Exact sum rules for spectral zeta functions of homogeneous 1D quantum oscillators, revisited

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1D Schrödinger operators: $\hat{H}_N = -\mathrm{d}^2/\mathrm{d}q^2 + |q|^N, \quad q \in \mathbb{R}, \quad N = 1, 2, \ldots$

- Spectral zeta functions $Z_N^\#(s)$ (convergent for Re $s > \frac{1}{2} + \frac{1}{N}$):

$$Z_N(s) = \sum_{k=0}^{\infty} \left(E_k^{[N]}\right)^{-s}, \quad Z_N^{\mathrm{P}}(s) = \sum_{k=0}^{\infty} (-1)^k \left(E_k^{[N]}\right)^{-s}, \quad Z_N^{\pm}(s) = \tfrac{1}{2}[Z_N \pm Z_N^{\mathrm{P}}](s),$$

- Spectral determinants $D_N^\#(\lambda) \stackrel{\mathrm{def}}{=} \det(\hat{H}_N + \lambda)^\#$ (zeta-regularized):

defined through their logarithms, e.g.,

$$\log\det(\hat{H}_N + \lambda) \stackrel{\mathrm{def}}{=} -\partial_s Z_N(s,\lambda)|_{s=0}, \qquad Z_N(s,\lambda) \stackrel{\mathrm{def}}{=} \sum_{k=0}^{\infty} \left(E_k^{[N]} + \lambda\right)^{-s}.$$

Fundamental (convergent) expansion:

$$\log D_N^\#(\lambda) \equiv -Z_N^{\#\prime}(0) - \sum_{n=1}^{\infty} \frac{Z_N^\#(n)}{n}(-\lambda)^n.$$

Our sum rules are *exact identities on the spectral-zeta values at* $n = 0, 1, 2, \ldots$, that mainly result from the *exact WKB method* via *exact functional equations for the spectral determinants.*

WKB: Wentzel–Kramers–Brillouin (initially asymptotic) method

Figure 1: My walk to and from high school took me every day - unknowingly at the time - alongside the house where L. Brillouin was born, in the city of Sèvres, France.

Exact WKB approaches

Exact WKB analysis, assuming 1D potentials analytic in the complex q-plane, sees the usual complex-WKB equations for the Schrödinger wave function $\psi(q)$ as an encoding of its analytical continuation in q, a *fully exact* operation.

Its growth around 1980 owes much to important earlier works:

- microlocal analysis in the *analytic* category: ramified Cauchy problem (Leray), analytic pseudodifferential operators (Boutet de Monvel–Krée), hyperfunctions and microfunctions (Sato–Kawai–Kashiwara);
- Borel-transform approaches to perturbative series (Bender–Wu, Zinn-Justin) and to semiclassical expansions in particular (Balian–Bloch);
- Dingle's treatment of asymptotic series using terminants;
- Sibuya's direct approach yielding exact functional equations for Stokes multipliers.

