac{3}{2}\zeta_3 + \mathrm{H}_{-2,0} + 2\mathrm{H}_{-1}\zeta_2 + \frac{10}{3}\mathrm{H}_{-1,0} + \mathrm{H}_{-1,0,0} - 2\mathrm{H}_{-1,2} - \frac{1}{2}\mathrm{H}_0\zeta_2 - \frac{5}{3}\mathrm{H}_{0,0} -$$

$$-(1-x)\Big[\frac{10}{9} + \frac{19}{18}\mathrm{H}_{0,0} - \frac{4}{3}\mathrm{H}_1 + \frac{2}{3}\mathrm{H}_{1,0} + \frac{4}{3}\mathrm{H}_2\Big] + (1+x)\Big[\frac{4}{3}\mathrm{H}_{-1,0} - \frac{25}{24}\mathrm{H}_0 +$$

$$+\frac{7}{9}\mathrm{H}_{0,0} + \frac{4}{3}\mathrm{H}_2 - \delta(1-x)\Big[\frac{23}{16} - \frac{5}{12}\zeta_2 - \frac{29}{30}\zeta_2{}^2 + \frac{17}{6}\zeta_3\Big]\Big) + 16\, {C_F}^3\Big(p_{\mathrm{qq}}(x)\Big[$$

$$+6\mathrm{H}_{-2}\zeta_2 + 12\mathrm{H}_{-2,-1,0} - 6\mathrm{H}_{-2,0,0} - \frac{3}{16}\mathrm{H}_0 - \frac{3}{2}\mathrm{H}_0\zeta_2 + \mathrm{H}_0\zeta_3 + \frac{13}{8}\mathrm{H}_{0,0} - 2\mathrm{H}_0$$

$$+12\mathrm{H}_1\zeta_3 + 8\mathrm{H}_{1,-2,0} - 6\mathrm{H}_{1,0,0} - 4\mathrm{H}_{1,0,0,0} + 4\mathrm{H}_{1,2,0} - 3\mathrm{H}_{2,0} + 2\mathrm{H}_{2,0,0} + 4\mathrm{H}_{2,1}$$

$$+4\mathrm{H}_{3,0} + 4\mathrm{H}_{3,1} + 2\mathrm{H}_4\Big] + p_{\mathrm{qq}}(-x)\Big[\frac{7}{2}\zeta_2{}^2 - \frac{9}{2}\zeta_3 - 6\mathrm{H}_{-3,0} + 32\mathrm{H}_{-2}\zeta_2 + 8\mathrm{H}_{-2}$$

$$-26\mathrm{H}_{-2,0,0} - 28\mathrm{H}_{-2,2} + 6\mathrm{H}_{-1}\zeta_2 + 36\mathrm{H}_{-1}\zeta_3 + 8\mathrm{H}_{-1,-2,0} - 48\mathrm{H}_{-1,-1}\zeta_2 + 40\mathrm{H}$$

3rd loop, and UFF

$$+48\mathrm{H}_{-1,-1,2} + 40\mathrm{H}_{-1,0}\zeta_2 + 3\mathrm{H}_{-1,0,0} - 22\mathrm{H}_{-1,0,0,0} - 6\mathrm{H}_{-1,2} - 4\mathrm{H}_{-1,2,0} - 32$$

$$-\frac{3}{2}\mathrm{H}_0\zeta_2 - 13\mathrm{H}_0\zeta_3 - 14\mathrm{H}_{0,0}\zeta_2 - \frac{9}{2}\mathrm{H}_{0,0,0} + 6\mathrm{H}_{0,0,0,0} + 6\mathrm{H}_2\zeta_2 + 3\mathrm{H}_3 + 2\mathrm{H}_{3,0}$$

$$+(1-x)\Big[2\mathrm{H}_{-3,0} - \frac{31}{8} + 4\mathrm{H}_{-2,0,0} + \mathrm{H}_{0,0}\zeta_2 - 3\mathrm{H}_{0,0,0,0} + 35\mathrm{H}_1 + 6\mathrm{H}_1\zeta_2 - \mathrm{H}_{1,}$$

$$+(1+x)\Big[\frac{37}{10}{\zeta_2}^2 - \frac{93}{4}\zeta_2 - \frac{81}{2}\zeta_3 - 15\mathrm{H}_{-2,0} + 30\mathrm{H}_{-1}\zeta_2 + 12\mathrm{H}_{-1,-1,0} - 2\mathrm{H}_{-1,0}$$

$$-24\mathrm{H}_{-1,2} - \frac{539}{16}\mathrm{H}_0 - 28\mathrm{H}_0\zeta_2 + \frac{191}{8}\mathrm{H}_{0,0} + 20\mathrm{H}_{0,0,0} + \frac{85}{4}\mathrm{H}_2 - 3\mathrm{H}_{2,0,0} - 2\mathrm{H}_3$$

$$-\mathrm{H}_4\Big] + 4\zeta_2 + 33\zeta_3 + 4\mathrm{H}_{-3,0} + 10\mathrm{H}_{-2,0} + \frac{67}{2}\mathrm{H}_0 + 6\mathrm{H}_0\zeta_3 + 19\mathrm{H}_0\zeta_2 - 25\mathrm{H}_{0,0}$$

$$-2\mathrm{H}_2 - \mathrm{H}_{2,0} - 4\mathrm{H}_3 + \delta(1-x)\left[\frac{29}{32} - 2\zeta_2\zeta_3 + \frac{9}{8}\zeta_2 + \frac{18}{5}{\zeta_2}^2 + \frac{17}{4}\zeta_3 - 15\zeta_5\right]\Bigg)$$

music of the spheres

2×2 anomalous dimension matrix occupies

1 st loop: $1/10$ page

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Moch, Vermaseren and Vogt

[waterfall of results launched
March 2004, and counting]