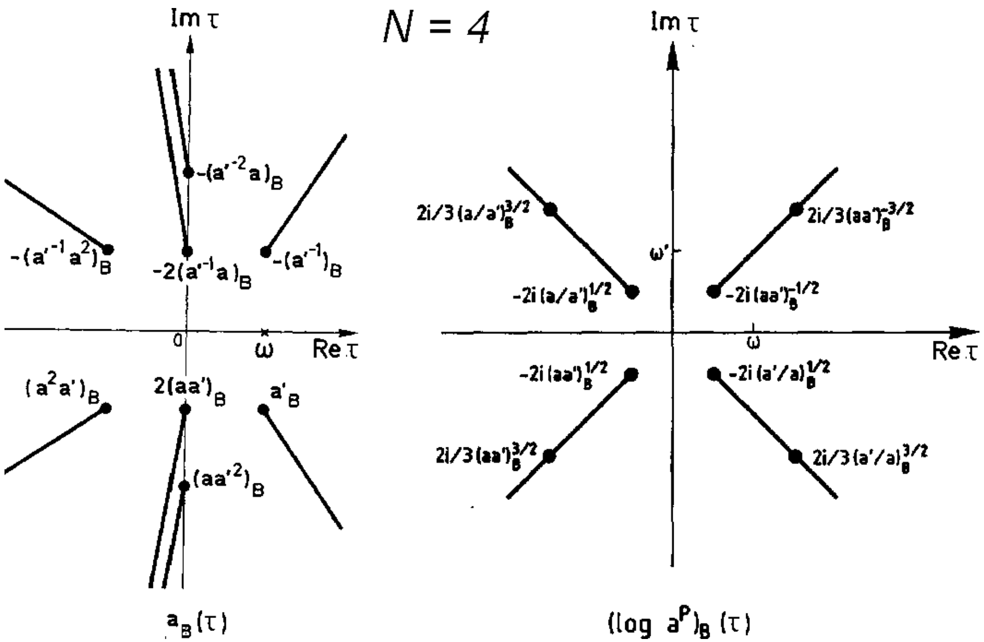
FIrst came partial analyticity results in the Borel plane for the pure quartic potential (Balian–Parisi–AV).

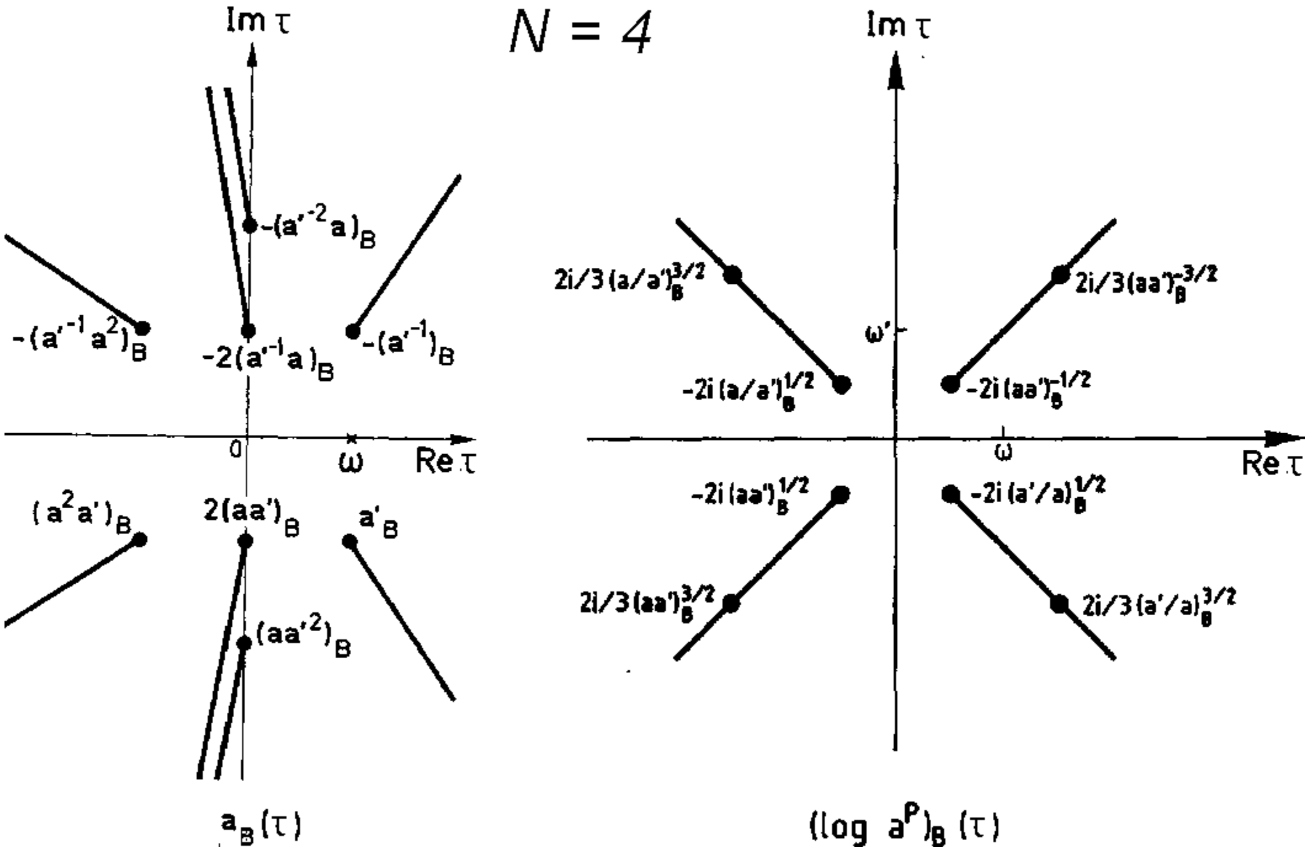
Asymptotic WKB results (large variable: $x \equiv \lambda^{\frac{1}{2}+\frac{1}{N}} \propto 1/\hbar \to +\infty$):

$$\log D(\lambda) - a_0\, x \stackrel{\text{def}}{=} \log a(\lambda) \sim \sum_{m=1}^{\infty} a_m\, x^{1-2m}$$

$$\log D^{\mathrm{P}}(\lambda) - \tfrac{1}{2}\log\lambda \stackrel{\text{def}}{=} \log a^{\mathrm{P}}(\lambda) \sim \sum_{m=1}^{\infty} a_m^{\mathrm{P}}\, x^{-Nm}.$$

Then, with Borel transform specified as $(x^{-\alpha})_{\mathrm{B}} \stackrel{\text{def}}{=} \tau^{\alpha}/\alpha!$, *exact WKB* analysis predicts (quartic example):





Borel resummation converts that information into a *functional relation* (now for general $N \neq 2$, with $\nu \stackrel{\text{def}}{=} \dfrac{1}{N+2}$),

$$\log a^{\mathrm{P}}(\mathrm{e}^{4\mathrm{i}\nu\pi}\,\lambda) - \log a^{\mathrm{P}}(\lambda) = -2\mathrm{i}\ \arcsin\,[D(\mathrm{e}^{4\mathrm{i}\nu\pi}\,\lambda)D(\lambda)]^{-1/2} \quad \Rightarrow$$

$$\sin\Big[\nu\pi + \tfrac{\mathrm{i}}{2}\big(\log D^{\mathrm{P}}(\mathrm{e}^{2\mathrm{i}\nu\pi}\,\lambda) - \log D^{\mathrm{P}}(\mathrm{e}^{-2\mathrm{i}\nu\pi}\,\lambda)\big)\Big] = [D(\mathrm{e}^{2\mathrm{i}\nu\pi}\,\lambda)D(\mathrm{e}^{-2\mathrm{i}\nu\pi}\,\lambda)]^{-1/2},$$

later reduced to an equivalent bilinear functional equation

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\nu\pi} D_N^+(\lambda) D_N^-(\mathrm{e}^{4\mathrm{i}\nu\pi}\lambda) - \mathrm{e}^{-\mathrm{i}\nu\pi} D_N^+(\mathrm{e}^{4\mathrm{i}\nu\pi}\lambda) D_N^-(\lambda) &= 2\mathrm{i} \qquad\qquad (N \neq 2) \\ &= 2\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\pi\lambda/4} \quad (N = 2). \end{aligned}$$

Enforcing the compatibility of such functional relations with the above expansions for $\log D_N^{\#}(\lambda)$ at $\lambda = 0$ generates *polynomial identities* on the zeta-values $Z_N'(0), Z_N^{\#}(n)$ $(n = 1, 2, \ldots)$: “exact-WKB sum rules”.

Even later, the above functional equation gave rise to:
1) exact quantization conditions,
2) the ODE/IM correspondence.

For neither of these further developments do we currently have Borel-plane (resurgent) equivalents.