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$$V \sim \begin{cases} 10^{\frac{N(N-1)}{2}-1} \\ 10^{2^{N-1}-2} \end{cases}$$

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not too encouraging a trend . . .

Fighting complexity

How to reduce complexity ?

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Guidelines

Higher Orders

Innovative Bookkeeping

Think
Extract
Solve

Fighting complexity

How to reduce complexity ?

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✓ exploit internal properties :

- Drell–Levy–Yan relation
- Gribov–Lipatov reciprocity

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✓ separate classical & quantum effects in the gluon sector

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An essential part of gluon dynamics is Classical. (F.Low)

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"Classical" does not mean "Simple".
However, it has a good chance to be Exactly Solvable.

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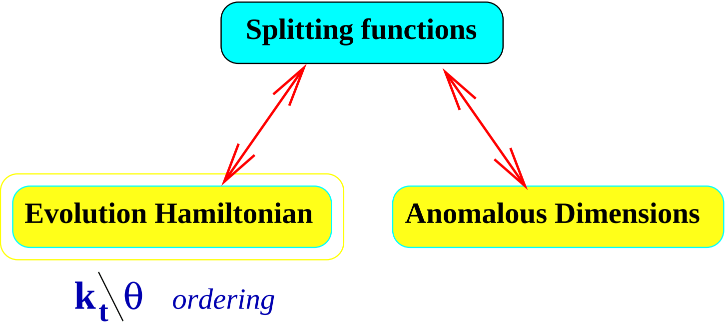
➥ A playing ground for theoretical theory: SUSY, AdS/CFT, . . .

Innovative Bookkeeping

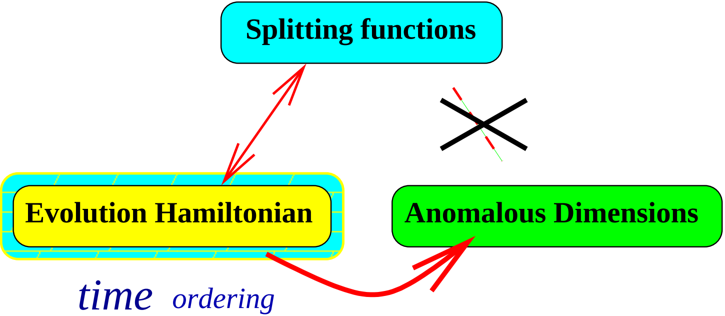
In the standard approach,



- parton splitting functions are equated with anomalous dimensions;
- they are different for DIS and e^+e^- evolution;
- "clever evolution variables" are different too

Innovative Bookkeeping

In the new approach,



- splitting functions are disconnected from the anomalous dimensions;
- the evolution kernel is identical for space- and time-like cascades (Gribov–Lipatov reciprocity relation true in all orders);
- unique evolution variable — parton fluctuation time

Innovative Bookkeeping

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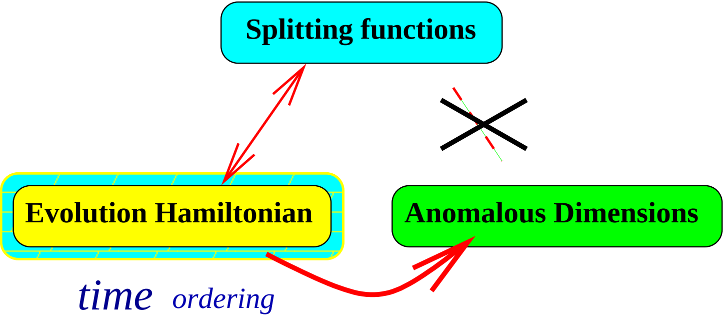

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old new evolution — Innovative Bookkeeping

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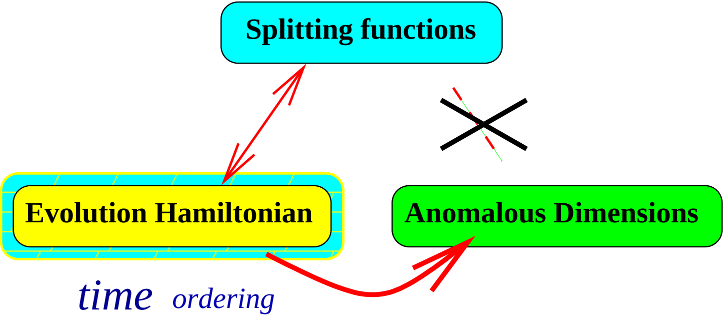