Explicit cases: $N=2$, $N=1$

$N=2$ (harmonic oscillator):

$\hat{H}_2$ has the explicit spectrum $\{E_k^{[N=2]} = 2k+1\}_{k=0,1,\ldots}$ giving spectral zeta- (= respectively Dirichlet lambda- and beta-) functions:

$$Z_2(s) = (1-2^{-s})\zeta(s), \qquad Z_2^{\mathrm{P}}(s) = \beta(s).$$

and the spectral determinants

$$D_2(\lambda) = 2^{-\lambda/2}\sqrt{2\pi}/\Gamma\left(\tfrac{1+\lambda}{2}\right), \qquad D_2^{\mathrm{P}}(\lambda) = 2\,\Gamma(\tfrac{3+\lambda}{4})/\Gamma(\tfrac{1+\lambda}{4}),$$

The functional equation for the determinants can be split as

$$D_2(\lambda)D_2(-\lambda) = 2\cos\tfrac{1}{2}\pi\lambda, \qquad D_2^{\mathrm{P}}(\lambda)/D_2^{\mathrm{P}}(-\lambda) = \cot\tfrac{1}{4}\pi(1-\lambda),$$

both expressing the *reflection formula for the Gamma function.*

If we now take the logarithms and expand around $\lambda=0$, then identify the resulting sides order by order:

- $\log\left[D_2(\lambda)D_2(-\lambda)\right] = \log\left[2\cos\tfrac{1}{2}\pi\lambda\right] \quad \Rightarrow$

$$2\left[-{Z_2}'(0) - \sum_{m=1}^{\infty}\frac{Z_2(2m)}{2m}\lambda^{2m}\right] \equiv \log 2 - \sum_{m=1}^{\infty}\frac{|G_{2m}|}{4m(2m)!}(\pi\lambda)^{2m},$$

by integrating $\tan\frac{1}{2}\pi\lambda = \sum_{m=1}^{\infty}\frac{|G_{2m}|}{(2m)!}(\pi\lambda)^{2m-1}$ (G_{2m}: Genocchi numbers);

- $\log\left[D_2^{\mathrm{P}}(\lambda)/D_2^{\mathrm{P}}(-\lambda)\right] = \log\left[\cot\tfrac{1}{4}\pi(1-\lambda)\right] \quad \Rightarrow$

$$2\sum_{m=0}^{\infty}\frac{Z_2^{\mathrm{P}}(2m+1)}{2m+1}\lambda^{2m+1} \equiv \sum_{m=0}^{\infty}\frac{|E_{2m}|}{(2m+1)!}(\tfrac{1}{2}\pi\lambda)^{2m+1},$$

by integrating $\sec\frac{1}{2}\pi\lambda = \sum_{m=0}^{\infty}\frac{|E_{2m}|}{(2m)!}(\frac{1}{2}\pi\lambda)^{2m}$ (E_{2m}: Euler numbers).

Then, overall: ${Z_2}'(0) = -\frac{1}{2}\log 2$, and (with B_{2m} : Bernoulli numbers)

$$Z_2(2m) = \frac{\pi^{2m}}{4(2m)!}|G_{2m}| \quad (m>0), \qquad Z_2^{\mathrm{P}}(2m+1) = \frac{(\frac{1}{2}\pi)^{2m+1}}{2(2m)!}|E_{2m}|$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$\zeta(2m) = \frac{(2\pi)^{2m}|B_{2m}|}{2(2m)!}, \qquad \beta(2m+1) = \frac{(\frac{1}{2}\pi)^{2m+1}}{2(2m)!}|E_{2m}|.$$

$N = 1$ (the potential $|q|$)

$\hat{H}_1$ has the spectral determinants $D_1^+(\lambda) = -2\sqrt{\pi}\mathrm{Ai}'(\lambda)$, $D_1^-(\lambda) = 2\sqrt{\pi}\mathrm{Ai}(\lambda)$.

A property exceptional to $N = 1$ determinants is that $\mathrm{Ai}(\lambda)$ obeys a *linear differential equation* (Airy's equation), making its Taylor coefficients at $\lambda = 0$ fully explicit: $\mathrm{Ai}^{(n)}(0) = 3^{(n-2)/3}\pi^{-1} \sin \frac{2}{3}(n+1)\pi\, \Gamma(\frac{n+1}{3})$. Thereby, $\log D_1^\pm(\lambda)$ also expand to any order in closed form:

$$
\begin{array}{rclrcll}
Z_1^{+\prime}(0) &=& \frac{1}{2}\log[\sqrt{3}/(2\rho)], & Z_1^{-\prime}(0) &=& \frac{1}{2}\log[\sqrt{3}\,\rho/2] & \text{(regularized sums)},\\
Z_1^+(1) &=& 0, & Z_1^-(1) &=& -\rho & (\quad \text{''} \qquad\qquad \text{''}\ ),\\
Z_1^+(2) &=& 1/\rho, & Z_1^-(2) &=& \rho^2, & \\
Z_1^+(3) &=& 1, & Z_1^-(3) &=& \frac{1}{2} - \rho^3, & \\
& \vdots & & & \vdots & &
\end{array}
$$

with $\rho \stackrel{\text{def}}{=} D_1^{\mathrm{P}}(0) = -\mathrm{Ai}'(0)/\mathrm{Ai}(0) = 3^{5/6}(2\pi)^{-1}\Gamma(2/3)^2$ (≈ 0.729011133).

Exact-WKB sum rules for the potentials $|q|^N$ $(N = 1, 2, \ldots)$:

countably many *exact identities* for the spectral zeta functions $Z_N^{\#}(s)$, specifically for the values at nonnegative integers $Z_N'(0)$ and $Z_N^{\#}(n)$ $(n = 1, 2, \ldots)$:

$$(n = 0 :) \qquad Z_N'(0) = \log \sin \nu\pi \qquad \left[\nu \stackrel{\text{def}}{=} \frac{1}{N+2}\right]$$

$$\begin{aligned}
-\cot \nu\pi \sin 2\nu\pi \, Z_N^{\mathrm{P}}(1) + \cos 2\nu\pi \, Z_N(1) &= 0 \quad [\text{indeterminacy for } N = 2] \\
-\cot \nu\pi \sin 4\nu\pi \, Z_N^{\mathrm{P}}(2) + \cos 4\nu\pi \, Z_N(2) &= -4 \cos^2 \nu\pi \, Z_N^{\mathrm{P}}(1)^2 \\
-\cot \nu\pi \sin 6\nu\pi \, Z_N^{\mathrm{P}}(3) + \cos 6\nu\pi \, Z_N(3) &= 4 \cos^2 \nu\pi \big[2 \cos^2 \nu\pi \, Z_N^{\mathrm{P}}(1)^3 \\
&\qquad - 3 \cos 2\nu\pi \, Z_N^{\mathrm{P}}(1) Z_N^{\mathrm{P}}(2)\big] \\
&\;\;\vdots \\
-\cot \nu\pi \sin 2n\nu\pi \, Z_N^{\mathrm{P}}(n) + \cos 2n\nu\pi \, Z_N(n) &= \mathcal{P}_{N,n}\{Z_N^{\mathrm{P}}(m)\}_{1 \le m < n} \\
&\;\;\vdots
\end{aligned}$$

with $\mathcal{P}_{N,n}$: homogeneous polynomials of degree n if $\deg[Z_N^{\mathrm{P}}(m)] \stackrel{\text{def}}{=} m$.