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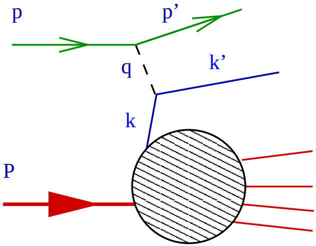
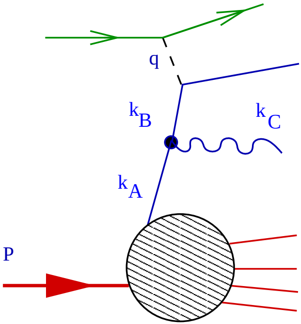
Long-living partons fluctuations



Kinematics of the parton splitting $A \to B + C$

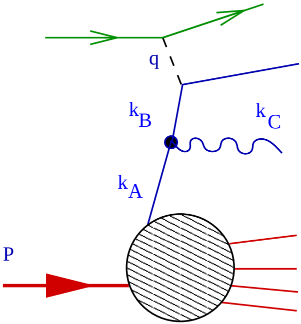
Long-living partons fluctuations



Kinematics of the parton splitting $A \to B + C$

$$k_B \simeq x \cdot P\,, \quad k_A \simeq \frac{x}{z} \cdot P$$

Long-living partons fluctuations



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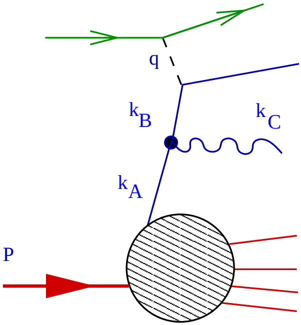
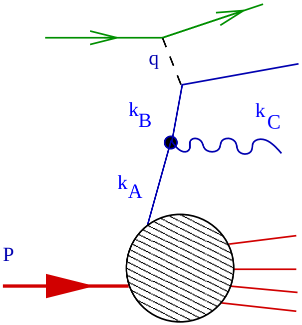


Kinematics of the parton splitting $A \to B + C$

$$k_B \simeq z k_A\,, \quad k_C \simeq (1-z) k_A$$

Long-living partons fluctuations

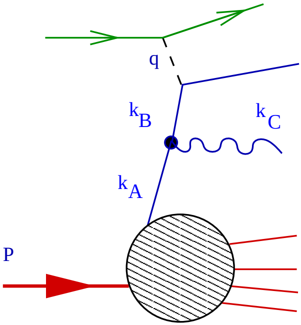


Kinematics of the parton splitting $A \to B + C$

$$k_B \simeq z k_A \,, \quad k_C \simeq (1-z) k_A$$

$$\frac{|k_B^2|}{z} = \frac{|k_A^2|}{1} + \frac{k_C^2}{1-z} + \frac{k_\perp^2}{z(1-z)}$$

Long-living partons fluctuations



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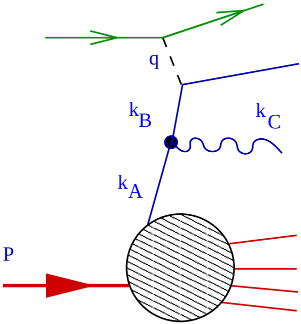
$$\frac{|k_B^2|}{z} = \frac{|k_A^2|}{1} + \frac{k_C^2}{1-z} + \frac{k_\perp^2}{z(1-z)}$$

Probability of the splitting process :

$$dw \propto \frac{\alpha_s}{\pi} \frac{dk_\perp^2 \, k_\perp^2}{(k_B^2)^2}$$

Long-living partons fluctuations



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$$k_B \simeq z k_A\,, \quad k_C \simeq (1-z) k_A$$

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Probability of the splitting process :

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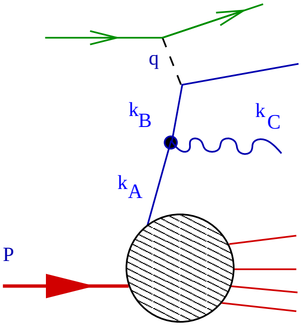
Long-living partons fluctuations



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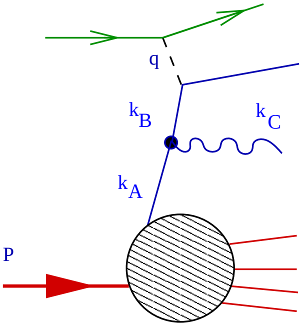

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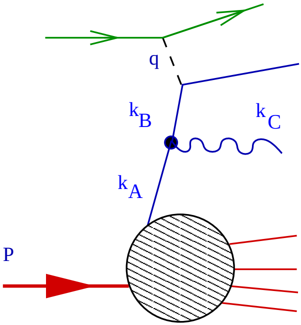

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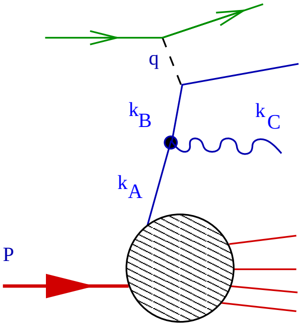

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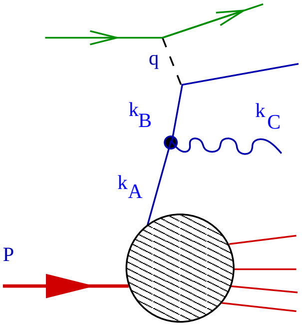

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strongly ordered *lifetimes* of successive parton fluctuations !