Classic closed forms for 1D potentials $|q|^N$ $(N = 1, 2, \ldots)$

For $n = 0$ and 1, *all* $Z_N^{\#}(n)$-values are known *in closed form*, thanks to a *known solution* of the particular ODE $\hat{H}_N\, u = 0 :\ u(q) = \sqrt{q}\, K_\nu(2\nu\, q^{1+N/2})$ (for $q > 0$, then continued) $\left[\nu \stackrel{\text{def}}{=} \dfrac{1}{N+2}\right]$

$$n = 0 : \quad (Z_N^{\mathrm{P}})'(0) = \log\left[\nu^{N\nu}\Gamma(\nu)/\Gamma(1-\nu)\right], \qquad Z_N'(0) = \log \sin \nu\pi\,;$$

(the exact-WKB method only catches $Z_N'(0)$);

$$n = 1 : \quad Z_N^{\mathrm{P}}(1) = \frac{\sqrt{\pi}}{2}(2\nu)^{2N\nu}\frac{\Gamma(2\nu)\Gamma(3\nu)}{\Gamma(1-\nu)\Gamma(2\nu+\frac{1}{2})}, \quad Z_N(1) = \frac{\tan 2\nu\pi}{\tan \nu\pi}\, Z_N^{\mathrm{P}}(1)$$

by Weber–Schafheitlin integral formulae;
(here the exact-WKB method only catches the *ratio* $Z_N(1)/Z_N^{\mathrm{P}}(1)$).

*

$n = 2 :$ now the exact-WKB method adds *new* closed-form results:

$$\cot \nu\pi \sin 4\nu\pi\, Z_N^{\mathrm{P}}(2) - \cos 4\nu\pi\, Z_N(2) = \frac{\pi(2\nu)^{4N\nu}}{4}\left[\frac{\Gamma(\nu)\Gamma(3\nu)}{\Gamma(1-2\nu)\Gamma(2\nu+\frac{1}{2})}\right]^2.$$

Table 1: N even: the left-hand sides in the exact-WKB sum rules give $Z_N(n)$ for $n = \ell(\frac{1}{2}N+1)$, $Z_N^{\mathrm{P}}(n)$ for $N \equiv 2 \bmod 4$, $n = \frac{1}{2}j(\frac{1}{2}N+1)$ (j odd).

$n \backslash N$	2	4	6	8	10	$\cdots$
0	Z_2', $Z_2^{\#\prime}$	Z_4', $Z_4^{\#\prime}$	Z_6', $Z_6^{\#\prime}$	Z_8', $Z_8^{\#\prime}$	Z_{10}', $Z_{10}^{\#\prime}$	$\cdots$
1	Z_2^{P}	$\longleftarrow$	$*, Z_N^{\#}$		$\longrightarrow$	$\cdots$
2	Z_2	$*$	Z_6^{P}	$*$	$*$	$*$
3	Z_2^{P}	Z_4	$*$	$*$	Z_{10}^{P}	$*$
4	Z_2	$*$	Z_6	$*$	$*$	$\ddots$
5	Z_2^{P}	$*$	$*$	Z_8	$*$	$\ddots$
6	Z_2	Z_4	Z_6^{P}	$*$	Z_{10}	$\ddots$
$\vdots$	$\vdots$	$\vdots$ $\ddots$	$\vdots$ $\ddots$	$\vdots$	$\vdots$	$\ddots$

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$n \backslash N$	2	4	6	8	10	$\cdots$
0	$Z_2', Z_2^{\#\prime}$	$Z_4', Z_4^{\#\prime}$	$Z_6', Z_6^{\#\prime}$	$Z_8', Z_8^{\#\prime}$	$Z_{10}', Z_{10}^{\#\prime}$	$\cdots$
1	Z_2^{P}	$\longleftarrow$		$*, Z_N^{\#}$	$\longrightarrow$	$\cdots$
2	Z_2	$*$	Z_6^{P}	$*$	$*$	$*$
3	Z_2^{P}	Z_4	$*$	$*$	Z_{10}^{P}	$*$
4	Z_2	$*$	Z_6	$*$	$*$	$\ddots$
5	Z_2^{P}	$*$	$*$	Z_8	$*$	$\ddots$
6	Z_2	Z_4	Z_6^{P}	$*$	Z_{10}	$\ddots$
$\vdots$	$\vdots$	$\vdots$ $\ddots$	$\vdots$ $\ddots$	$\vdots$	$\vdots$	$\ddots$

Table 2: N even. Top 4 rows: $n = 0,\ 1$ closed-form values; bottom 4 rows ($n = 0$ to 3): exact-WKB sum rules; the two sets share the middle 2 rows.