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Beyond the 1st loop, it starts to matter how does one order successive parton splittings that is, what one chooses for "parton evolution time".

The "clever choices" had been established quite some time ago:

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Transverse momentum ordering vs. angular ordering.

Each of these two clever choices — consequence of taking into full consideration soft gluon coherence in order to prevent explosively large terms $(\alpha_s \ln^2 x)^n$ from appearing in higher loop anomalous dimensions.

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Mark the different meaning of x in the two channels!

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Fluctuation time ordering : D-r (HERA, 1993)

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$\Longrightarrow$ the genuine $\mathcal{P}_2$ does not contain σ, is GLR respecting

GLR beyond the 1st loop

Examine the "reciprocity respecting equation" (RRE) by feeding in the one-loop parton "Hamiltonian", $\mathcal{P}(\alpha) \simeq \alpha P_1$:

$$\begin{aligned}\gamma[\alpha] &= \mathcal{P} + \dot{\mathcal{P}} \cdot (\sigma\gamma + \beta/\alpha) + \tfrac{1}{2}\ddot{\mathcal{P}} \cdot \left[\gamma^2 + \sigma(2\beta/\alpha\,\gamma + \beta\partial_\alpha\gamma) + \beta/\alpha\,\partial_\alpha\beta\right] + \ldots \\ &= \alpha P_1 + \alpha^2 \cdot (\sigma\, P_1 \dot{P}_1 + \beta_0 \; + \mathcal{P}_2) + \mathcal{O}(\alpha^3)\,.\end{aligned}$$

The difference between time- and space-like anomalous dimensions,

$$\tfrac{1}{2}\left[P^{(T)} - P^{(S)}\right] \;=\; \alpha^2 \cdot P_1 \dot{P}_1 + \mathcal{O}(\alpha^3)\,,$$

in the x-space corresponds to the convolution

$$\tfrac{1}{2}\left[P^{(2),T}_{qq} - P^{(2),S}_{qq}\right] = \int_0^1 \frac{dz}{z}\left\{P^{(1)}_{qq}\left(\frac{x}{z}\right)\right\}_+ \cdot\; P^{(1)}_{qq}(z)\ln z\,,$$

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More generally, a *renormalization scheme transformation* as a cure for/against GLR violation was proposed by Stratmann & Vogelsang (1996)

large x

Another important aspect of the RREE is the "double nature" of the perturbative expansion — in α_{phys} and, at the same time, in $(1-x)$:

$$\begin{aligned}\gamma[\alpha] = & \ \mathcal{P} + \dot{\mathcal{P}}\cdot\left(\sigma\gamma+\beta/\alpha\right)+\tfrac{1}{2}\ddot{\mathcal{P}}\cdot\left(\gamma^2+\sigma(2\beta/\alpha\,\gamma+\beta\partial_\alpha\gamma)+\beta/\alpha\,\partial_\alpha\beta\right)+\ldots \\ = & \ \alpha\ln N + \alpha^2\cdot\left(1/N\right)+\alpha^3\cdot\left(1/N^2\right)+\alpha^4\cdot\left(1/N^3\right)+\ldots\end{aligned}$$

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A gap between *classical radiation* (Low–Burnett–Kroll wisdom)

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and *quantum fluctuations*

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$D = -\sigma A\,B + \mathcal{O}(\beta)$ — another all-order relation

small x : 2 puzzles

RREE relates two long-standing puzzles :

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DIS (space-like evolution). Look at small x that is, $N \ll 1$

$$\text{BFKL} \quad : \quad \gamma_N = \frac{\alpha_s}{N} + \left(\frac{\alpha_s}{N}\right)^2 + \left(\frac{\alpha_s}{N}\right)^3 + \left(\frac{\alpha_s}{N}\right)^4 + \ldots$$

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$1 \to 1 + 2 + 3 + 4 \qquad \Longrightarrow \qquad (1 \to 1 + 2) \otimes (2 \to 2 + 3) \otimes (3 \to 3 + 4)$

so-called "Malaza puzzle"

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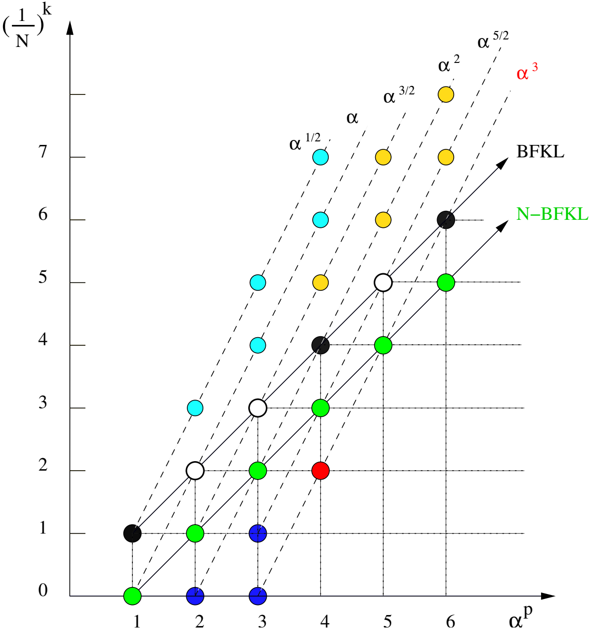
small x chart



Solid – BFKL (black) and N-BFKL (green) known in all orders.