$N:$	2	4	6
${Z_N^{\rm P}}'(0)$	$\log[2^{-3/2}\Gamma(\frac14)^2/\pi]$	$\log[2^{-7/3}3^{1/3}\Gamma(\frac13)^4/\pi^2]$	$\log[2^{-9/4}\Gamma(\frac18)/\Gamma(\frac78)]$
$Z_N^{\rm P}(1)$	$\pi/4$	$2^{-11/3}3^{-1/3}\Gamma(\frac13)^5/\pi^2$	$2^{-7/4}\,\pi\,\Gamma(\frac54)/\Gamma(\frac78)^2$
$Z_N'(0)$	$-\frac12\log 2$	$-\log 2$	$\frac12\log[(2-\sqrt2)/4]$
$Z_N(1)$	∞	$3\,Z_4^{\rm P}(1)$	$(1+\sqrt2)\,Z_6^{\rm P}(1)$
$n=2$	$Z_2(2) = 2Z_2^{\rm P}(1)^2$ $= \pi^2/8$	$3Z_4^{\rm P}(2)+Z_4(2) = 6Z_4^{\rm P}(1)^2$	$Z_6^{\rm P}(2) = \sqrt2\,Z_6^{\rm P}(1)^2$ $= \frac18\,[\pi\,\Gamma(\frac54)]^2/\Gamma(\frac78)^4$
$n=3$	$Z_2^{\rm P}(3) =$ $2Z_2^{\rm P}(1)^3$ $= \pi^3/32$	$Z_4(3) =$ $\frac92[-Z_4^{\rm P}(1)^3+Z_4^{\rm P}(1)Z_4^{\rm P}(2)]$ $= \frac16 Z_4(1)^3 - \frac12 Z_4(1)Z_4(2)$	$(\sqrt2+1)Z_6^{\rm P}(3)+Z_6(3) =$ $-(3\sqrt2+4)Z_6^{\rm P}(1)^3+$ $3(2+\sqrt2)Z_6^{\rm P}(1)Z_6^{\rm P}(2)$ $= \dfrac{(3+\sqrt2)}{2^{19/4}}\,\dfrac{[\pi\,\Gamma(\frac54)]^3}{\Gamma(\frac78)^6}$

Table 3: N odd: the left-hand sides in the exact-WKB sum rules give $Z_N(n)$ for $n = \ell(N+2)$, $Z_N^{\pm}(n)$ for $n = (\ell + \frac{1}{2})(N+2) \mp \frac{1}{2}$ (ℓ integer).

$n \backslash N$	1	3	5	7	9	$\cdots$
0	$Z_1', Z_1^{\#\prime}$	$Z_3', Z_3^{\#\prime}$	$Z_5', Z_5^{\#\prime}$	$Z_7', Z_7^{\#\prime}$	$Z_9', Z_9^{\#\prime}$	$\cdots$
1	$Z_1^+, Z_1^\#$	$\longleftarrow$	$*, Z_N^\#$		$\longrightarrow$	$\cdots$
2	$Z_1^-, Z_1^\#$	Z_3^+	$*$	$*$	$*$	$*$
3	$Z_1, Z_1^\#$	Z_3^-	Z_5^+	$*$	$*$	$*$
4	$Z_1^+, Z_1^\#$	$*$	Z_5^-	Z_7^+	$*$	$*$
5	$Z_1^-, Z_1^\#$	Z_3	$*$	Z_7^-	Z_9^+	$*$
6	$Z_1, Z_1^\#$	$*$	$*$	$*$	Z_9^-	$\ddots$
7	$Z_1^+, Z_1^\#$	Z_3^+	Z_5	$*$	$*$	$\ddots$
$\vdots$	$\vdots$	$\vdots$ $\ddots$	$\vdots$ $\ddots$			$\ddots$

Table 3: N odd: the left-hand sides in the exact-WKB sum rules give $Z_N(n)$ for $n = \ell(N+2)$, $Z_N^{\pm}(n)$ for $n = (\ell + \frac{1}{2})(N+2) \mp \frac{1}{2}$ (ℓ integer).