Dashed blue – γ_+ terms generated by α/N and α.

Yellow – unknown.

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inherited from previous loops !

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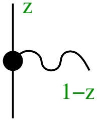
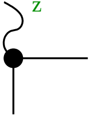
Recall an old hint from QCD ...

Relating parton splittings



$$= C_F \cdot \frac{1+z^2}{1-z}$$



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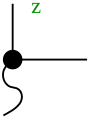
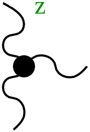


$$= T_R \cdot \left[\, z^2 + (1-z)^2 \,\right]$$

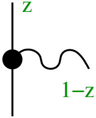


$$= N_c \cdot \frac{1+z^4+(1-z)^4}{z(1-z)}$$

Four “parton splitting functions”

$${}^{q[g]}_{q}(z)\,, \qquad {}^{g[q]}_{q}(z)\,, \qquad {}^{q[\bar{q}]}_{g}(z)\,, \qquad {}^{g[g]}_{g}(z)$$

Relating parton splittings



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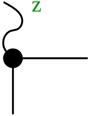

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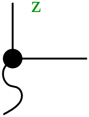

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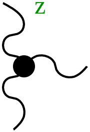

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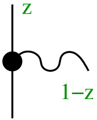
- Exchange the decay products : $z \to 1-z$

$\,^{q[g]}_{q}(z)$ $\,^{g[q]}_{q}(z)$ $\,^{q[\bar{q}]}_{g}(z)$ $\,^{g[g]}_{g}(z)$

Relating parton splittings



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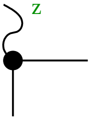

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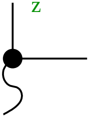

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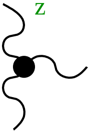

$$= N_c \cdot \frac{1+z^4+(1-z)^4}{z(1-z)}$$

- Exchange the decay products : $z \to 1-z$
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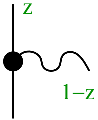

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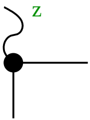

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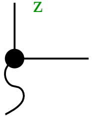

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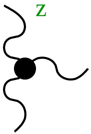

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Three (QED) "kernels" are inter-related; gluon self-interaction stays put :

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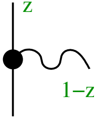

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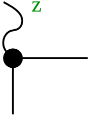

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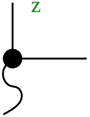

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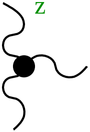

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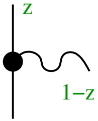

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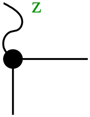

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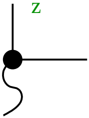

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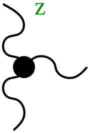

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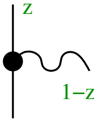

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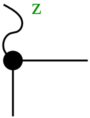

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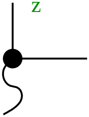

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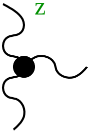

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from Bookkeeping to Solving

The integrability feature manifests itself already in *certain sectors* of QCD, in specific problems where one can *identify* QCD with SUSY-QCD :

✓ the Regge behaviour (large N_c)	Lipatov Faddeev & Korchemsky (1994)
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WHY and **WHAT FOR** ?

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And here we arrive at the second — Divide and Conquer — issue

LBK wisdom

Recall the diagonal first loop anomalous dimensions:

$$\begin{aligned}
\tilde{\gamma}_{q\to q(x)+g} &= \frac{C_F\alpha_s}{\pi}\left[\frac{x}{1-x}+(1-x)\cdot\frac{1}{2}\right],\\
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Let us look at the rôles these animals play on the QCD stage

Gluenatomy

Clagons :

✗ Classical Field

✓ infrared singular, $d\omega/\omega$

✓ define the physical coupling

✓ responsible for

- ➥ DL radiative effects,
- ➥ reggeization,
- ➥ QCD/Lund string (gluers)

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In addition,

- ✗ Tree multi-clagon (Parke–Taylor) amplitudes are *known exactly*
- ✗ It is clagons which dominate in all the *integrability cases*

N=4 SUSY Yang–Mills

Maximally super-symmetric YM field model:
Matter content = 4 Majorana fermions, 6 scalars;
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Euler–Zagier harmonic sums

In spite of having many states ($s = 0, \frac{1}{2}, 1$), the SYM-4 parton dynamics is built of a single "universal" anomalous dimension:

$\gamma_{\rm uni}(N+2)$, $\gamma_{\rm uni}(N+1)$, $\gamma_{\rm uni}(N)$, with the 1st loop given by

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In higher orders enter $m > 1$,

$$S_m(N) = \sum_{k=1}^{N}\frac{1}{k^m} = \frac{(-1)^m}{\Gamma(m)}\int_0^1 dx\, x^N\,\frac{\ln^{m-1}x}{1-x} + \zeta(m),$$

Euler–Zagier harmonic sums

In spite of having many states ($s = 0, \frac{1}{2}, 1$), the SYM-4 parton dynamics is built of a single "universal" anomalous dimension:

$\gamma_{\rm uni}(N+2)$, $\gamma_{\rm uni}(N+1)$, $\gamma_{\rm uni}(N)$, with the 1st loop given by

$$\gamma^{(1)}_{\rm uni}(N) = -S_1(N) \;=\; -\int_0^1 \frac{dx}{x}\left(x^N - 1\right)\cdot\frac{x}{x-1} \;\equiv\; \mathbf{M}\left[\frac{x}{(1-x)_+}\right].$$

Look upon S_1 as a "harmonic sum",

$$S_1(N) = \sum_{k=1}^{N} \frac{1}{k} \;=\; \psi(N+1) - \psi(1).$$

In higher orders enter $m > 1$,

$$S_m(N) = \sum_{k=1}^{N} \frac{1}{k^m} = \frac{(-1)^m}{\Gamma(m)} \int_0^1 dx\, x^N \frac{\ln^{m-1} x}{1-x} + \zeta(m),$$

as we as multiple indices — *nested sums*

$$S_{m,\vec{\rho}}(N) = \sum_{k=1}^{N} \frac{S_{\vec{\rho}}(k)}{k^m} \qquad (\vec{\rho} = (m_1, m_2, \ldots, m_i)),$$

negative indices

Starting from the 2nd loop, one encounters also *negative indices*,

$$S_{-m}(N) = \sum_{k=1}^{N} \frac{(-1)^k}{k^m}.$$

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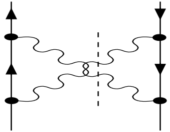
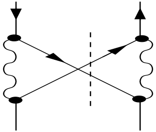


(a)



(b)

$(a) \leftrightarrow (b)$

$P \;\to\; -P$

$x \;\to\; -x$

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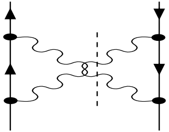

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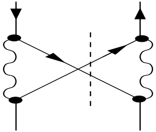

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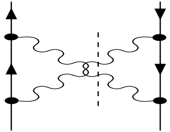

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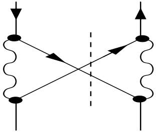

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$\frac{x}{1-x} \cdot \ln^2 x \to S_3(N) \qquad \frac{x}{1+x} \cdot \Phi_2(x) \to Y_{-3}(N)$

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RR evolution kernel

In terms of the perturbative expansion in the physical coupling,

$$a_{\rm ph} = a\left(1 - \frac{1}{2}\zeta_2 a + \frac{11}{20}\zeta_2^2 a^2 + \ldots\right),$$

$$\begin{aligned}
\mathcal{P}_1 &= -\ S_1; \\
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$$\frac{\text{clever 2nd loop}}{\text{clever 1st loop}} < 2\% \qquad \left(\begin{array}{c} \text{Heavy quark fragmentation} \\ \text{D-r, Khoze \& Troyan , PRD 1996} \end{array} \right)$$

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Employ $\mathcal{N}=4$ SYM to simplify the essential part of the QCD dynamics

Conclusions

- A steady progress in high order perturbative QCD calculations is worth accompanying by reflections upon the origin and the structure of higher loop correction effects
- Reformulation of parton cascades in terms of Gribov–Lipatov reciprocity respecting evolution equations (RREE)
  - reduces complexity by (at leat) one order of magnitude
  - improves perturbative series (less singular, better "converging")
  - links interesting phenomena in the DIS and e^+e^- annihilation channels
- The Low theorem should be part of theor.phys. curriculum, worldwide
- Complete solution of the $\mathcal{N}=4$ SYM QFT should provide us a *one-line-all-orders* description of the major part of QCD parton dynamics
- Long live perturbative QCD !

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Extras

Physical coupling

$$A = \sum_{1}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^n A_n, \qquad \boxed{\frac{A^{(g)}}{C_A} = \frac{A^{(q)}}{C_F}} \qquad P_{a\to a[x]+g}(x) = \frac{A(\alpha_s)}{1-x}$$

$$\begin{aligned}
\frac{A_1}{C} &= 4 \\
\frac{A_2}{C} &= 8\left[\left(\frac{67}{18} - \zeta_2\right) C_A - \frac{5}{9}\, n_f\right] \\
\frac{A_3}{C} &= 16 C_A^2 \left(\frac{245}{24} - \frac{67}{9}\,\zeta_2 + \frac{11}{6}\,\zeta_3 + \frac{11}{5}\,\zeta_2^2\right) \\
&\quad + 16 C_F n_f \left(-\frac{55}{24} + 2\,\zeta_3\right) \\
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$\alpha_s^{\text{physical}}$

$=$ *universal* magnitude of double-log enhanced contributions.

Enters in :

large-N asymptotics of anomalous dimensions *and* coefficient functions,

Sudakov quark and gluon form factors,

quark and gluon Regge trajectories,

- threshold resummation,
- singular ($x \to 1$) part of the Drell–Yan K-factor,
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non-diagonal transitions

Second loop $G \to G$ [quark box] $(n_f T_R C_F)$

$$P_G^{(S)} = 8x - 16 + \frac{20}{3}x^2 + \frac{4}{3}x^{-1} - (6 + 10x)\ln x - 2(1 + x)\ln^2 x,$$

$$P_G^{(T)} = 12x - 4 - \frac{164}{9}x^2 + \frac{92}{9}x^{-1} + (10 + 14x + \frac{16}{3}[x^2 + x^{-1}])\ln x + 2(1 + x)\ln^2 x;$$

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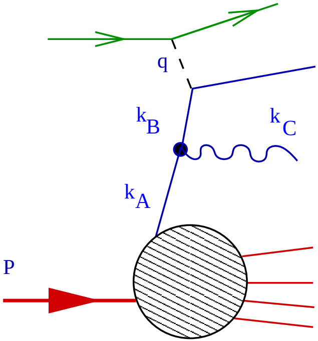
Parton Dynamics turned out to be extremely simple.

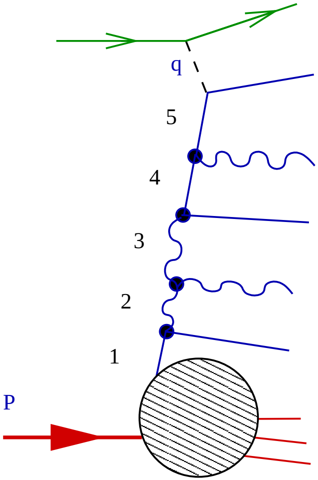
Have a deeper look at parton splitting probabilities
– our evolution Hamiltonian –
to fully appreciate the power of the probabilistic
interpretation of parton cascades

quark–gluon cascades



So long as probability of one extra parton emission is large, one has to consider and treat *arbitrary number* of parton splittings

quark–gluon cascades



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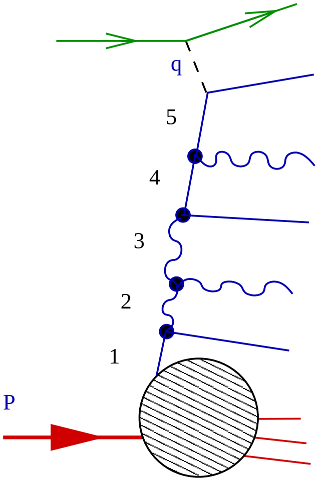

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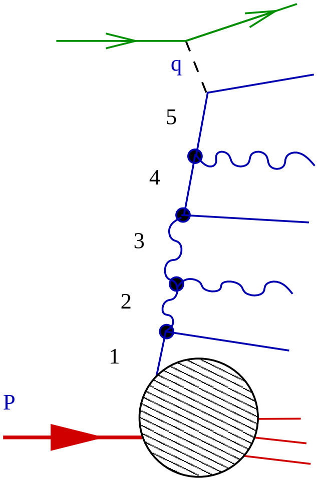

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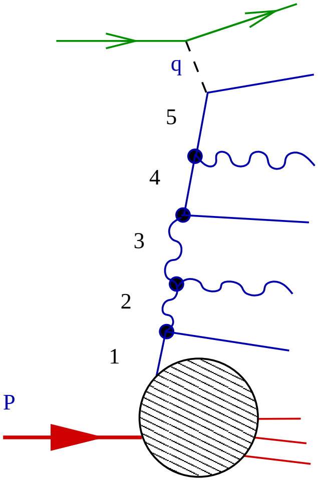

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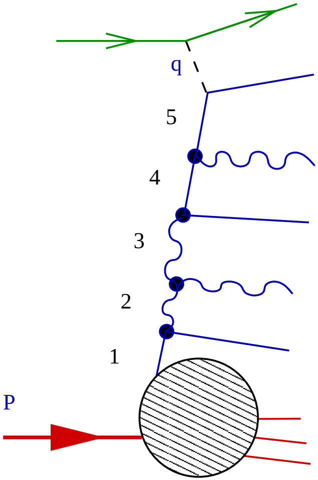

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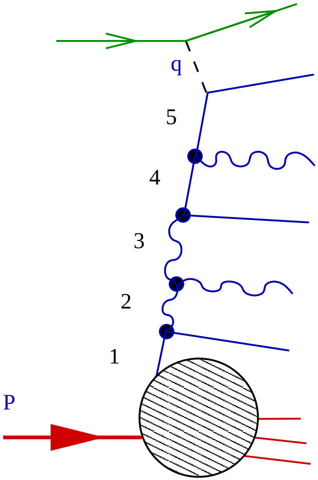

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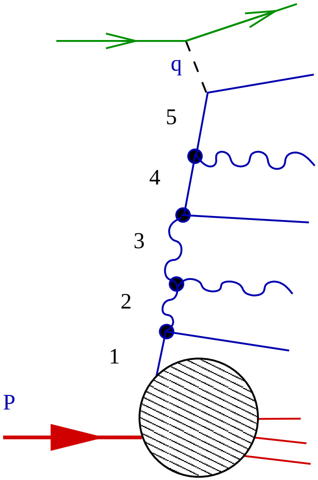

$$\mu^2 \ll k_{1\perp}^2 \ll k_{2\perp}^2 \ll k_{3\perp}^2 \ll k_{4\perp}^2 \ll k_{5\perp}^2 \ll Q^2$$

Four basic splitting processes :