$n \backslash N$	1	3	5	7	9	$\cdots$
0	$Z'_1, Z_1^{\#\prime}$	$Z'_3, Z_3^{\#\prime}$	$Z'_5, Z_5^{\#\prime}$	$Z'_7, Z_7^{\#\prime}$	$Z'_9, Z_9^{\#\prime}$	$\cdots$
1	$Z_1^+, Z_1^{\#}$	$\longleftarrow$	$*, Z_N^{\#}$		$\longrightarrow$	$\cdots$
2	$Z_1^-, Z_1^{\#}$	Z_3^+	$*$	$*$	$*$	$*$
3	$Z_1, Z_1^{\#}$	Z_3^-	Z_5^+	$*$	$*$	$*$
4	$Z_1^+, Z_1^{\#}$	$*$	Z_5^-	Z_7^+	$*$	$*$
5	$Z_1^-, Z_1^{\#}$	Z_3	$*$	Z_7^-	Z_9^+	$*$
6	$Z_1, Z_1^{\#}$	$*$	$*$	$*$	Z_9^-	$\ddots$
7	$Z_1^+, Z_1^{\#}$	Z_3^+	Z_5	$*$	$*$	$\ddots$
$\vdots$	$\vdots$	$\vdots$ $\ddots$	$\vdots$ $\ddots$			$\ddots$

Table 4: N odd. $N\!=\!1$ column: $\rho = \frac{3^{5/6}}{2\pi}\Gamma(\frac{2}{3})^2$. $N\!=\!3$ column: $\phi = \frac{1}{2}(\sqrt{5}+1)$.

N :	1	3
${Z_N^{\mathrm{P}}}'(0)$	$-\log\rho$	$\log[5^{-3/5}\Gamma(\frac{1}{5})/\Gamma(\frac{4}{5})]$
$Z_N^{\mathrm{P}}(1)$	ρ	$(\frac{2}{5})^{1/5}\phi^{-1}\sqrt{\pi}\,\Gamma(\frac{6}{5})/\Gamma(\frac{9}{10})$
$Z_N'(0)$	$\log[\sqrt{3}/2]$	$\frac{1}{2}\log[(5-\sqrt{5})/8]$
$Z_N(1)$	$-Z_1^{\mathrm{P}}(1)$	$(2+\sqrt{5})\,Z_3^{\mathrm{P}}(1)$
$n=2$	$Z_1^-(2) = Z_1^{\mathrm{P}}(1)^2$ $= \rho^2$	$Z_3^+(2) = \phi\,Z_3^{\mathrm{P}}(1)^2$ $= (\frac{2}{5})^{2/5}\phi^{-1}\pi\,[\Gamma(\frac{6}{5})/\Gamma(\frac{9}{10})]^2$
$n=3$	$Z_1(3) = \frac{1}{2}Z_1^{\mathrm{P}}(1)^3 + \frac{3}{2}Z_1^{\mathrm{P}}(1)Z_1^{\mathrm{P}}(2)$ $= \frac{5}{2}Z_1(1)^3 - \frac{3}{2}Z_1(1)Z_1(2)$ $= \frac{3}{2} - \rho^3$	$Z_3^-(3) =$ $-(\phi+\frac{1}{2})Z_3^{\mathrm{P}}(1)^3 + \frac{3}{2}Z_3^{\mathrm{P}}(1)Z_3^{\mathrm{P}}(2)$