"Hamiltonian" for parton cascades

$$\begin{aligned} P_q^q(z) &= C_F \cdot \frac{1+z^2}{1-z}\,, \\ P_q^g(z) &= C_F \cdot \frac{1+(1-z)^2}{z}\,, \\ P_g^q(z) &= T_R \cdot \left[\,z^2 + (1-z)^2\,\right], \\ P_g^g(z) &= N_c \cdot \frac{1+z^4+(1-z)^4}{z(1-z)} \end{aligned}$$

Logarithmic "evolution time" $\quad d\xi = \frac{\alpha_s}{2\pi}\frac{dk_\perp^2}{k_\perp^2}$

Integrability

1. anomalous dimensions $\Rightarrow$ eigenvalues of the dilatation operator
2. subset of composite operators su(2) = trace(XXXYYXYXXXYYY) can be mapped onto a spin 1/2 system (X = spin up, Y = spin down)
3. At one loop, it is the Hamiltonian of the integrable XXX spin 1/2 chain
4. At higher loops, a more complicated spin chain, but with spins interacting at neighbouring sites (up to a certain distance)
5. At all loops, there are conjectures for the all loop spin Hamiltonian, exploiting the string results, assuming AdS/CFT duality.
6. Integrability = an infinite number of invariants (conserved quantities).

gluons in-between-jets

2- and 3-prong colour antennae are sort of "trivial": coherence being taken care of, the answers turned out to be essentially additive

The case of $2 \to 2$ hard parton scattering is more involved (4 emitters), especially so for gluon–gluon scattering.
Here one encounters 6 (5 for $SU(3)$) colour channels that mix with each other under soft gluon radiation

The difficult quest of sorting out large angle gluon radiation in all orders in $(\alpha_s \log Q)^n$ was set up and solved by George Sterman and collaborators.

Recent (fall 2005) addition to the problem (G.Marchesini & YLD)

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Soft anomalous dimension ,

$$\frac{\partial}{\partial \ln Q} M \propto \left\{ -N_c \ln\left(\frac{t\,u}{s^2}\right) \cdot \hat{\Gamma} \right\} \cdot M, \qquad \hat{\Gamma}\, V_i = E_i V_i .$$

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Three "ain't-so-simple" ones were found to satisfy the cubic equation:

$$\boxed{\left[E_i - \frac{4}{3}\right]^3 - \frac{(1+3b^2)(1+3x^2)}{3}\left[E_i - \frac{4}{3}\right] - \frac{2(1-9b^2)(1-9x^2)}{27} \;=\; 0,}$$

where

$$x \;=\; \frac{1}{N}\,, \qquad b \equiv \frac{\ln(t/s) - \ln(u/s)}{\ln(t/s) + \ln(u/s)}$$

Puzzle of large angle Soft Gluon radiation

Soft anomalous dimension ,

$$\frac{\partial}{\partial \ln Q} M \propto \left\{ -N_c \ln\left(\frac{t\,u}{s^2}\right) \cdot \hat{\Gamma} \right\} \cdot M, \qquad \hat{\Gamma} V_i = E_i V_i .$$

6=3+3. Three eigenvalues are "simple".
Three "ain't-so-simple" ones were found to satisfy the cubic equation:

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Mark the *mysterious symmetry* w.r.t. to $x \to b$: interchanging internal (group rank) and external (scattering angle) variables of the problem . . .

"Malaza puzzle"

Ratio of parton multiplicities in gluon and quark jets in *three loops* :

$$R\frac{\mathcal{N}_g}{\mathcal{N}_q} = 1-\frac{\gamma_0}{6}\left\{1+T(1-2R)\right\}+\left(\frac{\gamma_0}{6}\right)^2\frac{(6-4R-16R^2)\,T^2+(58R-19)\,T-25}{8}$$

where (J.B. Gaffney and A.H. Mueller, 1985)

$$\gamma_0=\sqrt{2N_c\frac{\alpha_s}{\pi}}; \qquad R\equiv\frac{C_F}{N_c}, \quad T\equiv\frac{2n_f\,T_R}{N_c}.$$

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Follows "algebraically" from the one-loop Evolution Equations
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$$R = T = 1 \quad \Longrightarrow \quad \frac{\mathcal{N}_g}{\mathcal{N}_q} = 1$$

small-x ambitions

$$\sqrt{\alpha_s} \Longrightarrow \frac{\alpha_s}{N} + \frac{\alpha_s^2}{N^3} + \frac{\alpha_s^3}{N^5} + \frac{\alpha_s^4}{N^7} + \dots$$

$$\alpha_s \Longrightarrow \alpha_s + \boxed{\frac{\alpha_s^2}{N^2}} + \frac{\alpha_s^3}{N^4} + \frac{\alpha_s^4}{N^6} + \dots$$

$$\alpha_s^{3/2} \Longrightarrow 0 + \frac{\alpha_s^2}{N} + \boxed{\frac{\alpha_s^3}{N^3}} + \frac{\alpha_s^4}{N^5} + \frac{\alpha_s^5}{N^7} + \dots$$

$$\alpha_s^2 \Longrightarrow 0 + \alpha_s^2 + \frac{\alpha_s^3}{N^2} + \frac{\alpha_s^4}{N^4} + \frac{\alpha_s^5}{N^6} + \dots$$

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